Fibonacci-run graphs I: Basic properties

Ömer Eğecioğlu [a], Vesna Iršič [b,c,*]

[a] *Department of Computer Science, University of California Santa Barbara, Santa Barbara, CA 93106, USA*
[b] *Faculty of Mathematics and Physics, University of Ljubljana, Slovenia*
[c] *Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia*

ARTICLE INFO	ABSTRACT
	Among the classical models for interconnection networks are hypercubes and Fibonacci cubes. Fibonacci cubes are induced subgraphs of hypercubes obtained by restricting the vertex set to those binary strings which do not contain consecutive 1s, counted by Fibonacci numbers. Another set of binary strings which are counted by Fibonacci numbers are those with a restriction on the runlengths. Induced subgraphs of the hypercube on the latter strings as vertices define *Fibonacci-run graphs*. They have the same number of vertices as Fibonacci cubes, but fewer edges and different graph theoretical properties. We obtain properties of Fibonacci-run graphs including the number of edges, the analogue of the fundamental recursion, the average degree of a vertex, Hamiltonicity, special degree sequences, and the number of hypercubes they contain. A detailed study of the degree sequences of Fibonacci-run graphs is interesting in its own right and is reported in a companion paper. © 2021 Elsevier B.V. All rights reserved.

1. Introduction

The *n-dimensional hypercube* Q_n is the graph on the vertex set

$$\{0, 1\}^n = \{v_1 v_2 \ldots v_n \mid v_i \in \{0, 1\}\},$$

where two vertices $v_1 v_2 \ldots v_n$ and $u_1 u_2 \ldots u_n$ are adjacent if $v_i \neq u_i$ for exactly one index $i \in [n]$. In other words, vertices of Q_n are all possible strings of length n consisting only of 0s and 1s, and two vertices are adjacent if and only if they differ in exactly one coordinate or "bit". Clearly, $|V(Q_n)| = 2^n$, and $|E(Q_n)| = n2^{n-1}$.

A well studied subfamily of hypercubes are *Fibonacci cubes* Γ_n which were introduced by Hsu [6] as an alternate interconnection topology. The vertex set Γ_n consists of the *Fibonacci strings* of length n,

$$\mathcal{F}_n = \{v_1 v_2 \ldots v_n \in \{0, 1\}^n \mid v_i v_{i+1} = 0 , i \in [n-1]\} ,$$

and two vertices are adjacent if and only if they differ in exactly one coordinate. In other words, Γ_n is the subgraph of Q_n, induced by the vertices that do not contain consecutive 1s. This family of graphs has proved to have an interesting structure and was studied extensively.

As mentioned in the seminal paper, the Fibonacci cubes can be viewed as an interconnection topology and can be applied to fault-tolerant computing [6]. A survey of a variety of properties of Fibonacci cube graphs [10] presents not just

---

the fundamental decomposition of Fibonacci cubes, but also its Hamiltonian properties, and many enumeration results. For some very recent results, see [1,14,16,19]. Following similar ideas, other graph families have also been introduced and studied, such as Lucas cubes [15], generalized Fibonacci cubes [7], k-Fibonacci cubes [5] and daisy cubes [12].

Note that a Fibonacci cube can equivalently be defined by adding 00 to the end of the binary representation of every vertex. We can call such binary strings *extended Fibonacci strings*. Actually, in an extended Fibonacci string, the rightmost zero corresponds to the Fibonacci number f_0 and the second to last zero corresponds to f_1 in the encoding of the so called Zeckendorf or canonical representation of integers [21]. In this representation, f_0 is not needed since it is zero, and f_1 is not needed because f_2 is already 1, and the inclusion of another 1 would prevent uniqueness of the Zeckendorf expansion. We can call this representation with the additional 0s the *extended Zeckendorf representation*.

With this interpretation

$$V(\Gamma_n) = \{w00 \mid w \in \mathcal{F}_n\}$$

and two vertices are adjacent if they differ in exactly one coordinate. Recall that the *Hamming distance* $H(u, v)$ between strings $u, v \in \{0, 1\}^n$ is the number of coordinates in which u and v differ. Using this, the edge set of Γ_n can be described as

$$E(\Gamma_n) = \{\{u00, v00\} \mid H(u, v) = 1\} .$$

Observe that extended Fibonacci strings defined above, together with the null word λ and the singleton 0, are generated freely (as a monoid) by the infinite alphabet

$$F = 0, 100, 10100, 1010100, \dots$$

This means that every $v \in V(\Gamma_n)$ can be written uniquely as a concatenation of zero or more strings from F.

With this in mind we introduce a new family of graphs. Instead of considering extended Fibonacci strings as the vertex set to define our communication network, we consider *run-constrained binary strings*. These are strings of 0s and 1s, in which every run (sometimes called a *block*) of 1s appearing in the word is immediately followed by a strictly longer run of 0s. Such run-constrained strings, together with the null word λ and the singleton 0, are generated freely by the letters from the alphabet

$$R = 0, 100, 11000, 1110000, \dots \tag{1}$$

Note that run-constrained strings of length $n \geq 2$ must end with 00.

These strings allow us to define the *Fibonacci-run graph* $\mathcal{R}_n$, parametrized by $n \geq 0$, in the following fashion. The vertex set of $\mathcal{R}_n$ is

$$V(\mathcal{R}_n) = \{w00 \mid w00 \text{ is a run-constrained string of length } n + 2\} ,$$

and its edge set is

$$E(\mathcal{R}_n) = \{\{u00, v00\} \mid H(u, v) = 1\} .$$

The term Fibonacci-run graph makes sense, since $|V(\mathcal{R}_n)| = |V(\Gamma_n)|$ is the $(n + 2)$th Fibonacci number f_{n+2}, as we will later show in Section 3.

Clearly, $\mathcal{R}_n$ is a subgraph of Q_{n+2}, but it is more natural to see it as a subgraph of Q_n (after suppressing the tailing 00 in the vertices of $\mathcal{R}_n$). In fact, from now on, we will view the vertices of a Fibonacci-run graph without the trailing pair of zeros as

$$V(\mathcal{R}_n) = \{w \mid w00 \text{ is a run-constrained binary string of length } n + 2\} .$$

This is the same kind of a convention as viewing Γ_n as a subgraph of Q_{n+2} if we think of the vertices as extended Fibonacci strings, or as a subgraph of Q_n as usual by suppressing the trailing 00 of the vertex labels in the extended Zeckendorf representation.

Additionally, we define $\mathcal{R}_0$ to be isomorphic to K_1, and its only vertex corresponding to the label 00, which after the removal of the trailing pair of zeros corresponds to the null word.

We start the study of the Fibonacci-run graphs $\mathcal{R}_n$ by considering the basic properties of these graphs, such as the number of edges, diameter, decomposition, Hamiltonicity, asymptotic average degree, and compare these graph parameters with the known results for Γ_n.

The rest of the paper is organized as follows. After the preliminaries in Section 2, we consider the nature of the run-constrained binary strings in Section 3, and calculate the number of edges of $\mathcal{R}_n$. This is followed by the graph-theoretic decomposition of $\mathcal{R}_n$, which is actually what we make use of for our results on the number of edges. In Section 5, we consider the diameter of Fibonacci-run graphs, followed by asymptotic results in Section 6. In Section 7, we use the properties of run-constrained strings to calculate the generating functions of the distribution of vertices with special types of degrees in $\mathcal{R}_n$. In Section 8, we prove that Fibonacci-run graphs are partial cubes for only small values of n, and provide a count of the induced hypercubes Q_k contained in $\mathcal{R}_n$. Finally, in Section 9 we study the Hamiltonicity of $\mathcal{R}_n$, and conclude with conjectures and further directions for research. Figures of a few large Fibonacci-run graphs are given in the Appendix.

2. Preliminaries

In this section, we give definitions, provide notation, and indicate the known results that are needed in the paper. To avoid possible confusion, recall that *Fibonacci numbers* are defined as $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. They are closely related to the *golden ratio* $\varphi = (1 + \sqrt{5})/2$, as $\lim_{n\to\infty} f_{n+1}/f_n = \varphi$. In general, it can be shown that $\lim_{n\to\infty} f_{n+k}/f_n = \varphi^k$ for any integer k.

