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EXTREMAL SETS MINIMIZING DIMENSION-NORMALIZED BOUNDARY IN HAMMING GRAPHS*

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Abstract. We prove that the set of first k vertices of a Hamming graph in reverse-lexicographic order constitutes an extremal set minimizing the dimension-normalized edge-boundary over all k-vertex subsets of the graph. This generalizes a result of Lindsey and can be used to prove a tight lower bound for the isoperimetric number and the bisection width of arrays.

Key words. Hamming graph, product graph, array, isoperimetric number, bisection width, extremal set, partition, Schur-convexity, majorization

AMS subject classifications. 05C35, 05C40, 05D05

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1. Introduction.

We consider questions of the following general form: Given a graph G and a natural number k, what is the optimum value of a certain quantity in a set of k vertices of G? The desired quantity could be the number of edges between a set of k vertices and its complement (i.e., the size of the boundary) or the number of edges induced by a set of k vertices, etc. The sets achieving the optimum value are called *extremal sets*.

Specifically, we study extremal sets in Hamming graphs minimizing the size of the edge-boundary of a set of vertices of given size, where boundary edges along each dimension are *normalized* by a weight determined by that dimension, as shall soon be explained.

First, we introduce some notation and terminology. Given a graph G and a subset X of its vertices, let ∂X denote the *edge-boundary*, or simply *boundary*, of X. This is the set of edges connecting vertices in X with vertices not in X (i.e., the complement of X). A *d-dimensional Hamming graph* H^d is a graph with $k_1 \times k_2 \times \cdots \times k_d$ vertices, $k_1 \leq k_2 \leq \cdots \leq k_d$, each having a unique label $l = \langle l_1, l_2, \ldots, l_d \rangle$, where $0 \leq l_i \leq k_i - 1$. There is an edge between two vertices iff their labels differ in exactly one digit. A *d-dimensional array* A^d resembles H^d with the exception that two vertices are adjacent iff their labels differ in exactly one digit *and* the difference is exactly one. Examples of a two-dimensional Hamming graph and a two-dimensional array are shown in Figure 1.

The *Cartesian product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which vertices (u, v) and (u', v') are adjacent iff u is adjacent to u' in G and $v = v'$, or v is adjacent to v' in H and $u = u'$. The constituent graphs G and H are called *factors*. A Hamming graph can be characterized as the Cartesian product of a number of *complete graphs* of different sizes, i.e., $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$, where K_r is a complete graph on r vertices. Similarly, A^d can be characterized as the Cartesian product of a number of *path graphs* of varying length, i.e., $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$, where P_r is a path graph (chain) with r vertices.

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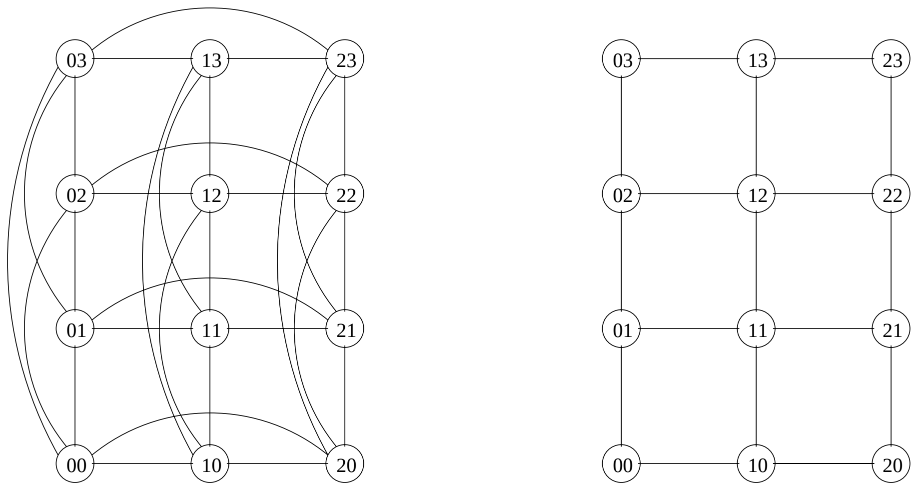

FIG. 1. *The two-dimensional Hamming graph $K_3 \times K_4$ and array $P_3 \times P_4$.*

Lindsey [19] proved that the set of first k vertices of a Hamming graph in *lexicographic order* constitutes an extremal set minimizing the boundary ∂X over all k-element subsets X. The lexicographic order is defined as follows: In the Hamming graph $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ with $k_1 \leq k_2 \leq \cdots \leq k_d$, vertex $x = \langle x_1, \ldots, x_d \rangle$ precedes vertex $y = \langle y_1, \ldots, y_d \rangle$ in lexicographic order iff there exists an index i such that $x_1 = y_1$, $x_2 = y_2$, $\ldots$, $x_{i-1} = y_{i-1}$ and $x_i < y_i$ holds. Intuitively, in lexicographic order, we traverse the Hamming graph in the direction of the *next largest factor* starting with the vertex labeled $\langle 0, 0, \ldots, 0 \rangle$. For instance, the vertices of the Hamming graph in Figure 1 in lexicographic order are labeled

$$00,\ 01,\ 02,\ 03,\ 10,\ 11,\ 12,\ 13,\ 20,\ 21,\ 22,\ 23.$$

Our aim in this paper is to determine and describe extremal sets of Hamming graphs minimizing the dimension-normalized boundary. This is defined next.

DEFINITION 1.1. *Given a Hamming graph $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ and a subset X of its vertices,* the dimension-normalized boundary $B(X)$ *of X is defined as*

$$B(X) = \frac{|\partial_1 X|}{c_1} + \frac{|\partial_2 X|}{c_2} + \cdots + \frac{|\partial_d X|}{c_d}, \tag{1.1}$$

where for $1 \leq i \leq d$, $\partial_i X$ is the set of boundary edges along dimension i and

$$c_i = \begin{cases} k_i^2 & \text{if } k_i \text{ is even,} \\ k_i^2 - 1 & \text{if } k_i \text{ is odd.} \end{cases} \tag{1.2}$$

We prove that the set of first k vertices in *reverse-lexicographic order* constitutes an extremal set minimizing the dimension-normalized boundary over all k-element subsets in a Hamming graph. The definition of the reverse-lexicographic order is similar to that of the lexicographic order: In the Hamming graph $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ with $k_1 \leq k_2 \leq \cdots \leq k_d$, vertex $x = \langle x_1, \ldots, x_d \rangle$ precedes vertex $y = \langle y_1, \ldots, y_d \rangle$ in reverse-lexicographic order iff there exists an index i such that $x_d = y_d$, $x_{d-1} = y_{d-1}$, $\ldots$, $x_{i+1} = y_{i+1}$ and $x_i < y_i$ holds. In other words, we move in the direction of the *next smallest factor* starting at the vertex labeled $\langle 0, 0, \ldots, 0 \rangle$. To illustrate, the vertices of the Hamming graph in the above example listed in reverse-lexicographic order are

$$00,\ 10,\ 20,\ 01,\ 11,\ 21,\ 02,\ 12,\ 22,\ 03,\ 13,\ 23.$$

We should point out that there are other sets of vertices which are structurally equivalent to the sets specified in our definitions of lexicographic or reverse-lexicographic orders. These are obtained by symmetries in the underlying graph. For instance, in Figure 1, another ordering structurally equivalent to the lexicographic ordering would be 23, 22, 21, 20, 13, etc. Similarly, the sets defined by the initial segments of the ordering 23, 13, 03, 22, 12, etc., give rise to sets structurally identical to those in reverse-lexicographic order.

