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Fibonacci-run graphs II: Degree sequences

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ABSTRACT

Fibonacci cubes are induced subgraphs of hypercube graphs obtained by restricting the vertex set to those binary strings which do not contain consecutive 1s. This class of graphs has been studied extensively and generalized in many different directions. Induced subgraphs of the hypercube on binary strings with restricted runlengths as vertices define Fibonacci-run graphs. These graphs have the same number of vertices as Fibonacci cubes, but fewer edges and different graph theoretical properties.

Basic properties of Fibonacci-run graphs are presented in a companion paper, while in this paper we consider the nature of the degree sequences of Fibonacci-run graphs. The generating function we obtain is a refinement of the generating function of the degree sequences, and has a number of corollaries, obtained as specializations. We also obtain several properties of Fibonacci-run graphs viewed as a partially ordered set, and discuss its embedding properties.

1. Introduction

The *n-dimensional hypercube* Q_n is the graph with all binary strings of length n as vertices, where two vertices $v_1v_2 \dots v_n$ and $u_1u_2 \dots u_n$ are adjacent if and only if $v_i \neq u_i$ for exactly one index $i \in [n]$. We have $|V(Q_n)| = 2^n$, and $|E(Q_n)| = n2^{n-1}$. *Fibonacci cubes* Γ_n are a family of subgraphs of Q_n, and were introduced by Hsu [7]. The vertices of Γ_n are the *Fibonacci strings* of length n,

$$\mathcal{F}_n = \{v_1v_2 \dots v_n \in \{0, 1\}^n \mid \; v_i \cdot v_{i+1} = 0 \;, i \in [n-1]\} \;,$$

and two vertices are adjacent if and only if they differ in exactly one coordinate. Shortly, Γ_n is the subgraph of Q_n, induced by the vertices that do not contain consecutive 1s. This family of graphs turned out to be interesting, and has been widely investigated, see for example [3,8,11,12,14].

Recall that a graph isomorphic to a Fibonacci cube is obtained by adding the string 00 to the end of every vertex. We call such binary strings *extended Fibonacci strings*. With this interpretation, we can set

$$V(\Gamma_n) = \{w00 \mid w \in \mathcal{F}_n\} \;,$$

and make two vertices adjacent if and only if they differ in exactly one coordinate.

Instead of considering extended Fibonacci strings as the vertex set, it is possible to consider *run-constrained binary strings*, which are used to define *Fibonacci-run graphs* introduced in [5]. Run-constrained binary strings are strings of 0s

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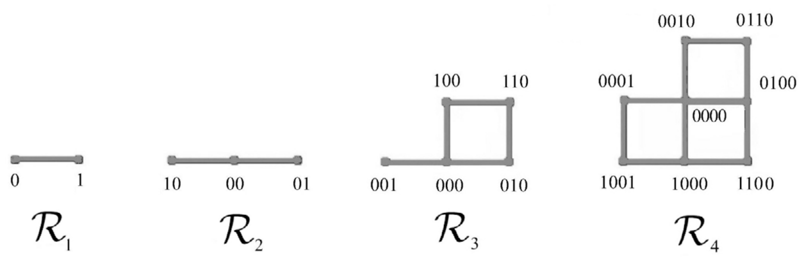

Fig. 1. Fibonacci-run graphs $\mathcal{R}_n$ for $n \in [4]$.

and 1s, in which every run of 1s appearing in the word is immediately followed by a strictly longer run of 0s. Run-constrained strings, together with the null word λ and the singleton 0, are generated freely (as a monoid) by the letters from the infinite alphabet

$$R = 0, 100, 11000, 1110000, \ldots \tag{1}$$

This means that every run-constrained binary string can be written uniquely as a concatenation of zero or more strings from R. Note that run-constrained strings of length $n \geq 2$ must end with 00.

For $n \geq 0$, the Fibonacci-run graph $\mathcal{R}_n$, has the vertex set

$$V(\mathcal{R}_n) = \{w00 \mid w00 \text{ is a run-constrained binary string of length } n+2\} ,$$

and edge set

$$E(\mathcal{R}_n) = \{\{u00, v00\} \mid H(u, v) = 1\} ,$$

where $H(u, v)$ is the Hamming distance between $u, v \in \{0, 1\}^n$, i.e. the number of coordinates in which u and v differ.

We take $\mathcal{R}_0$ to be the graph with a single vertex corresponding to the label 00, which after the removal of the trailing pair of zeros, corresponds to the null word. Clearly, $\mathcal{R}_n$ is a subgraph of Q_{n+2}. However it is more natural to see it as a subgraph of Q_n after suppressing the trailing 00 in the vertex labels of $\mathcal{R}_n$. In this way, we can view the vertices of $\mathcal{R}_n$ without the trailing pair of zeros as

$$V(\mathcal{R}_n) = \{w \mid w00 \text{ is a run-constrained binary string of length } n+2\} .$$

Note that here we use the same notation $\mathcal{R}_n$, even though the obtained graph is just isomorphic to $\mathcal{R}_n$. This is the same kind of convention as viewing Γ_n as a subgraph of Q_{n+2} if one thinks of the vertices as extended Fibonacci strings, or as a subgraph of Q_n as usual by suppressing the trailing 00 of the vertex labels. The graphs $\mathcal{R}_1$ – $\mathcal{R}_4$ with this truncated labeling of run-constrained strings are shown in Fig. 1.

Basic properties of Fibonacci-run graphs such as the number of vertices, the number of edges, diameter, the decomposition into lower dimensional Fibonacci-run graphs, Hamiltonicity and the nature of the asymptotic average degree are studied in [5].

The rest of the paper is organized as follows. After the general preliminaries in Section 2, we consider a decomposition of run-constrained binary strings and prove a result on special collections of words in Section 3. In Section 4, we consider the problem of keeping track of both the up-degree and the down-degree of a run-constrained string. Calculation of the generating function of the up–down degree enumerator polynomials is presented in Section 5. The proof is divided into a number of subsections. In Section 6, we derive a number of consequences of the generating function obtained in Section 5. Among these is the generating function for the degree enumerator polynomials of Fibonacci-run graphs. Following this, in Section 7, we consider a number of parameters for Fibonacci-run graphs as partially ordered sets. These include the rank generating polynomial, enumeration of the maximal elements, and the calculation of the Möbius function. A combinatorial aspect of run-constrained strings, namely the generating function by inversions is presented in Section 8. Embedding and related results are in Section 9, followed by conjectures, questions and further directions in Section 10.

2. Preliminaries

In this section, we present definitions and some known results which are needed in the paper. To avoid possible confusion that may arise due to the initial values, we reiterate that *Fibonacci numbers* are defined as $f_0 = 0, f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. The *Hamming weight* of a binary string u is the number of 1s in u, denoted by $|u|_1$. If a, b are strings, then ab denotes the concatenation of these two strings in that order. Similarly, for a set of strings B, we set $aB = \{ab \mid b \in B\}$. The distance between vertices u and v in a graph G is denoted by $d_G(u, v)$.

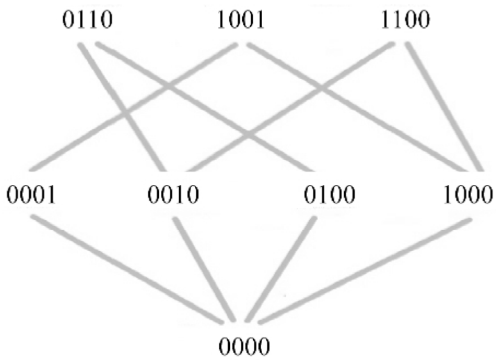

Fig. 2. The Hasse diagram of the Fibonacci-run graph $\mathcal{R}_4$ when viewed as a partially ordered set.