The *Hamming weight* of a binary string u is the number of 1s in u, denoted by $|u|_1$. If a, b are strings, then ab denotes the concatenation of those two strings. Similarly, for a set of strings B, we set $aB = \{ab \mid b \in B\}$.

The degree of a vertex v is the number of neighbors of v, denoted by $\deg(v)$. Since each neighbor of $v \in V(\mathcal{R}_n)$ is obtained either by changing a 0 in v into a 1 or by changing a 1 in v into a 0, we can distinguish between the *up-degree* $\deg_{up}(v)$, and *down-degree* $\deg_{down}(v)$ of the vertex v. The first is the number of neighbors of v which are obtained by switching a 0 in v into a 1, while the second is the number of neighbors obtained by switching a 1 in v into a 0. Clearly, $\deg(v) = \deg_{up}(v) + \deg_{down}(v)$. Note that in $\mathcal{R}_n$, $\deg_{down}(v)$ is not necessarily equal to the Hamming weight of v because of the constraints on the run-lengths that must hold.

We keep track of the degree sequences of our graphs $\mathcal{R}_n$ as the coefficients of a polynomial. This polynomial is called the *degree enumerator polynomial* of the graph. The coefficient of x^i in the degree enumerator polynomial is the number of vertices of degree i in $\mathcal{R}_n$. Similar polynomials are defined to keep track of the up- and down-degree sequences as well (see the beginning of Section 7 for an example).

The average degree of a graph G is

$$\overline{deg}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \deg(v) = \frac{2|E(G)|}{|V(G)|} .$$

The *diameter* of G is the maximum distance between pairs of vertices of the graph, that is

$$\mathrm{diam}(G) = \max\{d(u, v) \mid u, v \in V(G)\} .$$

We use the notation $d_G(u, v)$ when we need to emphasize that the distance is relative to G. An edge $\{u, v\} \in E(G)$ is sometimes denoted by $u \sim v$.

A *partial cube* is an isometric subgraph of a hypercube. A subgraph H of a graph G is an *isometric subgraph* if for every $u, v \in V(H)$ we have $d_H(u, v) = d_G(u, v)$. A graph G is a *median graph* if every triple u, v, w of its vertices has a unique median x, that is a vertex x with the properties $d(u, x) + d(x, v) = d(u, v)$, $d(v, x) + d(x, w) = d(v, w)$, and $d(u, x) + d(x, w) = d(u, w)$. Recall that hypercubes are median graphs, and that median graphs are partial cubes.

3. The number of edges

Lemma 3.1. *If $n \geq 1$, then $|V(\mathcal{R}_n)| = f_{n+2}$.*

Proof. Since we know that $|V(\Gamma_n)| = f_{n+2}$ [6,10], it suffices to provide a bijection between vertex sets of Γ_n and $\mathcal{R}_n$. For the purpose of this proof, we view both vertex sets with additional 00 at the end of each string.

An explicit bijection between Fibonacci strings and run-constrained strings is obtained by setting

$$\begin{aligned}
\Phi(0) &= 0 \\
\Phi(100) &= 100 \\
\Phi(10100) &= 11000 \\
\Phi(1010100) &= 1110000,
\end{aligned}$$

etc., a bijection between the alphabets F and R, and extending Φ to full words via the unique factorization. So for example, starting with a Fibonacci string $w = 10010100001010100100$, we find $\Phi(w)$ as

$$\begin{aligned}
\Phi(w) &= \Phi\Big((100)(10100)(0)(0)(1010100)(100)\Big) \\
&= (100)(11000)(0)(0)(1110000)(100) \\
&= 10011000001110000100.
\end{aligned}$$

This clearly yields a bijection, thus $|V(\mathcal{R}_n)| = |V(\Gamma_n)| = f_{n+2}$. $\quad\square$

Corollary 3.2. *The number of vertices in $\mathcal{R}_n$ of Hamming weight w, $0 \leq w \leq \lceil n/2 \rceil$ is*

$$\binom{n - w + 1}{w} .$$

Proof. The result follows from the bijection Φ between vertex sets of $\mathcal{R}_n$ and Γ_n (as it preserves the number of 1s in a string), and the classical result that the number of Fibonacci strings of length n and Hamming weight w is $\binom{n-w+1}{w}$ (see, for example [17]). $\quad\square$
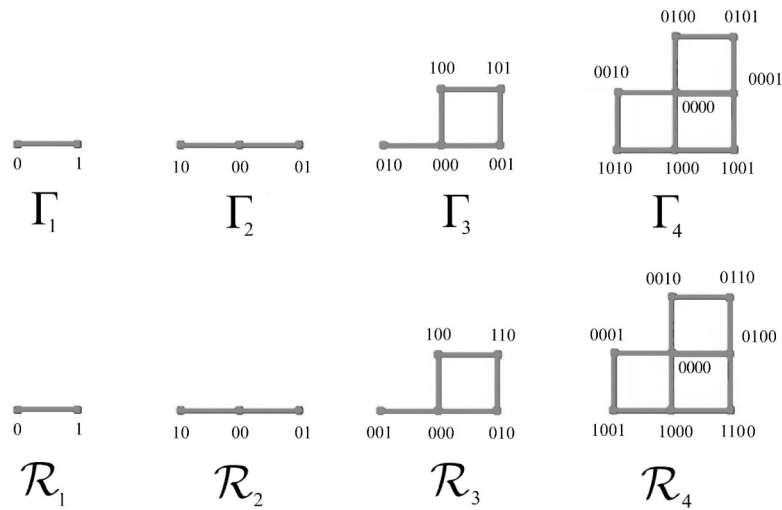
Fig. 1. Graphs Γ_n and $\mathcal{R}_n$ for $n \in [4]$.

Fig. 1 depicts the first four Fibonacci graphs Γ_n and the first four Fibonacci-run graphs $\mathcal{R}_n$. As noted, we omit the ending 00 of the vertex labels in both. Note that the graphs Γ_n and $\mathcal{R}_n$ are isomorphic for $n \in [4]$. However, the bijection Φ between the vertex sets of these graphs is not in general a graph isomorphism. For example, for the adjacent vertices $u = 101$ and $v = 001$ in Γ_3, we have $\Phi(u) = 110$, $\Phi(v) = 001$ with $H(\Phi(u), \Phi(v)) = 3$ in $\mathcal{R}_3$, so $\Phi(u)$ and $\Phi(v)$ are not adjacent in $\mathcal{R}_3$. However as it turns out the two graphs are isomorphic anyway (under a different graph isomorphism), see Fig. 1.

In the following, we prove that the graphs Γ_n and $\mathcal{R}_n$ are never isomorphic for $n \geq 5$. Recall [10] that the number of edges of Γ_n is given by

$$|E(\Gamma_n)| = \frac{1}{5}\left(2(n+1)f_n + nf_{n+1}\right),\tag{2}$$

with generating function

$$\sum_{n \geq 1}|E(\Gamma_n)|t^n = \frac{t}{(1-t-t^2)^2} = t + 2t^2 + 5t^3 + 10t^4 + 20t^5 + 38t^6 + \cdots\tag{3}$$

Lemma 3.3. *If $n \geq 5$, then*

$$|E(\mathcal{R}_n)| = |E(\Gamma_n)| - |E(\Gamma_{n-4})|,\tag{4}$$

with generating function

$$\sum_{n \geq 1}|E(\mathcal{R}_n)|t^n = \frac{t(1-t^4)}{(1-t-t^2)^2} = t + 2t^2 + 5t^3 + 10t^4 + 19t^5 + 36t^6 + \cdots$$

Proof. For the proof of the expression (4), see Section 4, Corollary 4.3.