We state our claim formally in the following theorem.

THEOREM 1.2. *Given a d-dimensional Hamming graph H^d, let X be any k-vertex subset of $V(H^d)$ and $\overline{X}$ be the set of first k vertices of H^d in reverse-lexicographic order. Then $B(\overline{X}) \leq B(X)$.*

Interestingly, when all factors of H^d have equal size, the lexicographic and reverse-lexicographic orders both result in structurally symmetric subsets and hence are equivalent with respect to extremal sets minimizing the boundary (dimension-normalized or otherwise). Therefore Theorem 1.2 is trivially true when $k_1 = k_2 = \cdots = k_d$ by Lindsey's result, since the denominators c_i in (1.1) will all be equal and minimizing $B(X)$ will be equivalent to minimizing $|\partial_1 X| + |\partial_2 X| + \cdots + |\partial_d X| = |\partial X|$, i.e., the size of the boundary of X.

In the next section, we describe the notion of the isoperimetric number, which is a quantity closely related to extremal sets. The isoperimetric number problem for special classes of graphs provides the basis of our motivation for this work.

1.1. Motivation. An important quantity in the theory of graphs is the *isoperimetric number* $i(G)$ of a graph G, defined as

$$i(G) = \min_{1 \leq |X| \leq \frac{|V(G)|}{2}} \frac{|\partial X|}{|X|}, \tag{1.3}$$

where $X \subseteq V(G)$. That is, the set of vertices of G is partitioned into two nonempty sets and the ratio of the number of edges between the two parts and the number of vertices in the smaller one is minimized. A subset X achieving the equality in (1.3) is called an *isoperimetric set*.

The notion of the isoperimetric number of a graph G serves as a measure of connectivity of G as it quantifies the minimal interaction between a set of vertices X and its complement $V(G) \setminus X$ in terms of the number of edges between them. In many instances, the isoperimetric number of a graph can be used to obtain a tight lower bound for its *bisection width* as well [18]. We refer the reader to Mohar [22] or Chung [12] for a discussion of basic results and various interesting properties of $i(G)$.

At present, the isoperimetric number of an array $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ is known only when either $k_1 = k_2 = \cdots = k_d$ (see Azizoğlu and Eğecioğlu [4]) or the size of the largest factor is even (see Azizoğlu and Eğecioğlu [5]). The latter is also implicit in [10] (see also [17]). See also [3] and [13]. The techniques used to obtain these results seem to fail in the general case. However, using the notion of extremal sets minimizing dimension-normalized boundary together with a result of Nakano [23], one can show that

$$i(A^d) = \min_i \frac{1}{\lfloor \frac{k_i}{2} \rfloor}.$$

The technique used involves embedding a Hamming graph into A^d and associating these extremal sets with isoperimetric sets of the array. We refer the reader to [6] for details.

1.2. A summary of previous results. There has been a significant amount of research in the area of isoperimetric bounds on various popular classes of graphs such as Hamming graphs, arrays, and tori. We shall only mention those results in this area which pertain to our discussion and refer the reader to Bezrukov [8] for a comprehensive survey and Bollobás [9] for a general discussion of this and related topics.

As mentioned before, an extremal set of a graph for a given k is, in a broad sense, a configuration of k vertices with

- minimum number of boundary edges or
- maximum number of spanned edges

among all such k-vertex subsets of the given graph. The problem of finding extremal sets of the first (or second) type is called *the minimum-boundary-edge problem* (or *the maximum-induced-edge problem*). It can be shown that the minimum-boundary-edge and the maximum-induced-edge problems are equivalent for regular graphs [11]. We remark that one can easily obtain the isoperimetric number of a given graph if the extremal sets of the first type are known (and the boundary is actually computable). Evidently, an extremal set X with $\lfloor |V(G)|/2 \rfloor$ vertices in a given graph G determines a bisection for G.

The maximum-induced-edge problem (hence the minimum-boundary-edge problem, because of its regularity) for the hypercube (d-dimensional binary Hamming graph) was solved by Harper [14] and extended by Lindsey [19] to the d-dimensional k-ary Hamming graph. In both instances, there is a nested structure of solutions, and the first k vertices in *lexicographic order* constitute an extremal set. The maximum-induced-edge problem for the d-dimensional k-ary array A_k^d was solved by Bollobás and Leader [11]. Since A_k^d is not regular, this result does not automatically give a solution to the minimum-boundary-edge problem. It was later extended to general arrays by Ahlswede and Bezrukov [1] who also gave a solution for $P_{k_1} \times P_{k_2}$ for the minimum-boundary-edge problem. The first nontrivial bounds on the minimum-boundary-edge problem for the d-dimensional k-ary arrays are in Bollobás and Leader [11]. Unfortunately, however, the bounds obtained are not tight enough to yield an exact formula for $i(A_k^d)$.

Similar problems have been studied in the literature for the vertex-boundary of a given configuration of vertices. For instance, for the d-dimensional k-ary torus, Bollobás and Leader [10] solved the vertex-boundary problem for even k. Riordan [24] later extended their result by giving an ordering of vertices on the d-dimensional even torus, which minimizes the number of vertices at shortest distance t from the vertices in the ordering. Wang and Wang [25] solved this problem for $P_\infty \times \cdots \times P_\infty$, i.e., the *$d$-dimensional infinite array*, where the minimum is taken over all nonempty finite subsets of vertices. In their result, each P_∞ may be infinite in both directions or in one direction only. They also gave a simple ordering of the vertices in which the first k vertices constitute an extremal set minimizing the vertex-boundary. In a recent paper, Harper [15] solved the vertex-boundary problem on Hamming graphs.