The number of vertices and the number of edges of $\mathcal{R}_n$ for $n \geq 5$ are given by

$$|V(\mathcal{R}_n)| = |V(\Gamma_n)| = f_{n+2} ,$$
$$|E(\mathcal{R}_n)| = |E(\Gamma_n)| - |E(\Gamma_{n-4})| = (3n+4)f_{n-6} + (5n+6)f_{n-5} ,$$

as proved in [5, Lemma 3.1] and [5, Corollary 4.3].

Fibonacci-run graphs can also be viewed as partially ordered sets whose structure is inherited from the Boolean algebra of subsets of $[n]$. The elements here correspond to all binary strings of length n and the covering relation is flipping a 0 to a 1. Therefore in $\mathcal{R}_n$ we have a natural distinction between up- and down-degree of a vertex, denoted by $\deg_{up}(v)$ and $\deg_{down}(v)$. Here $\deg_{up}(v)$ is the number of vertices u in $\mathcal{R}_n$ obtained by changing a 0 to a 1, and $\deg_{down}(v)$ is the number of vertices u in $\mathcal{R}_n$ obtained from v by changing a 1 to a 0. Clearly

$$\deg(v) = \deg_{up}(v) + \deg_{down}(v) .$$

Note that in $\mathcal{R}_n$, $\deg_{down}(v)$ is not necessarily equal to the Hamming weight of v because the vertices of the graph are restricted to be run-constrained binary strings.

The degree sequences, i.e. the nature of the vertices of a given degree in a graph, has been well studied for Fibonacci cubes [9]. Here we keep track of the degree sequences of our graphs $\mathcal{R}_n$ as the coefficients of a polynomial. This polynomial is called the *degree enumerator polynomial* of the graph denoted by $g_n(x)$. The coefficient of x^i in the degree enumerator polynomial is the number of vertices of degree i in $\mathcal{R}_n$. More precisely,

Definition 2.1. The degree enumerator polynomials $g_n(x)$ of $\mathcal{R}_n$ is defined for $n \geq 1$ by

$$g_n(x) = \sum_{v \in \mathcal{R}_n} x^{\deg(v)} . \tag{2}$$

Similar polynomials are defined to keep track of the up- and down-degree sequences as well. In particular the *down-degree enumerator polynomial* and the *up-degree enumerator polynomial* of $\mathcal{R}_n$ are defined as

$$\sum_{v \in \mathcal{R}_n} d^{\deg_{down}(v)} , \quad \text{and} \quad \sum_{v \in \mathcal{R}_n} u^{\deg_{up}(v)} ,$$

respectively.

The generating function of the sequence of down-degree enumerator polynomials of $\mathcal{R}_n$ is

$$\sum_{n \geq 1} t^n \sum_{v \in \mathcal{R}_n} d^{\deg_{down}(v)} .$$

The generating functions of the sequence of up-degree enumerator polynomials and the degree enumerator polynomials are defined similarly.

The nature of the distribution of the up-degrees and the down-degrees are most easily seen from the Hasse diagram of $\mathcal{R}_n$ for which $\deg_{up}(v)$ and $\deg_{down}(v)$ are simply the number of edges emanating up and down from $v \in \mathcal{R}_n$, respectively. Inspecting Fig. 2, we see that the down-degree, up-degree and the degree enumerator polynomials of $\mathcal{R}_4$ are given respectively by

$$1 + 4d + 3d^2, \quad 3 + 2u + 2u^2 + u^4, \quad 5x^2 + 2x^3 + x^4 . \tag{3}$$

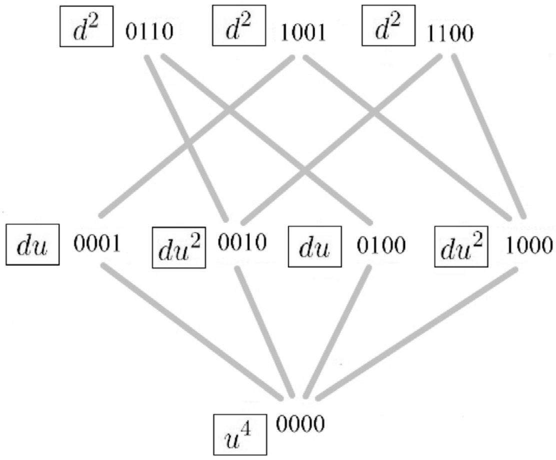

Fig. 3. The calculation of the bivariate up–down degree enumerator polynomial $3d^2 + 2du + 2du^2 + u^4$ of $\mathcal{R}_4$.

It was determined in [5, Proposition 7.1] and [5, Proposition 7.2] that generating function for down-degree enumerator polynomials of $\mathcal{R}_n$ is

$$\frac{t(1 + d + dt + (d^2 - 1)t^2 + d(d - 1)t^3 + d(d - 1)t^4)}{1 - t - t^2 - (d - 1)t^3 - d(d - 1)t^5}\ , \tag{4}$$

and the generating function for up-degree enumerator polynomials of $\mathcal{R}_n$ is

$$\frac{t(1 + u - (u - 2)t - 2ut^2 + t^3 - (u - 1)t^5 - (u - 1)t^6)}{1 - ut - 2t^2 + (2u - 1)t^3 + t^4 - (u - 1)t^5 + (u - 1)t^7}\ . \tag{5}$$

Our general aim in this paper is to study the generating function of the bivariate polynomials

$$\sum_{v \in \mathcal{R}_n} u^{\deg_{up}(v)} d^{\deg_{down}(v)} \tag{6}$$

that we refer to as the *up–down degree enumerator* of $\mathcal{R}_n$. For example, for $n = 4$, this polynomial is given by

$$3d^2 + 2du + 2du^2 + u^4\ , \tag{7}$$

as can be verified by inspecting $\mathcal{R}_4$ in Fig. 3, where the term contributed by each vertex is indicated in a box.

The up–down degree enumerator polynomial is a refinement of the both up- and down-degree enumerator polynomials and of the degree enumerator polynomial $g_n(x)$. For instance the polynomial in (7) specializes to the first polynomial in (3) for $u = 1$, to the second one for $d = 1$, and to the last for $u = d = x$. More generally, the generating function of the up–down degree enumerator polynomials for $\mathcal{R}_n$ specializes to the generating functions (4) and (5) as corollaries, and provide the generating function for the degree enumerator polynomials $\{g_n(x)\}_{n \geq 1}$ itself for $u = d = x$.

Example 2.2. For the graphs $\mathcal{R}_1$ through $\mathcal{R}_8$, the up–down degree enumerator polynomials are as follows:

$$\begin{aligned}
&d + u\\
&2d + u^2\\
&d + d^2 + 2du + u^3\\
&3d^2 + 2du + 2du^2 + u^4\\
&5d^2 + 2d^2u + 3du^2 + 2du^3 + u^5\\
&4d^2 + 2d^3 + 6d^2u + 2d^2u^2 + 4du^3 + 2du^4 + u^6\\
&3d^2 + 7d^3 + 5d^2u + 9d^2u^2 + 2d^2u^3 + 5du^4 + 2du^5 + u^7\\
&2d^2 + 10d^3 + d^4 + 4d^2u + 8d^3u + 7d^2u^2 + 12d^2u^3 + 2d^2u^4 + 6du^5 + 2du^6 + u^8
\end{aligned}$$

We define the generating function GF of the up–down degree enumerator polynomials of $\mathcal{R}_n$ formally as follows:

Definition 2.3. The generating function of the sequence of the up–down degree enumerator polynomials is defined as

$$GF = GF(u, d; t) = \sum_{n \geq 1} t^n \sum_{v \in \mathcal{R}_n} u^{\deg_{up}(v)} d^{\deg_{down}(v)}\ . \tag{8}$$

3. Decomposition of run-constrained strings and a preparatory result

In this section, we present some preliminary results about formal power series, which are needed for the computation of our up–down degree enumerators and their generating function GF.