The ordinary generating function for $|E(\mathcal{R}_n)|$ follows directly from the recursion, the properties of graphs $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3,$ R_4, and (3):

$$\sum_{n \geq 1}|E(\mathcal{R}_n)|t^n = t + 2t^2 + 5t^3 + 10t^4 + \sum_{n \geq 5}(|E(\Gamma_n)| - |E(\Gamma_{n-4})|)t^n$$

$$= t + 2t^2 + 5t^3 + 10t^4 + \sum_{n \geq 5}|E(\Gamma_n)|t^n - t^4\sum_{n \geq 1}|E(\Gamma_n)|t^n$$

$$= t + 2t^2 + 5t^3 + 10t^4 + \frac{t}{(1-t-t^2)^2}$$

$$- (t + 2t^2 + 5t^3 + 10t^4) - t^4\frac{t}{(1-t-t^2)^2}$$

$$= \frac{t - t^5}{(1-t-t^2)^2}. \quad \square$$

In particular, an analytic expression for the number of edges of $\mathcal{R}_n$ in terms of the Fibonacci numbers is given by (6) in Corollary 4.3.

Corollary 3.4. *If $n \geq 5$, then the graphs $\mathcal{R}_n$ and Γ_n are not isomorphic.*

Proof. If $n \geq 5$, then $|E(\Gamma_{n-4})| \geq 1$, and therefore from (4) in Lemma 3.3 it follows that $|E(\mathcal{R}_n)| < |E(\Gamma_n)|$, thus the graphs cannot be isomorphic. $\quad\square$

As subgraphs of hypercubes, Fibonacci-run graphs are bipartite. The parts of the bipartition are simply obtained by separating the vertices of odd and even Hamming weight. Thus we have the following easy inequality for the vertex independence number $\alpha(\mathcal{R}_n)$.

Lemma 3.5. *If $n \geq 1$, then $\alpha(\mathcal{R}_n) \geq \left\lceil \frac{f_{n+2}}{2} \right\rceil$.*

4. Decomposition of $\mathcal{R}_n$

It is well known that Fibonacci cubes have a simple and very useful decomposition [6]. Namely, Γ_n can be partitioned into subgraphs $0\Gamma_{n-1}$ and $10\Gamma_{n-2}$, with a perfect matching between Γ_{n-2} and its copy $0\Gamma_{n-2}$ in Γ_{n-1}. This immediately implies a formula for the quantities $|V(\Gamma_n)|$, $|E(\Gamma_n)|$, existence of a Hamiltonian path in Γ_n, and other interesting properties [6,10]. With this in mind, we aim to find a similar decomposition of the Fibonacci-run graphs.

Lemma 4.1. *The vertex set of a Fibonacci-run graph $\mathcal{R}_n$ can be partitioned into*

$$\bigcup_{k=0}^{\lceil n/2 \rceil - 1} 1^k 0^{k+1} V(\mathcal{R}_{n-(2k+1)}) \cup 1^{\lceil n/2 \rceil} 0^{\lfloor n/2 \rfloor} V(\mathcal{R}_0) \,.$$

Proof. The vertices of $\mathcal{R}_n$ are run-constrained strings of length $n + 2$, all ending with 00. Each of them starts either with a 0 or with a run of 1s (followed by a longer run of 0s). The maximal number of 1s in such a string is $\lceil n/2 \rceil$. Thus every string in $V(\mathcal{R}_n)$ can be uniquely written as $1^k 0^{k+1} w$, where $0 \leq k \leq \lceil n/2 \rceil$, and w is a run-constrained string of length $n + 1 - 2k$. If $k < \lceil n/2 \rceil$, then $w \in V(R_{n-(2k+1)})$, while the last sting can be viewed as $1^{\lceil n/2 \rceil} 0^{\lfloor n/2 \rfloor} V(\mathcal{R}_0)$. This yields the described decomposition. $\quad\square$

It follows from Lemma 4.1 that $\mathcal{R}_n$ contains the following graphs as subgraphs

$$0\mathcal{R}_{n-1}, \ 100\mathcal{R}_{n-3}, \ 11000\mathcal{R}_{n-5}, \ 1110000\mathcal{R}_{n-7}, \ldots$$

The last graph in this sequence is $1^{m-1}0^m\mathcal{R}_1$ if $n = 2m$, and $1^{m-1}0^m\mathcal{R}_0$ if $n = 2m - 1$. We augment this list by one more subgraph of $\mathcal{R}_n$ consisting of a single vertex: $1^m 0^m \mathcal{R}_0$ if $n = 2m$, and $1^m 0^{m-1}\mathcal{R}_0$ if $n = 2m - 1$.

If we denote this partition into subgraphs with $+$, then we obtain the following for $n \in [8]$:

$$\mathcal{R}_1 = 0\mathcal{R}_0 + 1\mathcal{R}_0$$
$$\mathcal{R}_2 = 0\mathcal{R}_1 + 10\mathcal{R}_0$$
$$\mathcal{R}_3 = 0\mathcal{R}_2 + 100\mathcal{R}_0 + 110\mathcal{R}_0$$
$$\mathcal{R}_4 = 0\mathcal{R}_3 + 100\mathcal{R}_1 + 1100\mathcal{R}_0$$
$$\mathcal{R}_5 = 0\mathcal{R}_4 + 100\mathcal{R}_2 + 11000\mathcal{R}_0 + 11100\mathcal{R}_0$$
$$\mathcal{R}_6 = 0\mathcal{R}_5 + 100\mathcal{R}_3 + 11000\mathcal{R}_1 + 111000\mathcal{R}_0$$
$$\mathcal{R}_7 = 0\mathcal{R}_6 + 100\mathcal{R}_4 + 11000\mathcal{R}_2 + 1110000\mathcal{R}_0 + 1111000\mathcal{R}_0$$
$$\mathcal{R}_8 = 0\mathcal{R}_7 + 100\mathcal{R}_5 + 11000\mathcal{R}_3 + 1110000\mathcal{R}_1 + 11110000\mathcal{R}_0$$

A schematic representation of the decomposition of graphs $\mathcal{R}_1, \ldots, \mathcal{R}_5$ is shown in Fig. 2.

Next, we consider the edges between the mentioned subgraphs of $\mathcal{R}_n$. Inside each subgraph $1^k 0^{k+1}\mathcal{R}_{n-(2k+1)}$ we inherit the edges from the graph $\mathcal{R}_{n-(2k+1)}$. Between subgraphs $0\mathcal{R}_{n-1}$ and $1^k 0^{k+1}\mathcal{R}_{n-(2k+1)}$ for $1 \leq k \leq \lceil n/2 \rceil - 1$, we have an edge if and only if the two strings differ in exactly one coordinate. This means that the edges are of the form $01^{k-1}0^{k+1}w \sim 1^k 0^{k+1}w$, where $w \in V(\mathcal{R}_{n-(2k+1)})$. Therefore each pair of subgraphs $0\mathcal{R}_{n-1}$ and $1^k 0^{k+1}\mathcal{R}_{n-(2k+1)}$ for $1 \leq k \leq \lceil n/2 \rceil - 1$ yields exactly $|V(\mathcal{R}_{n-(2k+1)})|$ edges in $\mathcal{R}_n$. On the other hand, between subgraphs $1^k 0^{k+1}\mathcal{R}_{n-(2k+1)}$ and $1^\ell 0^{\ell+1}\mathcal{R}_{n-(2\ell+1)}$ for $k < \ell$ and $1 \leq k, \ell \leq \lceil n/2 \rceil - 1$, we can have an edge only if $\ell = k+1$. To be precise, the edges between those two subgraphs are of the form $1^{k+1}0^{k+2}w \sim 1^k 0^{k+3}w$, where $w \in V(\mathcal{R}_{n-(2k+3)})$, so we get exactly $|V(\mathcal{R}_{n-(2k+3)})|$ edges between appropriate subgraphs. The only remaining edges to study are between the vertex $1^{\lceil n/2 \rceil}0^{\lfloor n/2 \rfloor}V(\mathcal{R}_0)$ and the other parts of the graph. But this vertex has exactly two neighbors in $\mathcal{R}_n$: $1^{\lceil n/2 \rceil-1}0^{\lfloor n/2 \rfloor+1}V(\mathcal{R}_0)$ and $01^{\lceil n/2 \rceil-1}0^{\lfloor n/2 \rfloor}V(\mathcal{R}_0)$.