1.3. Outline. The outline of the remainder of this paper is as follows. In section 2 we consider the case of two-dimensional Hamming graphs. First we define the terminology we use and state a number of basic facts on restricted integer partitions, majorization, and Schur-convexity. Then we identify potential extremal sets in H^2 as integer partitions inside a rectangle. The problem of showing that the set of first k vertices of H^2 in reverse-lexicographic order constitutes an extremal set minimizing the dimension-normalized edge-boundary over all k-vertex subsets becomes the problem of maximization of a certain function on partitions, which is a linear combination of two Schur-convex functions. However, the function itself is not Schur-convex, and the identification of the partition on which the maximum is achieved is actually done using an inductive argument. The main result of this section is Lemma 2.5. In section 3 we extend the proof to the higher-dimensional case. This is done by an induction on the number of dimensions, using the two-dimensional result as the base case. Finally, concluding remarks are given in section 4.

2. The two-dimensional case. Let $H^2 = K_m \times K_n$ be a given two-dimensional Hamming graph. Without loss of generality, we may assume that $m = k_1 \le k_2 = n$. Consider a subset X of vertices in H^2. Let X' be the subset of vertices of H^2 obtained by pushing (compressing) all the vertices in X as far downward and then to the left in H^2 as possible. It is easy to see (and proved in [19], [16]) that $B(X') \le B(X)$ since the number of boundary edges in either dimension will not increase as a result of this procedure. A subset X' in the compressed form corresponds to a *partition* of the integer $|X|$ contained in the $m \times n$ rectangle.

We give below the definitions and properties of partitions that we will use in our proof of Theorem 1.2. The reader is referred to [2] for further details.

Partitions. A *partition* λ of an integer N is a sequence $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of positive integers (called *parts*) satisfying $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\ell$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = N$. We put $|\lambda| = N$. The Ferrers diagram of λ is a two-dimensional array of unit cells (or nodes) in which row i from the bottom has λ_i cells and the rows are left justified. It is clear that, in our case, $\lambda = X'$ forms a partition of $|X|$ whose Ferrers diagram is contained in the $m \times n$ rectangle, i.e., $\lambda_i \le m$ for $1 \le i \le \ell$ (i.e., each part at most m) and $\ell \le n$ (i.e., number of parts at most n). We use $\mathbb{P}(m,n)$ to denote the set of these partitions. Thus we may assume that an extremal set is a partition $\lambda \in \mathbb{P}(m,n)$, and we use the symbol $\mathbb{P}(m,n)$ to refer to $H^2 = K_m \times K_n$ when we are not interested in the graph structure of H^2 but just the placement of the subset λ. We may augment partitions by adding parts of zero length and write $\sum_{i\ge 1} \lambda_i$ for $|\lambda|$. We also identify partitions with their diagrams when there is no confusion.

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, we may define a new partition $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_m)$ by choosing λ'_i as the number of parts of λ that are $\ge i$. The partition λ' is called the *conjugate* of λ. Geometrically, λ' is obtained from λ by reflection in the main diagonal (equivalently by counting the cells in successive columns of λ). For example, the conjugate of $(5,4,3,3,1,1)$ is $(6,4,4,2,1)$. Clearly $|\lambda| = |\lambda'|$, and if $\lambda \in \mathbb{P}(m,n)$, then $\lambda' \in \mathbb{P}(n,m)$.

Durfee square. Let $d = d(\lambda)$ denote the number of λ_i such that $\lambda_i \ge i$. Then d measures the largest square of cells contained in the partition λ, i.e., the number of cells on the main diagonal of λ, the cells with coordinates of the form (i,i). This square is called the *Durfee square*, and d is called the side of the Durfee square. For the partition $\lambda = (5,4,3,3,1,1)$, the side of the Durfee square is $d = 3$.

Frobenius notation. Suppose d is the side of the Durfee square of λ. Let $\alpha_i = \lambda_i - i$ be the number of cells in the ith row of λ to the right of (i,i) for $1 \le i \le d$, and let $\beta_i = \lambda'_i - i$ be the number of cells in the ith column of λ above (i,i) for $1 \le i \le d$. Then we have $\alpha_1 > \alpha_2 > \cdots > \alpha_d \ge 0$ and $\beta_1 > \beta_2 > \cdots > \beta_d \ge 0$. The

Frobenius notation for λ is

$$\lambda = (\alpha_1, \ldots, \alpha_d | \beta_1, \ldots, \beta_d) = (\alpha|\beta).$$

For example, if $\lambda = (5, 4, 3, 3, 1, 1)$, then $\alpha = (4, 2, 0)$ and $\beta = (5, 2, 1)$ as shown in Figure 2.

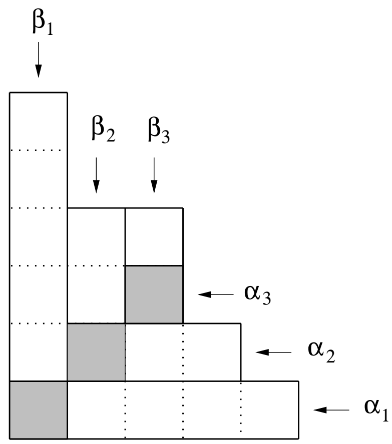

FIG. 2. *The main diagonal (cells in dark) of the* 3×3 *Durfee square and* $\alpha = (4, 2, 0)$, $\beta = (5, 2, 1)$ *of the Frobenius notation for the partition* $\lambda = (5, 4, 3, 3, 1, 1)$.

Reverse-lexicographic ordering on partitions. Given partitions λ and μ, μ *precedes* λ in reverse-lexicographic ordering, denoted by $\mu \geq \lambda$, if either $\lambda = \mu$ or else the first nonvanishing difference $\lambda_i - \mu_i$ is positive. Reverse-lexicographic ordering is a total order. For example, partitions of $N = 5$ are ordered by reverse-lexicographic ordering as

$$(5) \geq (4, 1) \geq (3, 2) \geq (3, 1, 1) \geq (2, 2, 1) \geq (2, 1, 1, 1) \geq (1, 1, 1, 1, 1),$$

the first (or the "smallest" one) being (5). The reason for this reversed notation is for consistency with the *dominance order* on partitions that we later define.

Majorization, Schur-convexity, and transfer. Given two partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ of N, λ is *majorized* by μ, written $\lambda \prec \mu$, if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \leq \mu_1 + \mu_2 + \cdots + \mu_k, \qquad k = 1, 2, \ldots, N.$$

Majorization is also referred to as the *dominance* or *natural* order [20, Chap. 1]. As soon as $N \geq 6$, majorization is not a total ordering. For example, the partitions $(3, 1, 1, 1)$ and $(2, 2, 2)$ of 6 are not comparable. However, reverse-lexicographic ordering on partitions is a linear extension of $\prec$. Thus

$$\lambda \prec \mu \;\Rightarrow\; \lambda \leq \mu.$$

Furthermore $\lambda \prec \mu \Leftrightarrow \mu' \prec \lambda'$ (see [20, (1.11)]). A real-valued function g defined on partitions of an integer N is said to be *Schur-convex* (see [21, Chap. 3]) if

$$\lambda \prec \mu \;\Rightarrow\; g(\lambda) \leq g(\mu).$$

We make use of the following special case of a result of Schur, 1923, and Hardy, Littlewood, and Polya, 1929 (see [21, Chap. 3, Prop. C.1]).