We also need another description of run-constrained binary strings. Define the set

$$S = \{100, 11000, 1110000, \dots\} ,$$

which is used in the rest of the paper. Every run-constrained binary string consists of words from S interspersed with runs of 0s, including a prefix and a suffix which may also be runs of 0s. Let s^* denote an arbitrary string of zero or more words from S, and s^+ denote an arbitrary string of one or more words from S. So we have $s^+ = ss^*$, and as another example, s^2s^* denotes all strings obtained by the concatenation of two or more words from S. Note that this notation is consistent with its usage in formal languages. Every non-trivial (i.e. not consisting only of 0s) run-constrained binary string can be written in the form

$$0^{i_0}s^+0^{i_1}s^+0^{i_2}\cdots 0^{i_k}s^+0^{i_{k+1}},$$

where $k \geq 0$, $i_0, i_{k+1} \geq 0$, $i_1, i_2, \dots, i_k \geq 1$. The runs $0^{i_1}, 0^{i_2}, \dots, 0^{i_k}$ are called *internal runs*, the initial string 0^{i_0} is the *pre-run*, and the final string $0^{i_{k+1}}$ is the *post-run* of the word. Note that for the latter two, we do not rule out the possibility that they have length zero (i.e. $i_0 = 0$ or $i_{k+1} = 0$.) So in that sense they are not "real" runs like the interior runs of the string, which are the portions in between the letters of S that appear in the word, and must have positive length.

Note that the up–down generating function G of the words in S itself is

$$G = dt^3 + d^2t^5 + d^2t^7 + \cdots = dt^3 + \frac{d^2t^5}{1-t^2} . \tag{9}$$

Here we are keeping the trailing pair of zeros in a run-constrained string into account as the exponent of t so that the exponent of t starts at 3. This in contrast with definition (8) in which the trailing zeros are not considered and the exponent of t starts at 1. The reason for this is that it is somewhat easier to explain the steps of the proof of our Theorem 5.1 if we keep the trailing pair of zeros in the representation of run-constrained strings. The generating function GFX we obtain this way differs from our target generating function GF in (8) in that it accounts for two extra words 0 and 00, and the exponent of t for all other words in GFX is 2 more than those in GF. Algebraically this is expressed in (20), and once we have GFX, we immediately obtain GF from it.

We make us of the following two preparatory results. Consider the alphabet $\Sigma = \{a, b\}$ and let Σ^* denote all words over Σ. The length of $u \in \Sigma^*$ is denoted by $|u|$. Let $|u|_a$ and $|u|_b$ denote the number of occurrences of a and b in u, respectively. Let also $|u|_{aa}, |u|_{ab}, |u|_{ba}, |u|_{bb}$ denote the number of appearances of the words aa, ab, ba, bb in u, respectively. For $n \geq 0$, let $a\Sigma^n a = \{awa \mid w \in \Sigma^*,\ |w| = n\}$. Similarly, we define the sets of strings $a\Sigma^n b$, $b\Sigma^n a$, and $b\Sigma^n b$.

For a word u with $|u| \geq 2$, define

$$m(u) = x^{|u|_a}y^{|u|_b}\alpha^{|u|_{aa}+|u|_{ba}}\beta^{|u|_{ab}+|u|_{bb}} . \tag{10}$$

Example 3.1. For the word $u = aababbaaa \in a\Sigma^7 a$, we have $|u| = 9$, $|u|_a = 6$, $|u|_b = 3$, $|u|_{aa} + |u|_{ba} = 5$, $|u|_{ab} + |u|_{bb} = 3$, and consequently

$$m(u) = x^6y^3\alpha^5\beta^3 .$$

Next, we prove the following proposition, to be used for the calculation of the generating function GF of the up–down degree enumerator polynomials.

Proposition 3.2. *Let $n \geq 0$ be an integer and let $w \in \Sigma^*$, where $\Sigma = \{a, b\}$. Then*

$$\sum_{|w|=n} m(awa) = \alpha x^2(\alpha x + \beta y)^n, \tag{11}$$

$$\sum_{|w|=n} m(awb) = \beta xy(\alpha x + \beta y)^n, \tag{12}$$

$$\sum_{|w|=n} m(bwa) = \alpha xy(\alpha x + \beta y)^n, \tag{13}$$

$$\sum_{|w|=n} m(bwb) = \beta y^2(\alpha x + \beta y)^n. \tag{14}$$

Proof. Consider the first identity. For $n = 0$, there is only one word aa, and both sides are αx^2 in this case. If $n > 0$ and $u = awa$, then note that the number $|u|_{aa} + |u|_{ba}$ is the number of a's in wa and $|u|_{ab} + |u|_{bb}$ is the number of b's in w. Given a word w with $|w|_a = k$ and $|w|_b = n - k$, we calculate $m(u)$ as

$$m(u) = x^{k+2}y^{n-k}\alpha^{k+1}\beta^{n-k} = \alpha x^2x^ky^{n-k}\alpha^k\beta^{n-k} .$$

Since there are $\binom{n}{k}$ such strings w, we obtain

$$\sum_{|w|=n} m(aua) = \alpha x^2 \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \alpha^k \beta^{n-k} = \alpha x^2 (\alpha x + \beta y)^n \ .$$

The proofs of the other identities are similar. □

By summing each identity in Proposition 3.2 over all nonnegative integers n, we obtain the following formulas.

Corollary 3.3. *Let $\Sigma = \{a, b\}$ and m be as defined in (10). Then*

$$\sum_{w \in \Sigma^*} m(awa) = \frac{\alpha x^2}{1-(\alpha x + \beta y)}, \tag{15}$$

$$\sum_{w \in \Sigma^*} m(awb) = \frac{\beta xy}{1-(\alpha x + \beta y)}, \tag{16}$$

$$\sum_{w \in \Sigma^*} m(bwa) = \frac{\alpha xy}{1-(\alpha x + \beta y)}, \tag{17}$$

$$\sum_{w \in \Sigma^*} m(bwb) = \frac{\beta y^2}{1-(\alpha x + \beta y)}. \tag{18}$$

4. Up–down degree polynomials

In the general case we aim to keep track of the terms

$$u^{\deg_{up}(v)} d^{\deg_{down}(v)} t^{|v|} \tag{19}$$

for every run-constrained binary string v. If we are to only keep track of the down-degree of a run-constrained binary string, then the problem is considerably simpler. In this case we are flipping 1s in the string to 0s, and the contribution of every word $s \in S$ in the observed run-constrained binary string v, independently of where it is located in v, is either 1 ($s = 100$) or 2 ($s \in S \setminus \{100\}$).