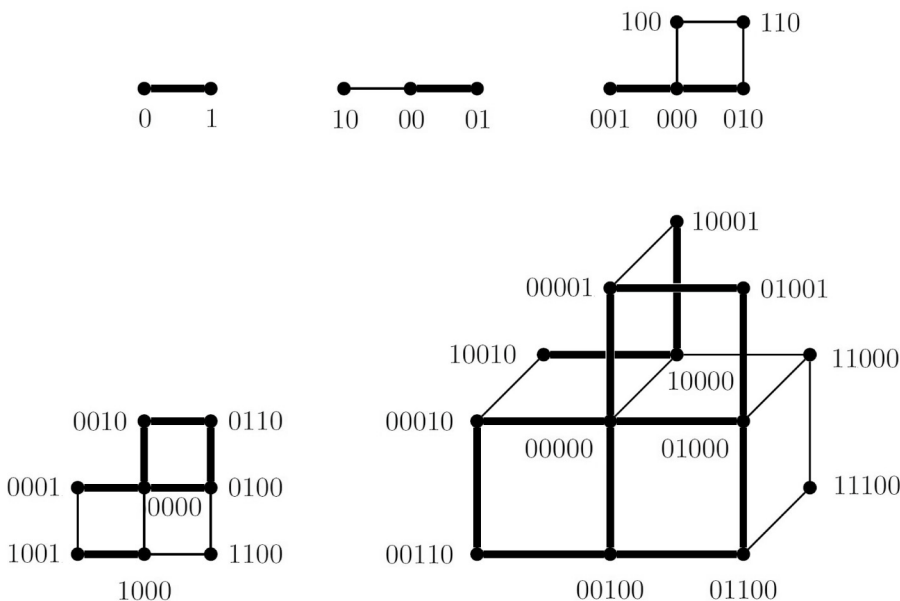As examples, see Fig. 3 for a schematic representation of the decomposition of $\mathcal{R}_6$ and $\mathcal{R}_7$.

Fig. 2. The graphs $\mathcal{R}_n$, for $n \in [5]$, with their decomposition schematically shown with thicker edges.
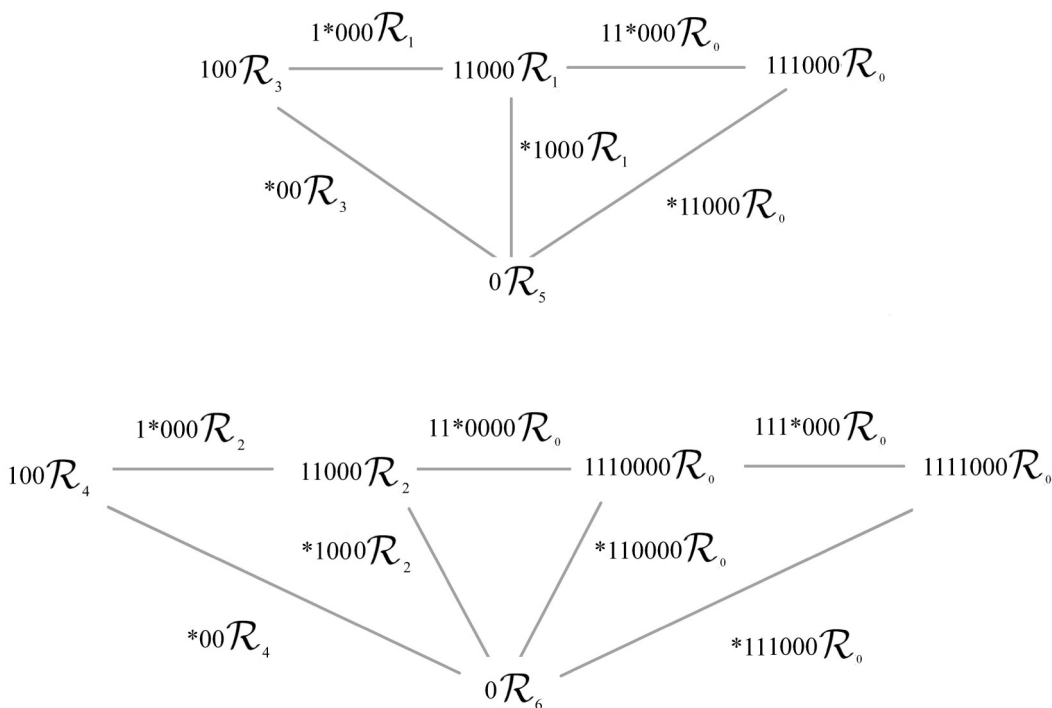

Fig. 3. The decomposition of $\mathcal{R}_6$ (above) and $\mathcal{R}_7$ (below). The edges are symbolically marked with the lines indicating the edges present between the parts.

We already know (see Fig. 1), that $|E(\mathcal{R}_1)| = 1$, $|E(\mathcal{R}_2)| = 2$, $|E(\mathcal{R}_3)| = 5$, and $|E(\mathcal{R}_4)| = 10$. For $n \geq 5$, we can use the above argument to obtain the following

$$|E(\mathcal{R}_n)| = \sum_{k=0}^{\lceil n/2 \rceil - 1} |E(\mathcal{R}_{n-(2k+1)})| + |V(\mathcal{R}_{n-3})| + 2 \sum_{k=2}^{\lceil n/2 \rceil - 1} |V(\mathcal{R}_{n-(2k+1)})| + 2. \tag{5}$$

Lemma 4.2. *If $n \geq 4$, then*

$$|E(\mathcal{R}_n)| = |E(\mathcal{R}_{n-1})| + |E(\mathcal{R}_{n-2})| + f_{n-1} + f_{n-3} .$$

Proof. We know from Lemma 3.1 that $|V(\mathcal{R}_n)| = f_{n+2}$. Set $e_n = |E(\mathcal{R}_n)|$. If $n \geq 7$, we can use the recursion from (5) for n and $n - 2$:

$$e_n = e_{n-1} + \sum_{k=1}^{\lceil n/2 \rceil - 1} e_{n-(2k+1)} + |V(\mathcal{R}_{n-3})| + 2|V(\mathcal{R}_{n-5})|$$

$$+ 2 \sum_{k=3}^{\lceil n/2 \rceil - 1} |V(\mathcal{R}_{n-(2k+1)})| + 2$$

$$= e_{n-1} + f_{n-1} + f_{n-3} + e_{n-2} .$$

For $n \in \{4, 5, 6\}$, we can determine the number of edges e_n from (5), and check by hand that the recursion $e_n = e_{n-1} + e_{n-2} + f_{n-1} + f_{n-3}$ gives the same result. □

Notice that the result of Lemma 4.2 can be simplified to $|E(\mathcal{R}_n)| = |E(\mathcal{R}_{n-1})| + |E(\mathcal{R}_{n-2})| + L_{n-2}$, where L_{n-2} is the Lucas number. This compares nicely to the expression $|E(\Gamma_n)| = |E(\Gamma_{n-1})| + |E(\Gamma_{n-2})| + f_n$, which follows from the fundamental decomposition of Fibonacci cubes.

Corollary 4.3. *If $n \geq 5$, then*

$$|E(\mathcal{R}_n)| = |E(\Gamma_n)| - |E(\Gamma_{n-4})| .$$

Proof. We can easily check that the equality holds for $n = 5$. Now let $n \geq 6$, and let $e_n = |E(\mathcal{R}_n)|$ and $g_n = |E(\Gamma_n)|$. By Lemma 4.2, the induction hypothesis, Eq. (2), and the relation $f_n = f_{n-1} + f_{n-2}$, it follows that

$$e_n = (g_{n-1} - g_{n-5}) + (g_{n-2} - g_{n-6}) + f_{n-1} + f_{n-3}$$
$$= (3n + 4)f_{n-6} + (5n + 6)f_{n-5} . \tag{6}$$

On the other hand, we can also calculate that

$$|E(\Gamma_n)| - |E(\Gamma_{n-4})| = (3n + 4)f_{n-6} + (5n + 6)f_{n-5} ,$$

which completes the proof. □

To conclude the section, we observe that from the decomposition it follows that each vertex in $\mathcal{R}_n \setminus 0\mathcal{R}_{n-1}$ has exactly one unique neighbor in $0\mathcal{R}_{n-1}$. More precisely, let $\varphi : \mathcal{R}_n \setminus 0\mathcal{R}_{n-1} \to 0\mathcal{R}_{n-1}$ be defined as

$$1^k 0^{k+1} w \mapsto 01^{k-1} 0^{k+1} w, \ w \in \mathcal{R}_{n-2k-1}, \ 1 \leq k \leq \lceil n/2 \rceil - 1,$$
$$1^{\lceil n/2 \rceil} 0^{\lfloor n/2 \rfloor} \mapsto 01^{\lceil n/2 \rceil - 1} 0^{\lfloor n/2 \rfloor}.$$

Clearly, the function φ is injective. Moreover, for every $u, v \in \mathcal{R}_n \setminus 0\mathcal{R}_{n-1}$ it satisfies $u \sim v \iff \varphi(u) \sim \varphi(v)$.