PROPOSITION 2.1. *Suppose ϕ is a real-valued convex function on $\mathbb{R}$ and N is a positive integer. Then the function*

$$g(\lambda) = \sum_{i\geq 1} \phi(\lambda_i)$$

is Schur-convex on partitions of N.

Given a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ with $\mu_i > \mu_j$, the transformation that takes μ to $\rho = (\rho_1, \rho_2, \ldots, \rho_N)$ defined by

$$\begin{aligned} \rho_i &= \mu_i - 1, \\ \rho_j &= \mu_j + 1, \\ \rho_k &= \mu_k, \qquad\qquad k \neq i, j, \end{aligned}$$

is called a *transfer* from i to j. By a result of Muirhead, if $\lambda \prec \mu$, then λ can be derived from μ by successive application of a finite number of transfers [21, Chap. 5, D.1], [20, (1.16)].

Now consider a partition $\lambda \in \mathbb{P}(m, n)$, where $k_1 = m \leq n = k_2$, which corresponds to a compressed set in $H^2 = K_m \times K_n$. Let $\partial_m \lambda$ and $\partial_n \lambda$ be sets of horizontal and vertical boundary edges of λ, respectively. Then we have

$$|\partial_m \lambda| = \sum_{\lambda_i > 0} \lambda_i (m - \lambda_i) \qquad \text{and} \qquad |\partial_n \lambda| = \sum_{\lambda'_j > 0} \lambda'_j (n - \lambda'_j).$$

After substituting these into (1.1) and eliminating constant terms, we see that finding a subset $\lambda \in K_m \times K_n$ minimizing $B(\lambda)$ is equivalent to *maximizing* the following function f:

$$\begin{aligned} f(\lambda) &= c_1 \sum_{i=1}^{\lambda'_1} \lambda_i^2 + c_2 \sum_{j=1}^{\lambda_1} \lambda_j'^2 \\ &= \gamma_n \sum_{i\geq 1} \lambda_i^2 + \gamma_m \sum_{j \geq 1} \lambda_j'^2 \end{aligned} \tag{2.1}$$

on $\mathbb{P}(m, n)$ $(m \leq n)$, where

$$\gamma_n = \begin{cases} n^2 & \text{if } n \text{ is even,} \\ n^2 - 1 & \text{if } n \text{ is odd} \end{cases} \tag{2.2}$$

in accordance with the definition of the weights c_i in (1.2). We prove the following equivalent formulation of Theorem 1.2 for $H^2 = K_m \times K_n$:

THEOREM 2.2. *When restricted to partitions of a fixed $N \leq mn$, the function f defined in* (2.1) *is maximized on $\mathbb{P}(m, n)$, $m \leq n$, by the reverse-lexicographically smallest partition of N in $\mathbb{P}(m, n)$.*

The proof of the main result of this paper, and consequently the proof of the formula for the isoperimetric number of arrays itself (see [6]) which uses this result, would be simplified by an independent proof of this fact. However, the function f is not Schur-convex. In other words, transfer operators [21, Chap. 5, D.1] or equivalently raising/lowering operators (see [20, (1.15)–(1.16)]) which move from a given λ to a smaller one in the linear order while keeping the value of f nondecreasing are insufficient to prove this fact. As an example take $m = 4$, $n = 8$, $\lambda = (4, 1, 1, 1, 1, 1, 1, 1)$, $\mu = (4, 2, 1, 1, 1, 1)$. Then $\lambda \prec \mu$ by a single transfer as shown in Figure 3, but $2240 = f(\lambda) > f(\mu) = 2208$. It can be shown that transfer arguments can be used to prove Theorem 2.2 in the special case when $n \geq m^2$.

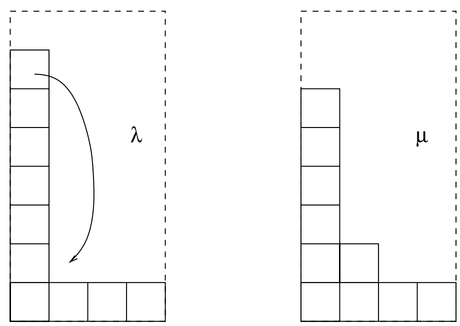

FIG. 3. *μ is obtained from λ by single transfer (lowering the indicated cell). Here $m = 4$, $n = 8$, $\lambda \prec \mu$ but $f(\lambda) = 2240$, whereas $f(\mu) = 2208$.*

We reformulate $f(\lambda)$ in (2.1) in a form which is more convenient for our characterization. Given a partition $\lambda \in \mathbb{P}(m, n)$, let $\widetilde{\lambda}$ be the partition in $\mathbb{P}(m-1, n-1)$ which is obtained by removing the bottommost row and the leftmost column from the $m \times n$ rectangle. Now consider the term $\gamma_n \sum \lambda_i^2$ of $f(\lambda)$ in (2.1). By taking the first term λ_1^2 out of the summation and putting $\lambda_i = (\lambda_i - 1) + 1$, we have

$$\begin{aligned}
\gamma_n \sum_{i\geq 1} \lambda_i^2 &= \gamma_n \left[\lambda_1^2 + \sum_{i=2}^{\lambda_1'} ((\lambda_i - 1) + 1)^2 \right] \\
&= \gamma_n \left[\lambda_1^2 + \sum_{i\geq 1} \widetilde{\lambda}_i^2 + 2 \sum_{i\geq 2} \lambda_i - \sum_{i=2}^{\lambda_1'} 1 \right] \\
&= \gamma_n \left[\lambda_1^2 + \sum_{i\geq 1} \widetilde{\lambda}_i^2 + 2(|\lambda| - \lambda_1) - (\lambda_1' - 1) \right].
\end{aligned}$$

Putting the second term $\gamma_m \sum_{j\geq 1} \lambda_j'^2$ of $f(\lambda)$ as above, we now have

$$\begin{aligned}
f(\lambda) = \gamma_n &\left[\lambda_1^2 + \sum_{i\geq 1} \widetilde{\lambda}_i^2 + 2(|\lambda| - \lambda_1) - (\lambda_1' - 1) \right] \\
&+ \gamma_m \left[\lambda_1'^2 + \sum_{j\geq 1} \widetilde{\lambda}_j'^2 + 2(|\lambda| - \lambda_1') - (\lambda_1 - 1) \right].
\end{aligned}$$