The difficulty with keeping track of the up-degree arises in the following situations. As an example consider the subword $s_1 0 s_1$ that appears somewhere in the string, where $s_1 = 100$, and use parentheses to highlight the words from S:

$\cdots(100)\,0\,(100)\cdots$

The 0 in the middle can be flipped to 1 in the case that the 100 on the right is followed by a 0:

$\cdots(100)\,0\,(100)\,0\cdots$

but not if the 100 on the right is followed by another word from S:

$\cdots(100)\,0\,(100)\,(11000)\cdots$

As another example, consider the last subword $s \in S$ in the word. It may be followed by zero or more 0s. For example, if $s = 100$,

$\cdots(100), \quad \cdots(100)\,0, \quad \cdots(100)\,00, \quad \cdots(100)\,000, \quad \cdots(100)\,0000, \ldots$

In the first two cases, the 0 immediately to the right of 1 cannot be flipped to a 1 but in all other cases it can be. Additionally, whenever we have $r \geq 3$ trailing 0s in a case like this, $r - 2$ of those can be flipped to a 1.

5. Calculation of *GF*

Theorem 5.1. *The generating function for the up–down degree enumerator polynomials of the graphs $\mathcal{R}_n$ is given by $GF = N_{u,d}/D_{u,d}$ where*

$$\begin{aligned} N_{u,d} = {} & (d+u)t - d(u-2)t^2 + (d^2 - d - 2u)t^3 - (d-2)d(u-2)t^4 \\ & - (d-1)(u-d+du)t^5 - d(d+u-2)t^6 + d(1-2d+2d^2+du-2d^2u)t^7 \\ & - 2(d-1)d^2(u-1)t^8 - (d-1)d^2(d+1)(u-1)t^9 - (d-1)^2 d^2 (u-1)t^{10} \\ & - (d-1)^2 d^2 (u-1) t^{11} \ , \end{aligned}$$

and

$$\begin{aligned} D_{u,d} = {} & 1 - ut - 2t^2 + (2u-d)t^3 + t^4 + (2d - d^2 - u)t^5 + d(du-1)t^7 \\ & + 2(d-1)d^2(u-1)t^9 + (d-1)^2 d^2 (u-1) t^{11} \ . \end{aligned}$$

Proof. We consider the cases according to the type of word from S that precedes the leftmost internal run of 0s, and the word from S that follows the rightmost internal run of 0s. There is also the case, where the run-constrained binary string does not have internal runs of 0s. Thus there are altogether five cases to consider. In the cases (A) through (D), $k \geq 1$, $i_0, i_{k+1} \geq 0$, and $i_1, i_2, \ldots, i_k > 0$. Note that s simply denotes a word from S, and this word may be different at different places in the below schematic description of the cases. Similarly, s^2s^* denotes a string of at least two words from S.

(A) The string is of the form $0^{i_0}\, s\, 0^{i_1} \cdots 0^{i_k}\, s\, 0^{i_{k+1}}$
(B) The string is of the form $0^{i_0}\, s^2s^*\, 0^{i_1} \cdots 0^{i_k}\, s\, 0^{i_{k+1}}$
(C) The string is of the form $0^{i_0}\, s\, 0^{i_1} \cdots 0^{i_k}\, s^2s^*\, 0^{i_{k+1}}$
(D) The string is of the form $0^{i_0}\, s^2s^*\, 0^{i_1} \cdots 0^{i_k}\, s^2s^*\, 0^{i_{k+1}}$
(E) The string has no internal runs of 0s.

Let $GF_A, GF_B, GF_C, GF_D, GF_E$ denote the generating functions of the classes of strings in the cases (A) through (E), respectively. Then the generating function for the run-constrained binary strings with the statistic (19) is

$$GFX = GF_A + GF_B + GF_C + GF_D + GF_E,$$

and the generating function GF for the same statistic for $V(\mathcal{R}_n)$ is (after getting rid of the strings 0 and 00, and removing the trailing 00 in all strings)

$$GF = (GFX - t - t^2)/t^2\ . \tag{20}$$

To simplify the notation in the remaining part of the proof, we define the following quantities:

$$G = dt^3 + \frac{d^2t^5}{1-t^2},$$
$$\alpha = \frac{ut}{1-ut}, \quad \beta = \frac{t}{1-ut},$$
$$x = G, \quad y = \frac{G^2}{1-G}\ .$$

Calculation of GF_E

We first consider the calculation of GF_E, which is the most straightforward. Since there are no internal runs of 0s in case (E), we can partition the strings in (E) into three classes and calculate the generating function for each.

(1) *The extended Fibonacci string is all 0s:*
The generating function for these is

$$t + \frac{t^2}{1-ut}\ . \tag{21}$$

(2) *The extended Fibonacci string has a single word from S:*
The generating function is

$$\left(1 + \frac{2t}{1-ut} + \frac{t^2}{(1-ut)^2}\right) G\ . \tag{22}$$

To see this, note that words in this set are of the form 0^is0^j with $i, j \geq 0$. For $j = 0$ the generating function is the product of G and

$$1 + t + t^2 + ut^3 + u^2t^4 + \cdots = 1 + t + \frac{t^2}{1-ut}\ .$$

For $j = 1$ it is the product of G and

$$t + ut^2 + ut^3 + u^2t^4 + \cdots = t + ut^2 + \frac{ut^3}{1-ut}\ .$$

For $j = 2$ it is the product of G and

$$ut^2 + u^2t^3 + u^2t^4 + u^3t^5 + \cdots = ut^2 + u^2t^3 + \frac{u^3t^5}{1-ut}\ ,$$

and so on. Adding the fractional expressions over j yields

$$\sum_{j\geq 0} \frac{u^jt^{j+2}}{1-ut} = \frac{t^2}{1-ut} \sum_{j\geq 0} u^jt^j = \frac{t^2}{(1-ut)^2}\ . \tag{23}$$

The sum of the pairs of terms $1+t$, $t+ut^2$, $ut^2+u^2t^3$, etc. over $j \geq 0$ is calculated to be

$$1+\frac{2t}{1-ut} \ . \tag{24}$$

Adding the contributions of (23) and (24) proves (22).

(3) *The extended Fibonacci string has two or more words from S*:
We prove that in this case the generating function is

$$\left(1+t+\frac{t+t^2}{1-ut}+\frac{t^2}{1-ut}+\frac{t^3}{(1-ut)^2}\right)\frac{G^2}{1-G} \ . \tag{25}$$

The strings here are of the form $0^i s^2 s^* 0^j$ with $i, j \geq 0$. The generating function of the strings of at least two words from S is $G^2/(1-G)$.
We again calculate the contribution to the generating function for $j = 0, 1, 2, \ldots$. For $j = 0$, we have

$$1+t+t^2+ut^3+u^2t^4+\cdots = 1+t+\frac{t^2}{1-ut},$$

for $j = 1$

$$t+t^2+t^3+ut^4+u^2t^5+\cdots = t+t^2+\frac{t^3}{1-ut},$$

for $j = 2$

$$ut^2+ut^3+ut^4+u^2t^5+\cdots = ut^2+ut^3+\frac{ut^4}{1-ut} \ ,$$

etc. Adding the fractional terms gives

$$\frac{t^2}{1-ut}+\sum_{j\geq 1}\frac{u^{j-1}t^{j+2}}{1-ut} = \frac{t^2}{1-ut}+\frac{t^3}{(1-ut)^2} \ .$$

The sum of the pairs of remaining terms that appear for each j is

$$1+\frac{t}{1-ut}+t+\frac{t^2}{1-ut}$$

and adding these up gives (25).