5. Diameter

The diameter of Fibonacci cubes is well known [6], and equals $\mathrm{diam}(\Gamma_n) = n$. Determining the diameter of Fibonacci-run graphs turns out to be a rather difficult task. Clearly, $\mathrm{diam}(\mathcal{R}_n) = n$ for $n \in [4]$. The exact values of the diameter computed by brute force are shown in Fig. 4 for $n \leq 30$. For general n we present a lower bound on $\mathrm{diam}(\mathcal{R}_n)$, and a conjecture on the actual value of the diameter.

We first prove the following lemma, which will be generalized later.

Lemma 5.1. *If $n = \frac{1}{2}(r^2 + 3r - 2)$, then $\mathrm{diam}(\mathcal{R}_n) \geq n - \lfloor \frac{r-1}{2} \rfloor$.*

Proof. It suffices to find two vertices $u, v \in V(\mathcal{R}_n)$ with $d(u, v) = n - \lfloor \frac{r-1}{2} \rfloor$.
   If r is even, take

$$u = 10^2 1^3 0^4 \ldots 1^{r-1} 0^r 1^{r/2} 0^{r/2+1},$$
$$v = 01^2 0^3 1^4 \ldots 0^{r-1} 1^r 0^{r+1},$$

where the trailing 00 is *not* omitted. In this case,

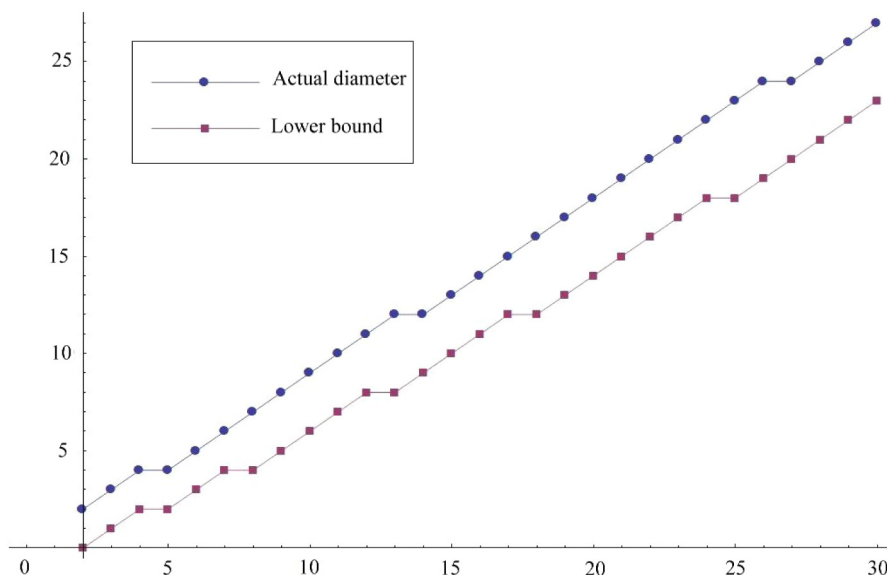$$n = 1 + 2 + \cdots + r + (r - 1) = \frac{1}{2}(r^2 + 3r - 2),$$

Fig. 4. The exact values and the lower bound given by Theorem 5.3 for diam($\mathcal{R}_n$) for $n \leq 30$.

$$d(u, v) = 1 + 2 + \cdots + r + \frac{r}{2} = n - \left(\frac{r}{2} - 1\right) = n - \left\lfloor \frac{r-1}{2} \right\rfloor.$$

If r is odd, then we take

$$u = 10^2 1^3 0^4 \ldots 1^r 0^{r+1},$$
$$v = 01^2 0^3 1^4 \ldots 0^r 1^{(r+1)/2-1} 0^{(r+1)/2+1},$$

where the last 00 is again not omitted. In this case, we get

$$n = 1 + 2 + \cdots + r + (r - 1) = \frac{1}{2}(r^2 + 3r - 2),$$
$$d(u, v) = 1 + 2 + \cdots + r + \frac{r-1}{2} = n - \left(\frac{r-1}{2}\right) = n - \left\lfloor \frac{r-1}{2} \right\rfloor. \quad \square$$

Corollary 5.2. *If $n = \frac{1}{2}(r^2 + 3r - 2)$, then* diam($\mathcal{R}_n$) $\geq n - \sqrt{n/2}$.

Proof. This can be verified simply by checking that $\sqrt{\frac{n}{2}} \geq \frac{r-1}{2} \geq \left\lfloor \frac{r-1}{2} \right\rfloor$. $\quad \square$

Theorem 5.3. *If $n \geq 1$, then* diam($\mathcal{R}_n$) $> n - \sqrt{2n}$.

Proof. If $n = \frac{1}{2}(r^2 + 3r - 2)$ for some integer r, then the result holds. Otherwise, we have

$$\frac{1}{2}(r^2 + 3r - 2) < n < \frac{1}{2}((r+1)^2 + 3(r+1) - 2),$$

which implies

$$n = \frac{1}{2}(r^2 + 3r - 2) + D, \quad \text{with } 1 \leq D \leq r + 1.$$

We again aim to construct vertices $u, v \in V(\mathcal{R}_n)$ which differ on at least $n - \sqrt{2n}$ coordinates. The idea is to extend the last run of $r + 1$ zeros in u or v from the proof of Lemma 5.1, and add the maximum allowed number of 1s followed by appropriate number of 0s in the other vertex.

If r is even, then let (without omitting the 00 at the tail end)

$$u = 10^2 1^3 0^4 \ldots 1^{r-1} 0^r 1^{\lceil (D+r-1)/2 \rceil} 0^{\lfloor (D+r+3)/2 \rfloor},$$
$$v = 01^2 0^3 1^4 \ldots 0^{r-1} 1^r 0^{D+r+1},$$

while if r is odd, let

$$u = 10^2 1^3 0^4 \ldots 1^r 0^{D+r+1},$$
$$v = 01^2 0^3 1^4 \ldots 0^r 1^{\lceil (D+r-1)/2 \rceil} 0^{\lfloor (D+r+3)/2 \rfloor}.$$

In both cases, we obtain

$$n = 1 + 2 + \cdots + r + (r + D - 1) = \frac{1}{2}(r^2 + 3r - 2) + D,$$

$$d(u, v) = 1 + 2 + \cdots + r + \left\lceil \frac{r+D-1}{2} \right\rceil = n - \left\lfloor \frac{r+D-1}{2} \right\rfloor.$$

Now, it suffices to prove that $n - \lfloor \frac{r+D-1}{2} \rfloor > n - \sqrt{2n}$. This holds, if we can prove that $\sqrt{2n} > \frac{r+D-1}{2}$, with $n = \frac{1}{2}(r^2 + 3r - 2) + D$ for some $r \geq 1$, and $1 \leq D \leq r + 1$. But since $D \geq 1$, we have $\sqrt{2n} \geq \sqrt{r^2 + 3r} > r$, and since $D \leq r + 1$, we get $\frac{r+D-1}{2} \leq r$. Thus $\sqrt{2n} > \frac{r+D-1}{2}$, and indeed $d(u, v) > n - \sqrt{2n}$. $\quad\square$

We have calculated by computer the exact values of diam($\mathcal{R}_n$) for $n \leq 30$. These values, together with the corresponding values of the lower bound of Theorem 5.3 are shown in Fig. 4. From the actual values of the diameter we see that in the range given, the values for $n = 4, 8, 13, 19, 26$ corresponding to $r = 2, 3, 4, 5, 6$ of Lemma 5.1 are actually exact.