Since $f(\widetilde{\lambda}) = \gamma_n \sum_{i\geq 1} \widetilde{\lambda}_i^2 + \gamma_m \sum_{j\geq 1} \widetilde{\lambda}_j'^2$, this is equivalent to

$$\begin{aligned}
f(\lambda) = f(\widetilde{\lambda}) &+ \gamma_n \left[\lambda_1^2 + 2(|\lambda| - \lambda_1) - (\lambda_1' - 1) \right] \\
&+ \gamma_m \left[\lambda_1'^2 + 2(|\lambda| - \lambda_1') - (\lambda_1 - 1) \right].
\end{aligned} \tag{2.3}$$

Suppose now $\lambda = (\alpha|\beta) = (\alpha_1, \ldots, \alpha_d|\beta_1, \ldots, \beta_d)$. Then the partition $\widetilde{\lambda}$ obtained from λ by deleting the leftmost column and the bottommost row of the $m \times n$ rectangle is a partition $(\widetilde{\alpha}|\widetilde{\beta})$ which has a Durfee square of size $d - 1$, where $\widetilde{\lambda} = (\widetilde{\alpha}|\widetilde{\beta}) = (\alpha_2, \ldots, \alpha_d|\beta_2, \ldots, \beta_d)$. Using this and $\lambda_1 = 1 + \alpha_1$, $\lambda'_1 = 1 + \beta_1$, equality (2.3) can be reformulated as

$$f(\alpha|\beta) = f(\widetilde{\alpha}|\widetilde{\beta}) + \gamma_n \alpha_1^2 - \gamma_m \alpha_1 + \gamma_m \beta_1^2 - \gamma_n \beta_1 + (\gamma_n + \gamma_m)(2|\lambda| - 1). \tag{2.4}$$

Since the last term $(\gamma_n + \gamma_m)(2|\lambda| - 1)$ is constant for all configurations with size $|\lambda|$, iterating this expression we have the following proposition.

PROPOSITION 2.3. *Over partitions $\lambda = (\alpha|\beta)$ in $\mathbb{P}(m, n)$ of a fixed integer $|\lambda|$ with Durfee square of size d, maximizing $f(\lambda)$ is equivalent to maximizing*

$$\gamma_n \left[\alpha_1^2 + \cdots + \alpha_d^2 - (\beta_1 + \cdots + \beta_d) \right] + \gamma_m \left[\beta_1^2 + \cdots + \beta_d^2 - (\alpha_1 + \cdots + \alpha_d) \right]. \tag{2.5}$$

Durfee-equivalence. Suppose $\lambda \in \mathbb{P}(m, n)$ has a Durfee square D of size d. Let $\nu = \nu(\lambda)$ denote the partition that lies north (on top) of D and $\eta = \eta(\lambda)$ the partition that lies east (to the right) of D. Then $\nu_1 \leq d$ and $\nu'_1 \leq n - d$, and $\eta'_1 \leq d$ and $\eta_1 \leq m - d$. Two partitions $\lambda, \mu \in \mathbb{P}(m, n)$ are *Durfee-equivalent* iff

1. $d(\lambda) = d(\mu)$,
2. $|\nu(\lambda)| = |\nu(\mu)|$ and $|\eta(\lambda)| = |\eta(\mu)|$.

We single out a special representative λ^* in the equivalence class of partitions Durfee-equivalent to λ. λ^* is the partition in which η is the largest in the dominance order in the $d \times (m - d)$ rectangle to the right of the Durfee square and ν' is the largest in the dominance order in the $(n - d) \times d$ rectangle to the top of the Durfee square. In other words, in λ^*, η^* is obtained by distributing $|\eta|$ cells into as many rows as possible of length $m - d$, followed by a (possibly null) partial row of size r. Similarly in λ^*, ν^* is obtained by distributing $|\nu|$ cells by first laying as many columns as possible of length $n - d$, followed by a (possibly null) partial column of size s. An example of this is shown in Figure 4.

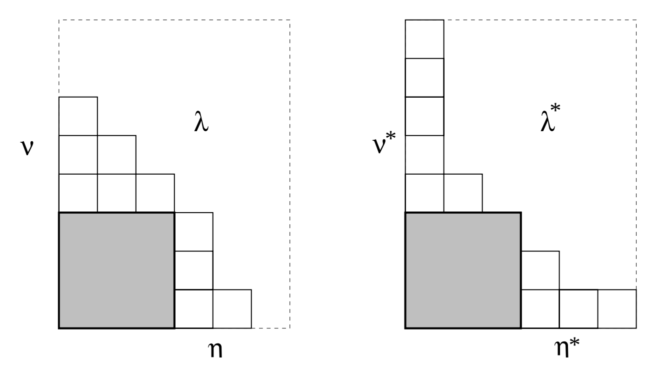

FIG. 4. *Partition $\lambda = (5, 4, 4, 3, 2, 1)$ is Durfee-equivalent to the special representative $\lambda^* = (6, 4, 3, 2, 1, 1, 1, 1)$ in $\mathbb{P}(6, 8)$.*

PROPOSITION 2.4. *Suppose $\lambda \in \mathbb{P}(m, n)$ and λ^* is the special representative of λ in the equivalence class of partitions Durfee-equivalent to λ. Then $f(\lambda^*) \geq f(\lambda)$.*

Proof. We use Proposition 2.3. Since in Durfee-equivalence $|\nu|$ and $|\eta|$ do not change, $\alpha_1 + \cdots + \alpha_d$ and $\beta_1 + \cdots + \beta_d$ are constant. Thus maximizing f over the

Durfee-equivalence class of λ is equivalent to maximizing

$$\gamma_n \left(\alpha_1^2 + \cdots + \alpha_d^2\right) + \gamma_m \left(\beta_1^2 + \cdots + \beta_d^2\right),$$

which is decoupled. Since the function $\phi(x) = x^2$ is convex on $\mathbb{R}$, applying the majorization result of Proposition 2.1 to each term separately, we obtain the proposition. $\square$

Remark. Proposition 2.4 allows us to restrict potential maximizers of the function $f(\lambda)$ on $\lambda \in \mathbb{P}(m, n)$ to partitions of the form shown in Figure 5. Here $|\lambda| = d^2 + w(m-d) + t(n-d) + r + s$.

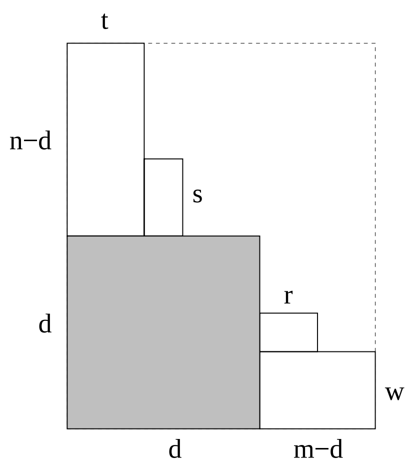

FIG. 5. *The form of special representatives of Durfee-equivalence classes of partitions in* $\mathbb{P}(m, n)$.