Finally, adding up and simplifying the contributions of (21), (22) and (25), we obtain

$$GF_E = \frac{t(1+(1-u)t)}{1-ut}+\frac{(1+(1-u)t)^2}{(1-ut)^2}G + \frac{(1+(1-u)t)(1+(1-u)t+(1-u)t^2)}{(1-ut)^2}\frac{G^2}{1-G} \ . \tag{26}$$

Calculation of GF_A

We first consider the generating function GF_A on an example. We will see that the observations made on it can be used on a general string. Take a word of the type

$$0^{i_0}s0^{i_1}s0^{i_2}s^2s^*0^{i_3}s^2s^*0^{i_4}s0^{i_5}s0^{i_6} \ . \tag{27}$$

In this example, $k = 5$, $i_0, i_6 \geq 0$, $i_1, i_2, i_3, i_4, i_5 > 0$. The contribution of the pre-run in such a word is the factor

$$1+ut+\frac{ut^2}{1-ut} \ .$$

Note that we are using the fact that the pre-run is followed by $s0$. The contribution of the first interior run 0^{i_1} is

$$\alpha = \frac{ut}{1-ut}$$

as it is located in the context $s0^{i_1}s0$. The contribution of the second interior run 0^{i_2} is

$$\beta = \frac{t}{1-ut}$$

as it appears in the context $s0^{i_2}s^2s^*$. Continuing, the contribution of the third interior run 0^{i_3} is

$$\beta = \frac{t}{1-ut}$$

because it appears in the context $s^2s^*0^{i_3}s^2s^*$. The contribution of the fourth interior run 0^{i_4} is

$$\alpha = \frac{ut}{1-ut}$$

as it appears in the context $s^2s^*0^{i_4}s$.

We can summarize the situation with the contribution of the internal runs as follows:

the internal runs located in the context of $s0^is0$ and $s^2s^*0^is0$ contribute α,
the internal runs located in the context of $s0^is^2s^*$ and $s^2s^*0^is^2s^*$ contribute β.

Of course each occurrence of s contributes $x = G$ and each occurrence of s^2s^* contributes $y = G^2/(1-G)$. For our example, this leaves the contribution of the last interior run and the post-run. Note that the last interior run contributes β if the post-run has length zero, and α if the post-run has positive length. In the first case the contribution of the post-run is 1, and in the second it is

$$\frac{t}{1-ut}\ .$$

Adding the two contributions, the last interior run and the post-run together contribute

$$\beta + \alpha\frac{t}{1-ut} = \alpha\frac{u^{-1}}{1-ut}\ .$$

Therefore we can "charge" α as the contribution of the last interior run if we make the contribution of the post-run equal to

$$\frac{u^{-1}}{1-ut}\ .$$

We can simplify the situation further. Encode the strings of type (27) by the word

aabbaa

over the two letter alphabet $\{a, b\}$, ignoring the runs of 0s altogether, and encoding s by a and s^2s^* by b. Since we are in case (A), these words start and end with the letter a. Then we have the following:

(1) The contribution of the pre-run is
$$1 + ut + \frac{ut^2}{1-ut}.$$
(2) The contribution of the post-run is
$$\frac{u^{-1}}{1-ut}\ . \tag{28}$$
(3) The contribution of each letter a is $x = G$.
(4) The contribution of each letter b is $y = G^2/(1-G)$.
(5) The contribution of each letter pair aa or ba is α.
(6) The contribution of each letter pair bb or ab is β.

In our example *aabbaa*, there are four a's, two b's, two aa's, one ba, one bb and one ab. So the generating function of the words encoded by *aabbaa* is

$$\left(1 + ut + \frac{ut^2}{1-ut}\right)\left(\frac{u^{-1}}{1-ut}\right)x^4y^2\alpha^3\beta^2\ .$$

Note that $x^4y^2\alpha^3\beta^2 = m(aabbaa)$ as defined in (10). By (11),

$$\sum_{|w|=4} m(awa) = \alpha x^2(\alpha x + \beta y)^4\ .$$

Generalizing these results and using (15), the generating function GF_A is given by

$$GF_A = \left(1 + ut + \frac{ut^2}{1-ut}\right)\left(\frac{u^{-1}}{1-ut}\right)\frac{\alpha x^2}{1-(\alpha x + \beta y)}. \tag{29}$$

Calculation of GF_B

Let us consider how this case differs from the computation of GF_A. This time the contribution of the pre-run is

$$1 + t + t^2 + ut^3 + u^2t^4 + \cdots = 1 + t + \frac{t^2}{1-ut}\ .$$

Again the contribution of the post-run is taken to be (28). Each adjacent pair *aa* or *ba* contributes α, and each pair *bb* or *ab* contributes β. In this case the encoding words start with *b* and end with *a*. Using (12) and (16), we find

$$\begin{aligned} GF_B &= \left(1+t+\frac{t^2}{1-ut}\right)\left(\frac{u^{-1}}{1-ut}\right)\sum_{n\geq 0}\alpha xy(\alpha x+\beta y)^n \\ &= \left(1+t+\frac{t^2}{1-ut}\right)\left(\frac{u^{-1}}{1-ut}\right)\frac{\alpha xy}{1-(\alpha x+\beta y)}\ . \end{aligned} \tag{30}$$

Calculation of GF_C

Here the contribution of the pre-run is as in case (*A*), but the contribution of the post-run is

$$1+\frac{t}{1-ut}\ .$$

The encoding words over $\{a, b\}$ start with *a* and end with *b*. Therefore by (15)

$$GF_C = \left(1+ut+\frac{ut^2}{1-ut}\right)\left(1+\frac{t}{1-ut}\right)\frac{\beta xy}{1-(\alpha x+\beta y)}\ . \tag{31}$$

Calculation of GF_D

In this case the contribution of the pre-run is such as in case (*B*), and the contribution of the post-run is as in case (*C*). The encoding words over $\{a, b\}$ start and end with *b*. Therefore by (17), we have

$$\left(1+t+\frac{t^2}{1-ut}\right)\left(1+\frac{t}{1-ut}\right)\frac{\beta y^2}{1-(\alpha x+\beta y)}\ . \tag{32}$$

Finally, adding up the contributions from (29), (30), (31), (32), (26) (by Mathematica) gives the generating function *GFX* and via (20), the generating function *GF* given in the theorem. □

6. Consequences of the up–down degree enumerator

The degree enumerator polynomials $g_n(x)$ for $\mathcal{R}_1$ through $\mathcal{R}_{10}$ (computed by Mathematica) are as follows:

$$\begin{aligned} g_1(x) &= 2x \\ g_2(x) &= x^2+2x \\ g_3(x) &= x^3+3x^2+x \\ g_4(x) &= x^4+2x^3+5x^2 \\ g_5(x) &= x^5+2x^4+5x^3+5x^2 \\ g_6(x) &= x^6+2x^5+6x^4+8x^3+4x^2 \\ g_7(x) &= x^7+2x^6+7x^5+9x^4+12x^3+3x^2 \\ g_8(x) &= x^8+2x^7+8x^6+12x^5+16x^4+14x^3+2x^2 \\ g_9(x) &= x^9+2x^8+9x^7+15x^6+22x^5+24x^4+14x^3+2x^2 \\ g_{10}(x) &= x^{10}+2x^9+10x^8+18x^7+30x^6+32x^5+39x^4+10x^3+2x^2 \\ g_{11}(x) &= x^{11}+2x^{10}+11x^9+21x^8+39x^7+48x^6+57x^5+42x^4+10x^3+2x^2 \\ g_{12}(x) &= x^{12}+2x^{11}+12x^{10}+24x^9+49x^8+68x^7+81x^6+84x^5+46x^4+8x^3+2x^2 \end{aligned} \tag{33}$$

Making the substitutions $u \to x$ and $d \to x$ in the generating function of the up–down degree enumerator polynomials, we find the generating function of the degree enumerator polynomials of the $\mathcal{R}_n$ to be as follows.