Based on this fact and the values in Fig. 4, we conjecture that the diameter is given by

$$\text{diam}(\mathcal{R}_n) = n - \left\lfloor \sqrt{1 + \frac{n}{2}} - \frac{3}{4} \right\rfloor . \tag{7}$$

6. Asymptotic results

We have seen that $|E(\mathcal{R}_n)| < |E(\Gamma_n)|$ holds for $n \geq 5$, so it is natural to consider the asymptotic behavior of the quotient $|E(\mathcal{R}_n)|/|E(\Gamma_n)|$. From the recursion in Corollary 4.3, expression in Eq. (2) for $|E(\Gamma_n)|$, and the asymptotic behavior of Fibonacci numbers it follows that

$$\lim_{n \to \infty} \frac{|E(\mathcal{R}_n)|}{|E(\Gamma_n)|} = 1 - \lim_{n \to \infty} \frac{|E(\Gamma_{n-4})|}{|E(\Gamma_n)|} = \frac{1}{2}(3\sqrt{5} - 5) \approx 0.854. \tag{8}$$

So asymptotically, $\mathcal{R}_n$ has about four-fifths the number of edges of the Fibonacci cube Γ_n.

The asymptotic average degree of Fibonacci cube Γ_n is known [11] to be

$$\lim_{n \to \infty} \frac{\overline{deg}(\Gamma_n)}{n} = \left(1 - \frac{1}{\sqrt{5}}\right) \approx 0.55. \tag{9}$$

Using (9) and (8), we obtain the asymptotic average degree of the Fibonacci-run graph $\mathcal{R}_n$ as

$$\lim_{n \to \infty} \frac{\overline{deg}(\mathcal{R}_n)}{n} = 2\left(\sqrt{5} - 2\right) \approx 0.47 ,$$

a slightly lower value.

7. Up–down degree sequences

The degree sequences, i.e. the nature of the vertices of a given degree in a graph, has been well studied for Fibonacci cubes [13]. Fibonacci-run graphs can also be viewed as partially ordered sets whose structure is inherited from the Boolean algebra of subsets of $[n]$. The elements here correspond to all binary strings of length n and the covering relation is flipping a 0 to a 1. Therefore in $\mathcal{R}_n$ we have a natural distinction between up- and down-degree of a vertex, denoted by $\deg_{up}(v)$ and $\deg_{down}(v)$: $\deg_{up}(v)$ is the number of vertices u in $\mathcal{R}_n$ obtained by changing a 0 to a 1, and $\deg_{down}(v)$ is the number of vertices u in $\mathcal{R}_n$ obtained from v by changing a 1 to a 0. We have $\deg(v) = \deg_{up}(v) + \deg_{down}(v)$.

The nature of the distribution of the up-degrees and the down-degrees is most easily seen from the Hasse diagram of $\mathcal{R}_n$ for which $\deg_{up}(v)$ and $\deg_{down}(v)$ are simply the number of edges emanating up and down from $v \in \mathcal{R}_n$, respectively.

From Fig. 5, the down-degree enumerator polynomial of $\mathcal{R}_4$ is given by

$$1 + 4d + 3d^2 , \tag{10}$$

and its up-degree enumerator polynomial by

$$3 + 2u + 2u^2 + u^4 . \tag{11}$$

From Fig. 5 we also find that the degree-enumerator polynomial of $\mathcal{R}_4$ is

$$5x^2 + 2x^3 + x^4 . \tag{12}$$

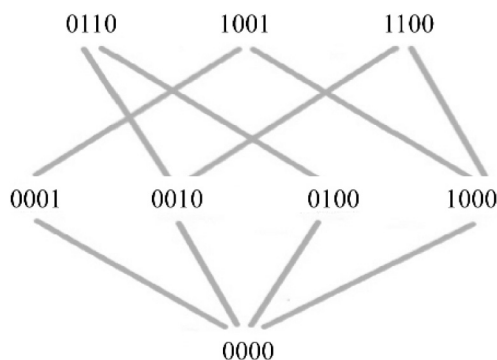First, we consider the generating function for the down-degree enumerator polynomials for $\mathcal{R}_n$.

Fig. 5. The Hasse diagram of $\mathcal{R}_4$.

Proposition 7.1. *The generating function for down-degree enumerator polynomial in Fibonacci-run graphs is*

$$\sum_{n\geq 1} t^n \sum_{v\in\mathcal{R}_n} d^{\deg_{down}(v)} = \frac{t(1+d+dt+(d^2-1)t^2+d(d-1)t^3+d(d-1)t^4)}{1-t-t^2-(d-1)t^3-d(d-1)t^5}.$$

Proof. We first consider the contributions of the strings from the alphabet R of (1). The string 100 contributes dt^3, but longer strings 1^k0^{k+1} of length n contribute d^2t^n, since both the first and the last 1 appearing can be switched to 0. Thus, keeping track of the down-degree by the exponent of d and the total length as the exponent of t, the strings from R give

$$t + dt^3 + d^2t^5 + d^2t^7 + \cdots = t + dt^3 + \frac{d^2t^5}{1-t^2}.$$

Therefore the generating function of the free monoid, which includes vertices $V(\mathcal{R}_n)$, $n \geq 1$, is

$$\frac{1}{1-t-dt^3-\frac{d^2t^5}{1-t^2}}.$$

From this we need to subtract the terms $1, t, t^2$, which correspond to the null word, 0 and 00 respectively. This gives

$$\frac{1}{1-t-dt^3-\frac{d^2t^5}{1-t^2}} - 1 - t - t^2 = \frac{t^3(1+d+dt+(d^2-1)t^2+d(d-1)t^3+d(d-1)t^4)}{1-t-t^2-(d-1)t^3-d(d-1)t^5}.$$

Finally, we divide by t^2 to effectively shorten the length by 2 and get rid of the contribution of the last two zeros in each run-constrained string generated to obtain the desired result. $\square$

First few terms given by the generating function in Proposition 7.1 are

$$(1+d)t + (1+2d)t^2 + (1+3d+d^2)t^3 + (1+4d+3d^2)t^4 + (1+5d+7d^2)t^5$$
$$+ (1+6d+12d^2+2d^3)t^6 + (1+7d+19d^2+7d^3)t^7 + \cdots$$

Note that since the sum of down-degrees of the vertices in $\mathcal{R}_n$ is the total number of edges, we can obtain the generating function of the number of edges in $\mathcal{R}_n$ by differentiating the result of Proposition 7.1 with respect to d, and then setting $d = 1$. This again gives the generating function obtained in Lemma 3.3.

Along similar lines to the proof of Proposition 7.1, we also obtain the generating function of the up-degree enumerator polynomials for $\mathcal{R}_n$.

Proposition 7.2. *The generating function for up-degree enumerator polynomials of Fibonacci-run graphs is*

$$\sum_{n\geq 1} t^n \sum_{v\in\mathcal{R}_n} u^{\deg_{up}(v)} = \frac{t(1+u-(u-2)t-2ut^2+t^3-(u-1)t^5-(u-1)t^6)}{1-ut-2t^2+(2u-1)t^3+t^4-(u-1)t^5+(u-1)t^7}.$$

First few terms of the power series expansion of the generating function in Proposition 7.2 are

$$(1+u)t + (2+u^2)t^2 + (2+2u+u^3)t^3 + (3+2u+2u^2+u^4)t^4$$
$$+ (5+2u+3u^2+2u^3+u^5)t^5 + (6+6u+2u^2+4u^3+2u^4+u^6)t^6 + \cdots$$

It is desirable to study the generating function of the bivariate up–down degree enumerator polynomials

$$\sum_{v\in\mathcal{R}_n} u^{\deg_{up}(v)}d^{\deg_{down}(v)}.$$

For example, for $n = 4$, this polynomial is

$$3d^2 + 2du + 2du^2 + u^4 \, , \tag{13}$$

as can be verified by inspecting $\mathcal{R}_4$ in Fig. 5. The polynomial in (13) specializes to (10) for $u = 1$, to (11) for $d = 1$, and to (12) for $u = d = x$. More generally, the generating function of the up–down degree enumerator polynomials for $\mathcal{R}_n$ would give Propositions 7.1 and 7.2 as corollaries, and provide the generating function for the degree enumerator polynomials itself for $u = d = x$. The derivation of this general case is of independent interest and is studied in detail in a companion paper [4].