2.1. Extremal sets for the two-dimensional Hamming graph. Now we are ready to prove the two-dimensional case, which is stated using the terminology of Hamming graphs in the following lemma.

LEMMA 2.5. *Given a two-dimensional Hamming graph* $H^2 = K_m \times K_n$ *with* $m \leq n$, *let* λ *be any* k-*vertex subset of* $V(H^2)$ *and* $\overline{\lambda}$ *be the set of first* k *vertices of* H^2 *in reverse-lexicographic order. Then* $f(\overline{\lambda}) \geq f(\lambda)$. *That is,*

$$\gamma_n \sum_{i\geq 1} \overline{\lambda}_i^2 + \gamma_m \sum_{j \geq 1} \overline{\lambda}_j'^2 \geq \gamma_n \sum_{i \geq 1} \lambda_i^2 + \gamma_m \sum_{j \geq 1} \lambda_j'^2. \tag{2.6}$$

Proof. We give the proof only for n and m both even. The other cases are similar. By Proposition 2.4, we can assume that $\lambda = \lambda^*$ is the special representative in the Durfee-equivalence class of λ and is characterized by the parameters r, s, w, t, m, d, n as shown in Figure 5 with $|\lambda| = d^2 + w(m-d) + t(n-d) + r + s$. Using the original definition (2.1) of f, we compute

$$\gamma_n \sum \lambda_i^2 = n^2 \Big[wm^2 + (d+r)^2 + (d-w-1)d^2 + s(t+1)^2 + (n-d-s)t^2 \Big],$$

$$\gamma_m \sum_{j \geq 1} \lambda_j'^2 = m^2 \Big[tn^2 + (d+s)^2 + (d-t-1)d^2 + r(w+1)^2 + (m-d-r)w^2 \Big].$$

For simplicity, assume that m divides $|\lambda|$. Then $\overline{\lambda}$ consists of $|\lambda|/m$ rows of length m each. Thus

$$\gamma_n \sum \overline{\lambda}_i^2 = n^2 m \left[d^2 + w(m-d) + t(n-d) + r + s\right],$$
$$\gamma_m \sum_{j\geq 1} \overline{\lambda}_j'^2 = m \left[d^2 + w(m-d) + t(n-d) + r + s\right]^2.$$

Let $g(r,s,w,t,m,d,n) = f(\overline{\lambda}) - f(\lambda)$. Then

$$\begin{aligned} (2.7)\quad & g(r,s,w,t,m,d,n) \\ &= n^2 m \left[d^2 + w(m-d) + t(n-d) + r + s\right] \\ &\quad + m \left[d^2 + w(m-d) + t(n-d) + r + s\right]^2 \\ &\quad - n^2 \left[wm^2 + (d+r)^2 + (d-w-1)d^2 + s(t+1)^2 + (n-d-s)t^2\right] \\ &\quad - m^2 \left[tn^2 + (d+s)^2 + (d-t-1)d^2 + r(w+1)^2 + (m-d-r)w^2\right]. \end{aligned}$$

g is a polynomial of total degree 5 in the integer variables r, s, w, t, m, d, n, which is quadratic as a polynomial in r, s, w, t, and m, cubic in n, and quartic in d. Let R be region defined by the inequalities

$$\begin{aligned} & 0 \leq r \leq m-d, \\ & 0 \leq s \leq n-d, \\ (2.8)\qquad & 0 \leq w \leq d-1, \\ & 0 \leq t \leq d-1, \\ & d \leq m \leq n \end{aligned}$$

that we read off from Figure 5. Now we show that $g(r,s,w,t,m,d,n) \geq 0$ on R where g is as in (2.7) and R is the region defined in (2.8). Rewrite the inequalities in R in the form

$$\begin{aligned} & r_0 \leq r \leq r_1, \\ & s_0 \leq s \leq s_1, \\ & w_0 \leq w \leq w_1, \\ & t_0 \leq t \leq t_1, \\ & m_0 \leq m \leq m_1 \end{aligned}$$

with $r_0 = 0$, $r_1 = m-d$, and $s_0 = 0$, $s_1 = n-d$, etc., up to $m_0 = d$, $m_1 = n$. The idea of the proof is simple in theory: As a quadratic in r, we calculate that the leading coefficient is $m-n^2 \leq 0$. If, in addition, we can show that $g(r_0,s,w,t,m,d,n) \geq 0$ and $g(r_1,s,w,t,m,d,n) \geq 0$ on R, then we would be done. But this requires that we solve two subproblems: We need to show $g(r_0,s,w,t,m,d,n) \geq 0$ and $g(r_1,s,w,t,m,d,n) \geq 0$. Both of these are quadratic in s. If we can show that the leading coefficient in each is ≤ 0 on R *and* if each one evaluated in $s = s_0$ and $s = s_1$ is ≥ 0 on R, then we would be done. Iterating this argument, to prove the claim about the nonnegativity of g on R, it suffices to verify the following two assertions:

1. $g(r_{i_1}, s, w, t, m, d, n)$ has leading coefficient ≤ 0 on R as a polynomial in s, $g(r_{i_1}, s_{i_2}, w, t, m, d, n)$ has leading coefficient ≤ 0 on R as a polynomial in w, $g(r_{i_1}, s_{i_2}, w_{i_3}, t, m, d, n)$ has leading coefficient ≤ 0 on R as a polynomial in t, $g(r_{i_1}, s_{i_2}, w_{i_3}, t_{i_4}, m, d, n)$ has leading coefficient ≤ 0 on R as a polynomial in m for all 0-1 vectors (i_1, i_2, i_3, i_4),
2. $g(r_{i_1}, s_{i_2}, w_{i_3}, t_{i_4}, m_{i_5}, d, n)$ is ≥ 0 on R for each 0-1 vector $(i_1, i_2, i_3, i_4, i_5)$.

TABLE 1

Leading coefficients of the quadratic terms in g. For example, the entry in row 011 indicates that the quadratic $g(r_0, s_1, w_1, t, m, d, n) = g(0, n-d, d-1, t, m, n, d)$ in t has the expression $-(n-d)(n^2-mn+dm) \leq 0$ as the coefficient of t^2.