Theorem 6.1. *The generating function for the degree enumerator polynomials of Fibonacci-run graphs*

$$f(t,x) = \sum_{n\geq 1} g_n(x)t^n$$

is given in closed form by N_x/D_x, where

$$\begin{aligned} N_x = xt\Big(&2-(x-2)t+(x-3)t^2-(x-2)^2t^3-x(x-1)t^4-2(x-1)t^5 \\ &-(x-1)(2x^2-x+1)t^6-2x(x-1)^2t^7-x(x-1)^2(x+1)t^8 \\ &-x(x-1)^3t^9-x(x-1)^3t^{10}\Big) \end{aligned}$$

and

$$D_x = 1 - xt - 2t^2 + xt^3 + t^4 - x(x-1)t^5 + x(x-1)(x+1)t^7 + 2x^2(x-1)^2t^9 + x^2(x-1)^3t^{11}.$$

Remark 6.2. To find the generating function of the number of vertices with degree k, we can take $\frac{d^k}{dx^k}f(t,x)$, and then set $x = 0$. The resulting series divided by $k!$ is then the generating function of the number of vertices of degree k in $\mathcal{R}_n$. We can denote this series by

$$\frac{1}{k!}D^k f(t,x)\mid_{x=0} . \tag{34}$$

In other words, the coefficient of t^n in (34) is the number of vertices of degree k in $\mathcal{R}_n$.

In the following examples we calculate the number of vertices of small degree in Fibonacci-run graphs.

Example 6.3. We differentiate the generating function $f(t,x)$ of Theorem 6.1 with respect to x twice using Mathematica, and then set $x = 0$. Dividing the resulting expression by 2 gives

$$\frac{1}{2}D^2 f(t,x)\mid_{x=0} = t^2 + 3t^3 + 5t^4 + 5t^5 + 4t^6 + 3t^7 + \frac{2t^8}{1-t} ,$$

confirming that for $n \geq 8$, $\mathcal{R}_n$ has exactly two vertices of degree 2. If n is even, then these two vertices are $01^{n/2}0^{n/2+1}$ and $1^{n/2}0^{n/2+2}$. If n is odd, then they are $001^{\lfloor n/2\rfloor}0^{\lceil n/2\rceil}$ and $1^{\lceil n/2\rceil}0^{\lfloor n/2\rfloor+2}$.

Example 6.4. Continuing computing with Mathematica, we find

$$\frac{1}{6}D^3 f(t,x)\mid_{x=0} = t^3 + 2t^4 + 5t^5 + 8t^6 + 12t^7 + 14t^8 + 14t^9 + 10t^{10} + \frac{10t^{11}}{1-t^2} + \frac{8t^{12}}{1-t^2}$$

which means that for $n \geq 11$, the number of vertices of degree 3 in $\mathcal{R}_n$ is 10 if n is odd, and 8 if n is even.

Example 6.5. For $k = 4$, we get

$$\frac{1}{24}D^4 f(t,x)\mid_{x=0} = t^4 + 2t^5 + 6t^6 + 9t^7 + 16t^8 + 24t^9 + 39t^{10} + 42t^{11} + 46t^{12} + 39t^{13} + 43t^{14} + \frac{t^{15}(39 + 45t - 35t^2 - 42t^3)}{(1-t^2)^2}$$

from which we compute that for $n \geq 15$, the number of vertices of degree 4 in $\mathcal{R}_n$ is $2n + 9$ if n is odd and $3n/2 + 21$ if n is even.

Example 6.6. Similar calculations give that the number of vertices of degree 5 in $\mathcal{R}_n$ for $n \geq 18$ is $9n - 1$ if n is odd, and $12n - 48$ if n is even.

The presented examples lead to the following conjecture.

Conjecture 6.7. *For a given k, the generating function of the number of vertices of degree k in $\mathcal{R}_n$ is of the form*

$$\frac{p_k(t)}{(1-t^2)^{k+1}}$$

where $p_k(t)$ is a polynomial of degree $\frac{1}{2}(15k+8)$ if k is even, and of degree $\frac{1}{2}(15k+7)$ if k is odd.

We already have the first few degree enumerator polynomials as given in (33). From the denominator of their generating function in Theorem 6.1, we get the following result.

Theorem 6.8. *If $g_n = g_n(x)$ is the degree enumerator polynomial for $\mathcal{R}_n$ as defined in Definition 2.1 with the initial values given by (33), then for $n \geq 12$*

$$g_n = xg_{n-1} + 2g_{n-2} - xg_{n-3} - g_{n-4} + x(x-1)g_{n-5} - x(x^2-1)g_{n-7} - 2x^2(x-1)^2g_{n-9} - x^2(x-1)^3g_{n-11} .$$

Two other specializations of the generating function of the up–down degree enumerator polynomials are obtained in the following way.

(1) Setting $u = 1$, we obtain the generating function of the down-degree enumerator polynomials in (4), which we had already computed [5, Proposition 7.1].
(2) Setting $d = 1$, we obtain the generating function of the up-degree enumerator polynomials shown in (5), also already computed in [5, Proposition 7.2].

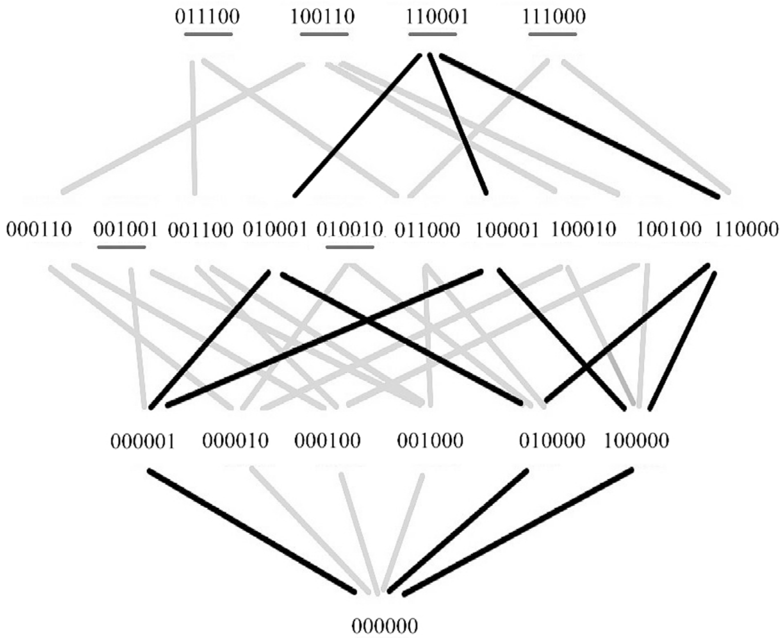

Fig. 4. The Hasse diagram of the graph $\mathcal{R}_6$ as a poset. The trailing 00 of the vertex labels have been truncated. The dark lines show the interval [000000, 110001] which is isomorphic to the 3-dimensional hypercube Q_3. The six maximal elements of the poset are underlined.

Note that Theorem 5.1 can be used to recalculate another result. The down-degree enumerator generating function differentiated with respect to d is

$$\frac{t(1-t^2)(1+(2d-1)t^2)}{(1-t-t^2-(d-1)t^3-d(d-1)t^5)^2}$$

which for $d = 1$ gives the generating function of the number of edges of $\mathcal{R}_n$, already determined in [5] as

$$\frac{t(1-t^4)}{(1-t-t^2)^2} \, . \tag{35}$$

Clearly, this result could also be obtained by differentiating the up-degree enumerator generating function with respect to u, and then evaluating it at $u = 1$.