8. Relation to hypercubes

Recall that partial cubes are an important and well-studied class of graphs. Since Fibonacci-run graphs are subgraphs of hypercubes, it is natural to ask whether they can be isometrically embedded in a hypercube.

Proposition 8.1. *Fibonacci-run graph $\mathcal{R}_n$ is a partial cube if and only if $n \leq 6$.*

Proof. We can check with a computer that graphs $\mathcal{R}_n$ for $n \leq 6$ are partial cubes.

Let $n \geq 7$. Winkler [20] proved that a connected graph is a partial cube if and only if it is bipartite and the Djoković–Winkler relation θ is transitive. Recall that for edges xy, uv of a bipartite graph it holds $xy\,\theta\,uv \iff d(x, u) = d(y, v)$ and $d(x, v) = d(y, u)$. Let a, b be arbitrary run-constraint strings of appropriate lengths and take vertices in $\mathcal{R}_n$:

$$x = a111100000b, \quad y = a111000000b,$$
$$p = a011100000b, \quad q = a011000000b,$$
$$u = a100100000b, \quad v = a100000000b.$$

It can be checked that $xy, pq, uv \in E(\mathcal{R}_n)$, and that $xy\,\theta\,pq$ and $pq\,\theta\,uv$, but not $xy\,\theta\,uv$. Thus θ is not transitive and $\mathcal{R}_n$ is not a partial cube. $\square$

Additionally, we remark that the Fibonacci-run graphs are in general not median graphs. They are median for $n \leq 4$, as those graphs are isomorphic to Γ_n, which are known to be median [9]. However, $\mathcal{R}_n$ is not a median graph for $n \geq 5$. Consider the following triple of vertices:

$$u = 111000^{n-5},$$
$$v = 100000^{n-5},$$
$$w = 001000^{n-5} \, .$$

Clearly, $u, v, w \in V(\mathcal{R}_n)$. Since $\mathcal{R}_n$ is an isometric subgraph of a hypercube, the unique candidate for a median of u, v, w is obtained in the same way as in a hypercube – using the majority rule [8]. With this rule, a coordinate of the median vertex equals the value that appears at least in two of the vertices u, v, w on that coordinate. Thus the unique candidate for a median is $x = 101000^{n-5} \notin V(\mathcal{R}_n)$, hence $\mathcal{R}_n$ is not a median graph.

Next, we consider an analogous question. Instead of observing Fibonacci-run graphs as subgraphs of hypercubes, we consider the possible hypercubes which are subgraphs of a Fibonacci-run graph. A similar question has been studied for Fibonacci cubes as well, cf. results about cube polynomial listed in [10], and generalizations in [18].

Proposition 8.2. *If $h_{n,d,k}$ denotes the number of k-dimensional hypercubes Q_k in $\mathcal{R}_n$ whose distance to the all zero vertex 0^n is d, then the generating function $\sum_{n \geq 1} t^n \sum_{d,k \geq 0} h_{n,d,k} q^d x^k$ equals*

$$\frac{t(1 + q + x + (q + x)t + ((q + x)^2 - 1)t^2 + (q + x)(q + x - 1)t^3 + (q + x)(q + x - 1)t^4)}{1 - t - t^2 - (q + x - 1)t^3 - (q + x)(q + x - 1)t^5} \, .$$

Proof. Let us keep track of all hypercubes Q_k in $\mathcal{R}_n$. To each such subgraph, we associate the monomial $q^d x^k$, where d is the distance of the Q_k to the all zero vertex $0^n \in V(\mathcal{R}_n)$. For every vertex $v \in \mathcal{R}_n$, select k 1s in its string representation that can be replaced with a 0. Note that the number of such 1s is at most $\deg_{down}(v)$ (which is not necessarily equal to $\deg(v)$). By flipping these 1s to 0s in all possible ways, we obtain the vertices of a copy of Q_k in $\mathcal{R}_n$. The distance of this hypercube to 0^n is $w - k$, where $w = |v|_1$ is the Hamming weight of the vertex v. So from every vertex of Hamming weight w and down-degree r, we obtain $\binom{r}{k}$ different copies of Q_k, with the associated monomial $q^{w-k} x^k$ for each.

The generating function can be obtained from Proposition 7.1 by replacing each d^r that appears in the series expansion, by $(q + x)^r$. The reason for this is that the term d^r arises from a vertex v with down-degree r, and its contribution to the monomials making up the generating function is

$$\sum_{k=0}^{r} \binom{r}{k} q^{r-k} x^k = (q + x)^r \, .$$

Table 1

Graphs $\mathcal{R}_n$ for $1 \leq n \leq 12$ which have a Hamiltonian cycle or a Hamiltonian path.

n	1	2	3	4	5	6	7	8	9	10	11	12
cycle?	no	no	no	✓	no	no	✓	no	no	✓	no	no
path?	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Thus this replacement is the same as if we replace d by $q + x$ in the generating function in Proposition 7.1, and this substitution gives the generating function of the proposition. $\square$

The coefficients of the generating function in Proposition 8.2 for $1 \leq n \leq 6$ are

$$1 + q + x,$$
$$1 + 2q + 2x,$$
$$1 + 3q + q^2 + (3 + 2q)x + x^2,$$
$$1 + 4q + 3q^2 + (4 + 6q)x + 3x^2,$$
$$1 + 5q + 7q^2 + (5 + 14q)x + 7x^2,$$
$$1 + 6q + 12q^2 + 2q^3 + (6 + 24q + 6q^2)x + (12 + 6q)x^2 + 2x^3.$$

So for example, the term $(12 + 6q)x^2$ in the last polynomial in this list indicates that in $\mathcal{R}_6$, there are 12 squares (Q_2's) that contain the all zero vertex, and 6 squares whose distance to the all zero vertex is one.

9. Hamiltonicity and further directions

It is well known and follows from the fundamental decomposition of Fibonacci cubes, that Γ_n has a Hamiltonian path for every $n \geq 0$ [3]. The corresponding result for Fibonacci-run graphs is more difficult. Properties of the graphs $\mathcal{R}_n$ for $1 \leq n \leq 12$ are presented in Table 1.

Since the graphs $\mathcal{R}_n$ are bipartite (vertices with odd/even Hamming weight form the two parts), the graph can only be Hamiltonian if both parts are of equal size. Let Δ_n denote the number of vertices of even Hamming weight minus the number of vertices of odd Hamming weight in $\mathcal{R}_n$. Using Corollary 3.2, we calculate that

$$\Delta_n = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 3, \\ -1 & \text{if } n \equiv 2, 3 \pmod 6, \\ 1 & \text{if } n \equiv 0, 5 \pmod 6. \end{cases}$$

Therefore we have

Lemma 9.1. *If $n \not\equiv 1$ (mod 3), then $\mathcal{R}_n$ does not contain a Hamiltonian cycle.*

Even though this lemma does not solve the Hamiltonicity of Fibonacci-run graphs completely, based on the decomposition in Fig. 3, and the computational values we have obtained in Table 1, we conjecture that $\mathcal{R}_n$ is Hamiltonian if and only if $n \equiv 1$ (mod 3) and that $\mathcal{R}_n$ always has a Hamiltonian path. We remark that the decomposition of $\mathcal{R}_n$ in terms of smaller dimensional Fibonacci-run graphs as indicated in Fig. 3 indicates various approaches for the construction of the Hamiltonian paths inductively. The decomposition would require the entry and the exit vertices of the Hamiltonian path for each smaller dimensional Fibonacci-run graph in a uniform manner. Such a construction is a topic of further interest.

However, it turns out that the problem can be approached using Gray codes, and has recently been resolved by Baril, Kirgizov, and Vajnovszki [2].