$i_1i_2i_3i_4$	Coefficient of the leading term
ϵ	$-(n^2-m)$
0	$-m(m-1)$
1	$-m(m-1)$
00	$-dm(m-d)$
01	$-dm(m-d)$
10	$-dm(m-d)$
11	$-dm(m-d)$
000	$-(n-d)(n^2-mn+dm)$
001	$-(n-d)(n^2-mn+dm)$
010	$-(n-d)(n^2-mn+dm)$
011	$-(n-d)(n^2-mn+dm)$
100	$-(n-d)(n^2-mn+dm)$
101	$-(n-d)(n^2-mn+dm)$
110	$-(n-d)(n^2-mn+dm)$
111	$-(n-d)(n^2-mn+dm)$
0000	$-d^3$
0001	$-d^2-(d-1)n^2$
0010	$-d$
0011	$-(d-1)(n-d+1)^2-1$
0100	$-d^2(d-1)-n^2$
0101	$-dn^2$
0110	$-(d-1)-(n-d+1)^2$
0111	$-d(n-d+1)^2$
1000	$-d(d-1)^2$
1001	$-(d-1)(n^2-2n+d)$
1010	0
1011	$-(d-1)(n-d)^2$
1100	$-d((d-1)(d-2)+1)-n(n-2)$
1101	$-d(n-1)^2$
1110	$-(n-d)^2$
1111	$-d(n-d)^2$

This is a job best suited to a symbolic algebra package. The expressions proving this proposition are given in Tables 1 and 2. They were calculated by a short Mathematica program. $\square$

3. The higher-dimensional case. In this section, we prove Theorem 1.2 for an arbitrary number of dimensions d. The main idea of the proof is based on that of [16]; hence our notation is similar to the notation therein.

Proof. The proof is by induction on d with $d = 2$, already proved in Lemma 2.5, being the base case. We assume $k_1 \leq k_2 \leq \cdots \leq k_d$ for $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ and vertices are labeled by d-tuples $\langle l_1, l_2, \ldots, l_d \rangle$, where $0 \leq l_i \leq k_i - 1$.

The idea is to transform a given arbitrary configuration into one in reverse-lexicographic order so as not to increase the normalized boundary. To aid the readability of the proof, Figure 6 provides a three-dimensional Hamming graph $H^3 = K_4 \times K_5 \times K_{10}$, which illustrates the transformation process.

Given an arbitrary configuration X in H^d, we permute the k_d $(d-1)$-dimensional Hamming subgraphs along dimension d such that successive subgraphs have fewer elements of X. Now we apply the induction hypothesis to each of these subgraphs. Phase (i) in Figure 6 illustrates a configuration obtained after this step. Note that applying this procedure cannot increase $B(X)$ since $|\partial_d X|$ cannot increase and by the

TABLE 2

The values of the specializations of the quadratic terms in g. For example, the entry in row 01101 *indicates that* $g(r_0, s_1, w_1, t_0, m_1, d, n) = g(0, n-d, d-1, 0, n, n, d) = 2n^2(d-1)(n-d) \geq 0$.

$i_1 i_1 i_3 i_4 i_5$	$g(r_{i_1}, s_{i_2}, w_{i_3}, t_{i_4}, m_{i_5}, d, n)$
00000	0
00001	$d^2n(n-d)^2$
00010	$(d-1)(n-d)(n^2-dn+d^2)$
00011	$dn(n-1)(n-d)$
00100	0
00101	$dn(n-1)(n-d)$
00110	$(d-1)(n-d)(n^2-dn+d^2)$
00111	$(d-1)^2n(n-d)^2+2n^2(d-1)(n-d)$
01000	$(d-1)(n-d)(n^2-dn+d^2)$
01001	$d(d-1)n(n-d)(n-d+1)$
01010	0
01011	0
01100	$(d-1)(n-d)(n^2-dn+d^2)$
01101	$2n^2(d-1)(n-d)$
01110	0
01111	$d(d-1)n(n-d)(n-d+1)$
10000	0
10001	$d(d-1)n(n-d)(n-d+1)$
10010	$(d-1)(n-d)(n^2-dn+d^2)$
10011	$2(d-1)n^2(n-d)$
10100	0
10101	0
10110	$(d-1)(n-d)(n^2-dn+d^2)$
10111	$d(d-1)n(n-d)(n-d+1)$
11000	$(d-1)(n-d)(n^2-dn+d^2)$
11001	$(d-2)^2n(n-d)^2+2n^2(d-1)(n-d)$
11010	0
11011	$dn(n-1)(n-d)$
11100	$(d-1)(n-d)(n^2-dn+d^2)$
11101	$dn(n-1)(n-d)$
11110	0
11111	$d^2n(n-d)^2$

induction hypothesis

$$\frac{|\partial_1 X_i|}{c_1} + \cdots + \frac{|\partial_{d-1} X_i|}{c_{d-1}}$$

is smallest for each subgraph i, where X_i is the set of elements of X that are in subgraph i. Not surprisingly, this means that candidate extremal sets in higher dimensions are among *higher-dimensional partitions* (see [2, Chap. 11]), which are contained in the d-dimensional parallelepiped $k_1 \times k_2 \times \cdots \times k_d$. Now we repeat the same steps for subgraphs along dimension $d-1$ as well. This step is illustrated by phase (ii) in Figure 6.

Consider the $(d-2)$-dimensional Hamming subgraphs of H^d when dimensions d and $d-1$ are fixed. We call each such subgraph "complete" iff its vertices are completely contained in X, "incomplete" iff there exists some (but not all) contained in X, and "empty" iff none is in X. We shall show that if there are more than one incomplete subgraph, then these can be combined without increasing $B(X)$. The result of this step is shown by phase (iii) in Figure 6. To this end we give some definitions and develop proper notation.

First, suppose $P^*_{p,q}$ and $P^*_{r,s}$ are sets of vertices of two such incomplete $(d-2)$-

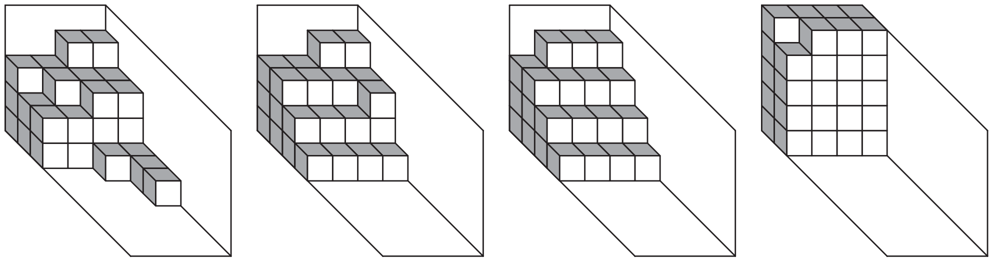

FIG. 6. *Conversion into the reverse-lexicographic order in three dimensions.*

dimensional Hamming subgraphs, where p, r and q, s are coordinates of dimension d and $d-1$, respectively, with $0 \leq p, r \leq k_d - 1$ and $0 \leq q, s \leq k_{d-1} - 1$. Without loss of generality, assume

$$\frac{p}{c_d} + \frac{q}{c_{d-1}} \geq \frac{r}{c_d} + \frac{s}{c_{d-1}}. \tag{3.1}$$

Next, let $P_{p,q} = P^*_{p,q} \cap X$, $P_{r,s} = P^*_{r,s} \cap X$, and $Y = X \setminus (P_{p,q} \cup P_{r,s})$. In our example, there are exactly two such subgraphs with $p = 0$, $q = 3$ and $r = 2$, $s = 2$, i.e., $P_{0,3}$ and $P_{2,2}$, which satisfies the assumption given by the inequality above.