7. Fibonacci-run graphs as partially ordered sets

Naturally, one might view the vertices of a Fibonacci-run graph as a partially ordered set (*poset* for short). This poset is defined as $(\mathcal{R}_n, \leq)$, where the covering relation is given as follows: for $u, v \in \mathcal{R}_n$, v covers u if and only if v is obtained from u by flipping a 0 in u to a 1. The binary relation "$\leq$" is the transitive closure of this covering relation. We have shown the Hasse diagram of $\mathcal{R}_4$ in Fig. 2. Fig. 4 depicts the Hasse diagram of $\mathcal{R}_6$.

Note that the number of vertices v that cover a vertex $u \in \mathcal{R}_n$ is precisely the up-degree $\deg_{up}(u)$. The number of vertices $u \in \mathcal{R}_n$ that are covered by v is the down-degree $\deg_{down}(v)$ of the vertex v, as indicated in Fig. 3.

Notice that $(\mathcal{R}_n, \leq)$ is a ranked poset, where the rank of $u \in \mathcal{R}_n$ is its Hamming weight $|u|_1$. Since the number of vertices in $\mathcal{R}_n$ of weight w is given by

$$\binom{n-w+1}{w} \tag{36}$$

for $0 \leq w \leq \lceil n/2 \rceil$ [5, Corollary 3.2], the rank generating polynomial of the poset $\mathcal{R}_n$ is

$$F(\mathcal{R}_n, x) = \sum_{k=0}^{\lceil n/2 \rceil} \binom{n-k+1}{k} x^k \, . \tag{37}$$

Next we consider the maximal elements of $\mathcal{R}_n$ and determine its Möbius function.

7.1. Maximal elements

The maximal elements of a poset are those elements which are not smaller than any other element of the set. The following table lists the run-constrained binary strings of lengths $3, 4, \ldots, 8$ which correspond to maximal elements of $\mathcal{R}_1$ through $\mathcal{R}_6$, viewed as a poset.

$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$	$\mathcal{R}_6$
100	0100	00100	011000	0001100	00100100
	1000	11000	100100	0100100	01001000
			110000	1000100	01110000
				1001000	10011000
				1110000	11000100
					11100000

The sequence $\{M_n\}_{n\geq 1}$ of the number of maximal elements of $\mathcal{R}_n$ starts as

$$1, 2, 2, 3, 5, 6, 10, 13, 20, 27, 40, 56, 80, \ldots$$

We can obtain the generating function of this sequence by setting $d = 1$ and $u = 0$ in the up–down degree enumerator generating function in Theorem 5.1. This specialization gives the following result.

Corollary 7.1. *If M_n denotes the number of maximal elements of the poset of $\mathcal{R}_n$, then the generating function of the sequence $\{M_n\}_{n\geq 1}$ is given by*

$$\sum_{n\geq 1} M_n t^n = \frac{t(1+2t-2t^3+t^5+t^6)}{1-2t^2-t^3+t^4+t^5-t^7} \,.$$

7.2. The Möbius function

We can easily compute the Möbius function μ of the poset $(\mathcal{R}_n, \leq)$, since every interval $[u, v]$ is isomorphic to a cube (Boolean algebra) and therefore has the same Möbius function. Denoting the weights of u and v by $|u|_1$ and $|v|_1$, we have

$$\mu(u, v) = \begin{cases} (-1)^{|v|_1-|u|_1} & \text{if } u \leq v, \\ 0 & \text{if } u \not\leq v. \end{cases}$$

8. Inversion generating function

In this section, we present an observation about a combinatorial property of run-constrained binary strings that we use to define Fibonacci-run graphs. For $n \geq 1$, consider the inversion enumerator polynomial $Q_n(x, q)$ defined by

$$Q_n(x, q) = \sum_{w\in V(\mathcal{R}_n)} x^{|w|_1} q^{\mathrm{inv}(w)}, \tag{38}$$

where $\mathrm{inv}(w)$ denotes the number of inversions of w and $|w|_1$ is the Hamming weight, or the rank of w. Recall that for $w = w_1 w_2 \cdots w_n$, $\mathrm{inv}(w)$ is the number of pairs $1 \leq i < j \leq n$ with $w_i > w_j$. Note that in (38), the trailing pair of zeros are not taken in the representation of the vertices $V(\mathcal{R}_n)$, as indicated in Fig. 1. A few of these polynomials are as shown below:

$$\begin{aligned}
Q_1(x, q) &= 1 + x, \\
Q_2(x, q) &= 1 + (q + 1)x, \\
Q_3(x, q) &= 1 + \left(q^2 + q + 1\right) x + x^2, \\
Q_4(x, q) &= 1 + \left(q^3 + q^2 + q + 1\right) x + \left(2q^2 + 1\right) x^2, \\
Q_5(x, q) &= 1 + \left(q^4 + q^3 + q^2 + q + 1\right) x + \left(2q^4 + q^3 + 2q^2 + 1\right) x^2 + x^3 \,.
\end{aligned}$$

Proposition 8.1. *Set $Q_{-2}(x, q) = Q_{-1}(x, q) = Q_0(x, q) = 1$ with $Q_{-n}(x, q) = 0$ for $n \geq 3$. Then for $n \geq 1$*

$$Q_n(x, q) = \sum_{k\geq 0} x^k Q_{n-1-2k}(xq^{k+1}, q) \,. \tag{39}$$

Proof. The proof is a consequence of the fundamental decomposition of Fibonacci-run graphs given in Theorem [5, Lemma 4.1]. Note that the terms in the sum vanish for $k \geq \lceil \frac{n+1}{2} \rceil$. $\square$

Define

$$H(x,q;t)=\sum_{n\geq 1} Q_n(x,q)t^n \ .$$

Then a consequence of (39) is the functional identity

$$H(x,q;t)=\frac{t(1+x+xt)}{1-xt^2}+t\sum_{k\geq 0} x^k t^{2k} H(xq^{k+1},q;t)\ . \tag{40}$$

This identity can be proved by multiplying both sides of (39) by t^n for $n \geq 1$, summing over n, and changing the order of summation on the right hand side. We omit the details.

We also note that substituting $q = 1$ in $Q_n(x, q)$ gives the rank generating polynomial $F(\mathcal{R}_n, x)$ of $\mathcal{R}_n$ defined in (37). In other words, the coefficient of x^k becomes the number of words in $V(\mathcal{R}_n)$ with weight k as given by (36). For instance, for $n = 7$ and $q = 1$, we have

$$1+7x+15x^2+10x^3+x^4,$$

for which the coefficients are

$$\binom{8}{0}=1,\ \binom{7}{1}=7,\ \binom{6}{2}=15,\ \binom{5}{3}=10,\ \binom{4}{4}=1.$$

Indeed, taking $q = 1$ in (40), we see that H satisfies

$$H(x,1;t)=\frac{t(1+x+xt)}{1-xt^2}+\frac{tH(x,1;t)}{1-xt^2}\ ,$$

so that

$$H(x,1;t)=\frac{t(1+x+xt)}{1-t-xt^2}\ .$$

We thus get the generating function of the rank generating polynomials $F(\mathcal{R}_n, x)$ (see (37)) of Fibonacci-run graphs as a poset as

$$\begin{aligned}\frac{t(1+x+xt)}{1-t-xt^2} &= \sum_{n\geq 1} F(\mathcal{R}_n,x)t^n \\ &= (1+x)t+(1+2x)t^2+(1+3x+x^2)t^3+(1+4x+3x^2)t^4+\cdots\end{aligned}$$

Inversions and the major index statistics of a similar flavor for a class of related Fibonacci strings can be found in [4].