Theorem 9.2 ([2, Corollary 8]). *Every Fibonacci-run graph has a Hamiltonian path.*

In conjunction with the lower bound in Lemma 3.5 we then have

Corollary 9.3. *For $n \geq 1$, $\alpha(\mathcal{R}_n) = \left\lceil \frac{f_{n+2}}{2} \right\rceil$.*

It would be satisfying to find a proof of the expression in Lemma 3.3 for the number of edges combinatorially, without using the decomposition of $\mathcal{R}_n$, and also prove the conjecture on the exact value of the diameter given by the expression in (7). Among many graph theoretical parameters of Fibonacci-run graphs that can be further studied, we can also mention the problem of the determination of the radius of $\mathcal{R}_n$.
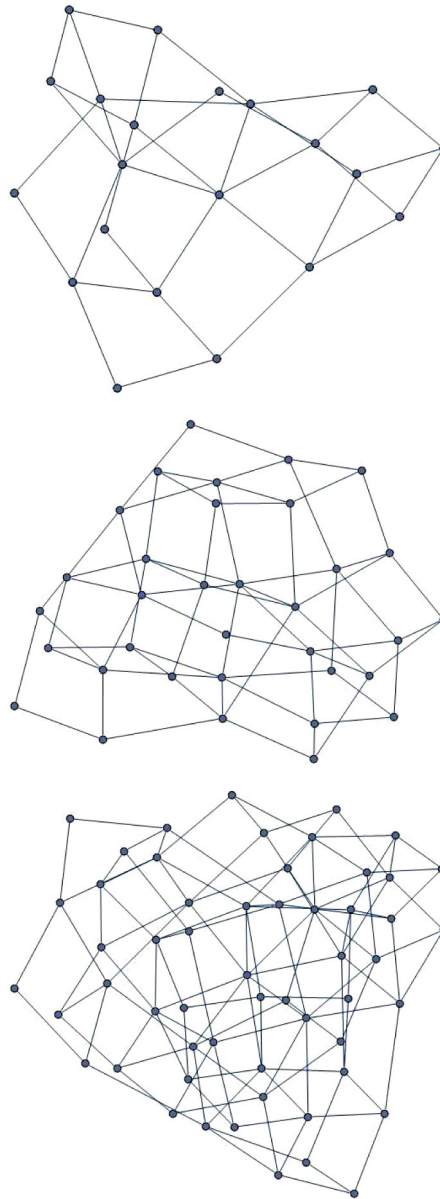
Fig. 6. Fibonacci-run graphs $\mathcal{R}_6$, $\mathcal{R}_7$ and $\mathcal{R}_8$.

Acknowledgments

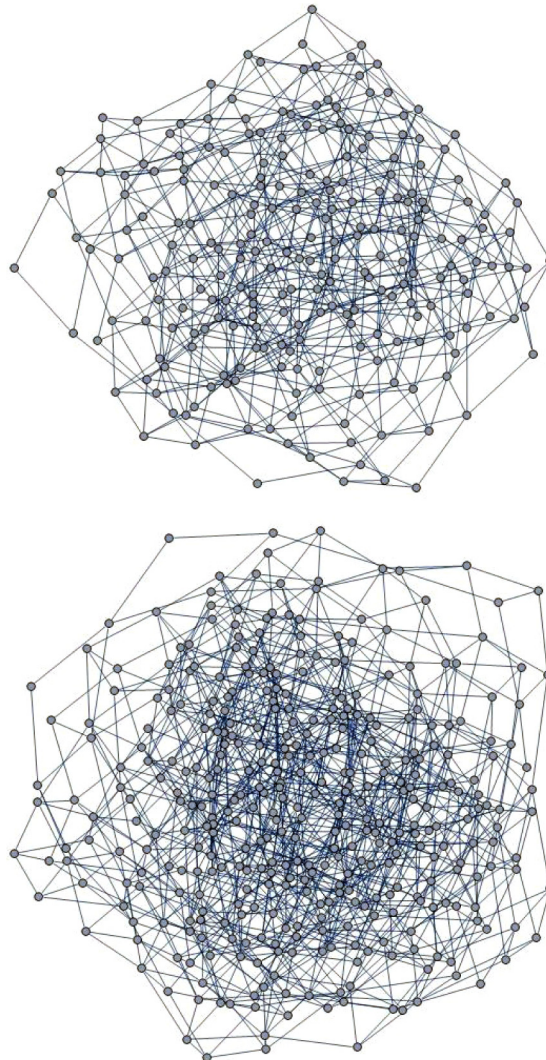## Appendix. Figures of some Fibonacci-run graphs

See Figs. 6 and 7.

Fig. 7. Fibonacci-run graphs $\mathcal{R}_{11}$ and $\mathcal{R}_{12}$.

References

[1] J. Azarija, S. Klavžar, Y. Rho, S. Sim, On domination-type invariants of Fibonacci cubes and hypercubes, Ars Math. Contemp. 14 (2) (2018) 387–395.
[2] J.-L. Baril, S. Kirgizov, V. Vajnovszki, Gray codes for Fibonacci q-decreasing words, 2020, arXiv:2010.09505 [cs.DM].
[3] B. Cong, S.Q. Zheng, S. Sharma, On simulations of linear arrays, rings and 2d meshes on Fibonacci cube networks, in: [1993] Proceedings Seventh International Parallel Processing Symposium, 1993, pp. 748–751.
[4] O. Eğecioğlu, V. Iršič, Fibonacci-run graphs II: degree sequences, 2020, arXiv:2010.05521v1 [math.CO].
[5] O. Eğecioğlu, E. Saygı, Z. Saygı, k-Fibonacci cubes: A family of subgraphs of Fibonacci cubes, Internat. J. Found. Comput. Sci. 31 (5) (2020) 639–661.
[6] W.J. Hsu, Fibonacci cubes - new interconnection topology, Parallel Distrib. Syst. IEEE Trans. 4 (1993) 3–12.
[7] A. Ilić, S. Klavžar, Y. Rho, Generalized Fibonacci cubes, Discrete Math. 312 (1) (2012) 2–11.
[8] W. Imrich, S. Klavžar, Product graphs, structure and recognition 01, 2000.
[9] S. Klavžar, On median nature and enumerative properties of Fibonacci-like cubes, Discrete Math. 299 (1–3) (2005) 145–153.
[10] S. Klavžar, Structure of Fibonacci cubes: A survey, J. Comb. Optim. 25 (4) (2013) 505–522.
[11] S. Klavžar, M. Mollard, Asymptotic properties of Fibonacci cubes and Lucas cubes, Ann. Comb. 18 (3) (2014) 447–457.
[12] S. Klavžar, M. Mollard, Daisy cubes and distance cube polynomial, European J. Combin. 80 (2019) 214–223.
[13] S. Klavžar, M. Mollard, M. Petkovšek, The degree sequence of Fibonacci and Lucas cubes, Discrete Math. 311 (14) (2011) 1310–1322.
[14] M. Mollard, Non covered vertices in Fibonacci cubes by a maximum set of disjoint hypercubes, Discrete Appl. Math. 219 (2017) 219–221.
[15] E. Munarini, C. Perelli Cippo, N. Zagaglia Salvi, On the Lucas cubes, Fibonacci Quart. 39 (1) (2001) 12–21.
[16] K.S. Savitha, A. Vijayakumar, Some diameter notions of Fibonacci cubes, Asian-Eur. J. Math. 13 (3) (2020) 2050057, 9.
[17] E. Saygı, On the domination number and the total domination number of Fibonacci cubes, Ars Math. Contemp. 16 (1) (2019) 245–255.

[18] E. Saygı, O. Eğecioğlu, q-cube enumerator polynomial of Fibonacci cubes, Discrete Appl. Math. 226 (2017) 127–137.
[19] E. Saygı, O. Eğecioğlu, Boundary enumerator polynomial of hypercubes in Fibonacci cubes, Discrete Appl. Math. 266 (2019) 191–199.
[20] P.M. Winkler, Isometric embedding in products of complete graphs, Discrete Appl. Math. 7 (2) (1984) 221–225.
[21] E. Zeckendorf, Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, Bull. Soc. Roy. Sci. Liège 41 (1972) 179–182.