9. Embedding related results

We can encode a binary string of length n as a run-constrained binary string of length $3n+1$. Let $s_i = 1^i 0^{i+1}$ for $i \geq 1$. Given a binary string with k runs of 1s,

$$w = 0^{j_0}1^{i_1}0^{j_1}1^{i_2}\cdots 1^{i_k}0^{j_k},$$

we first encode the runs as

$$0s_{j_0}s_{i_1}s_{j_1}\cdots s_{i_k}s_{j_k}$$

if $j_0 > 0$ (i.e. the binary string starts with 0), and by

$$s_{i_1}s_{j_1}\cdots s_{i_k}s_{j_k}$$

if $j_0 = 0$ (i.e. the binary string starts with 1). Then, if necessary, we append 0s at the end to make the length of the encoding $3n + 1$.

For $n = 3$, this works as follows

000	→	$0s_3$	→	01110000 00
001	→	$0s_2s_1$	→	01100010 0 0
010	→	$0s_1s_1s_1$	→	0100100100
100	→	s_1s_2	→	10011000 00
011	→	$0s_1s_2$	→	01001100 0 0
101	→	$s_1s_1s_1$	→	10010010 0 0
110	→	s_2s_1	→	11000100 00
111	→	s_3	→	1110000 000

This encoding can be carried out for arbitrary n, resulting in an embedding of the hypercube Q_n into $\mathcal{R}_{3n-1}$ (the trailing 00 is not included in the vertex labels here).

Question 9.1. *What is the dilation of this embedding, i.e. the value of*

$$\max_{uv\in E(Q_n)} d_{\mathcal{R}_{3n-1}}(u', v') ,$$

where the prime denotes the image under the encoding described above?

We note that although it is intuitive, the embedding described above does not seem to give the smallest dimensional Fibonacci-run graph in which it is possible to embed Q_n.

The study of hypercubes of various dimensions which are subgraphs of a Fibonacci-run graphs are interesting in its own right. An analogous question has been studied for Fibonacci cubes, cf. results about cube polynomial listed in [8], and generalizations in [13].

Let $h_{n,k}$ denote the number of k-dimensional hypercubes Q_k in $\mathcal{R}_n$. A corollary of [5, Proposition 8.2], obtained by taking $q = 1$ in that proposition is the following. The generating function of the cube polynomials of $\mathcal{R}_n$ is given by

$$\sum_{n\geq 1} t^n \sum_{k\geq 0} h_{n,k}x^k = \frac{t(2+x+(x+1)t+x(x+2)t^2+x(x+1)t^3+x(x+1)t^4)}{1-t-t^2-xt^3-x(x+1)t^5} . \tag{41}$$

In the series expansion of this generating function in powers of t, the largest m for which the term x^m appears as a coefficient of t^n, gives the dimension of the largest hypercube Q_m that can be embedded in $\mathcal{R}_n$. Calculations on (41), using high order derivatives with respect to x (with Mathematica) suggest that for $m \geq 0$, the hypercube Q_{2m+1} embeds in $\mathcal{R}_{5m+1}$ and the hypercube Q_{2m+2} embeds in $\mathcal{R}_{5m+3}$, and these are the smallest possible Fibonacci-run graphs with this property. So it appears that the hypercube graph Q_n can be embedded into $\mathcal{R}_{\lceil (5n-4)/2\rceil}$, and this is the smallest possible run graph with this property.

Conjecture 9.2. *The smallest m for which the hypercube graph Q_n can be embedded into $\mathcal{R}_m$ is $m = \left\lceil \frac{5n-4}{2} \right\rceil$.*

10. Further directions

In the final section, we list various questions and conjectures, which are of interest in the further study of Fibonacci-run graphs. These are in addition to Conjecture 9.2 on the determination of the smallest dimensional Fibonacci-run graph that contains Q_n, Question 9.1 on the dilation of the mapping described at the start of Section 9, and Conjecture 6.7 on the form of the generating function of the number of vertices of a given degree in $\mathcal{R}_n$.

For Conjecture 6.7, the extraction of the coefficients in the generating function of the degree enumerator polynomials $f(t, x)$ of Theorem 6.1, one may use the mechanism described in Remark 6.2, and the examples presented afterwards, along with the Leibniz formula for higher derivatives and the reciprocal differentiation result in [10].

Considering the results obtained in Examples 6.3–6.6 in Section 6, the following general question arises.

Question 10.1. *What is the number of vertices of degree k in $\mathcal{R}_n$?*

The analysis and the conjecture on the diameter of $\mathcal{R}_n$ can be found in [5]. In relation to this, a natural question that arises is the following:

Question 10.2. *What is the radius of $\mathcal{R}_n$?*

It may be possible to consider Question 10.2 in conjunction with the analysis for the exact diameter of $\mathcal{R}_n$ in [5, Section 5].

An irregularity measure of graphs was defined by Albertson [1], and recently studied for various families of graphs in [2], and generalized to a polynomial enumerator in [6].

Question 10.3. *What is the irregularity, or more generally the irregularity polynomial, of $\mathcal{R}_n$, as defined in [6]?*

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References

[1] M.O. Albertson, The irregularity of a graph, Ars Combin. 46 (1997) 219–225.

[2] Y. Alizadeh, E. Deutsch, S. Klavžar, On the irregularity of π-permutation graphs, Fibonacci cubes, and trees, Bull. Malays. Math. Sci. Soc. 43 (6) (2020) 4443–4456.

[3] J. Azarija, S. Klavžar, Y. Rho, S. Sim, On domination-type invariants of Fibonacci cubes and hypercubes, Ars Math. Contemp. 14 (2) (2018) 387–395.
[4] O. Eğecioğlu, Statistics on restricted Fibonacci words, Trans. Comb. 10 (1) (2021) 31–42.
[5] O. Eğecioğlu, V. Iršič, Fibonacci-run graphs i: basic properties, Discrete Appl. Math. 295 (2021) 70–84.
[6] O. Eğecioğlu, E. Saygı, Z. Saygi, The irregularity polynomials of Fibonacci and Lucas cubes, Bull. Malays. Math. Sci. Soc. 44 (2) (2021) 753–765.
[7] W.J. Hsu, Fibonacci cubes - new interconnection topology, IEEE Trans. Parallel Distrib. Syst. 4 (1993) 3–12.
[8] S. Klavžar, Structure of fibonacci cubes: A survey, J. Comb. Optim. 25 (2013) 4.
[9] S. Klavžar, M. Mollard, M. Petkovšek, The degree sequence of Fibonacci and Lucas cubes, Discrete Math. 311 (14) (2011) 1310–1322.
[10] R.A. Leslie, How not to repeatedly differentiate a reciprocal, Amer. Math. Monthly 98 (8) (1991) 732–735.
[11] M. Mollard, Non covered vertices in Fibonacci cubes by a maximum set of disjoint hypercubes, Discrete Appl. Math. 219 (2017) 219–221.
[12] K.S. Savitha, A. Vijayakumar, Some diameter notions of Fibonacci cubes, Asian-Eur. J. Math. 13 (3) (2020) 2050057, 9.
[13] E. Saygı, O. Eğecioğlu, q-cube enumerator polynomial of Fibonacci cubes, Discrete Appl. Math. 226 (2017) 127–137.
[14] E. Saygı, O. Eğecioğlu, Boundary enumerator polynomial of hypercubes in Fibonacci cubes, Discrete Appl. Math. 266 (2019) 191–199.