Now given two disjoint subsets S and T of $V(H^d)$, let

$$B(S,T) = \frac{|\partial_1(S,T)|}{c_1} + \frac{|\partial_2(S,T)|}{c_2} + \cdots + \frac{|\partial_d(S,T)|}{c_d}, \tag{3.2}$$

where c_i is as defined before and $\partial_i(S,T)$ is the set of edges in dimension i having one end in S and the other in T. Note that, in this notation, $B(X) = B(X, V(H^d) \setminus X)$.

Note that the following holds:

$$\begin{aligned} B(X) = {} & B(Y) + B(P_{p,q}) + B(P_{r,s}) \\ & - 2(B(Y, P_{p,q}) + B(Y, P_{r,s})) - 2B(P_{p,q}, P_{r,s}). \end{aligned} \tag{3.3}$$

We claim that if as many elements in $P_{r,s}$ as possible are moved to $P^*_{p,q}$ preserving the reverse-lexicographic order, then $B(X)$ does not increase. To this end, consider the terms in (3.3). First, we remark that, by virtue of the reverse-lexicographic order, we must have $p \neq r$ and $q \neq s$, and therefore $B(P_{p,q}, P_{r,s}) = 0$ in (3.3). Furthermore, because of inequality (3.1), $B(Y, P_{p,q}) + B(Y, P_{r,s})$ cannot decrease by this move, and $B(Y)$ is constant. Finally, we claim that $B(P_{p,q}) + B(P_{r,s})$ does not increase.

To prove this, note that any vertex $v \in (P_{p,q} \cup P_{r,s})$ is adjacent to $k_d - 1$ and $k_{d-1} - 1$ vertices in dimensions d and $d-1$, respectively. Thus, moving vertices from $P_{r,s}$ to $P^*_{p,q}$ does not change the boundary along dimensions d and $d-1$. Therefore, it suffices to prove

$$B'(P_{p,q}) + B'(P_{r,s}) \geq B'(P'_{p,q}) + B'(P'_{r,s}), \tag{3.4}$$

where

$$B'(X) = \frac{|\partial_1 X|}{c_1} + \cdots + \frac{|\partial_{d-2} X|}{c_{d-2}}$$

and $P'_{p,q}$ and $P'_{r,s}$ are the new subsets corresponding to $P_{p,q}$ and $P_{r,s}$ respectively, after elements are moved from $P_{r,s}$ to $P^*_{p,q}$.

To prove inequality (3.4), first suppose that all of $P_{r,s}$ fits in the complement of $P_{p,q}$ with respect to $P^*_{p,q}$. Thus we can place elements of $P_{r,s}$ into $P^*_{p,q} \setminus P_{p,q}$ in a set structurally identical to the one given by reverse-lexicographic order, i.e., starting with vertex $\langle k_1-1, k_2-1, \ldots, k_{d-2}-1, q, p\rangle$ of H^d and expanding in the direction of the smallest factor of the Hamming graph. That is,

$$\langle k_1-1, k_2-1, \ldots, k_{d-3}-1, k_{d-2}-1, q, p\rangle \to \langle k_1-1, k_2-1, \ldots, k_{d-3}-1, k_{d-2}-2, q, p\rangle$$

$$\to \cdots \to \langle k_1-1, k_2-1, \ldots, k_{d-3}-2, k_{d-2}-1, q, p\rangle$$

$$\to \langle k_1-1, k_2-1, \ldots, k_{d-3}-2, k_{d-2}-2, q, p\rangle \to \cdots$$

and so on. This is shown in Figure 7.

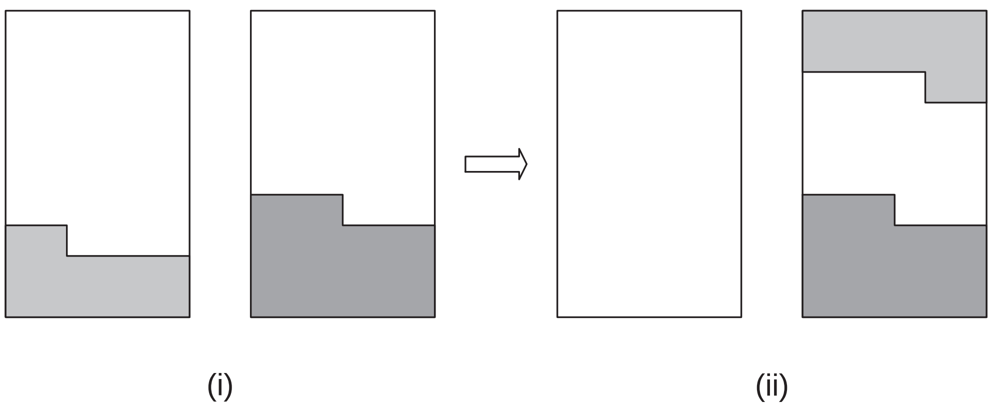

FIG. 7. *Combining two incomplete subgraphs where the elements can fit into one.* (i) *Subgraphs before, and* (ii) *after.*

In this case, we have $B'(P'_{r,s}) = 0$ since $P'_{r,s} = \phi$ and $B'(P'_{p,q})$ can be written as $B'(P'_{p,q}) = B'(P_{p,q}) + B'(P_{r,s}) - 2B'(P_{p,q}, P_{r,s})$. Substituting these values into inequality (3.4), it suffices to prove that

$$B'(P_{p,q}) + B'(P_{r,s}) \geq B'(P_{p,q}) + B'(P_{r,s}) - 2B'(P_{p,q}, P_{r,s}),$$

which obviously holds since $B'(P_{p,q}, P_{r,s}) \geq 0$. We remark that $P'_{p,q}$ is not in reverse-lexicographic order at this point since it consists of two subsets, each of which is structurally in reverse-lexicographic order. Nevertheless, by an easy application of the induction hypothesis, we can convert it to the reverse-lexicographic order without increasing $B'(P'_{p,q})$.

Now assume that not all elements of $P_{r,s}$ fit into $P^*_{p,q}$. First take $|P^*_{p,q} \setminus P_{p,q}|$ vertices in reverse-lexicographic order in $P^*_{r,s}$. These vertices are in $P_{r,s}$. Call this set of vertices Y_2 and set $Y_1 = P_{r,s} \setminus Y_2$. After moving all vertices in Y_2 to $P^*_{p,q}$, we put Y_1 in reverse-lexicographic order $\overline{Y}_1$ within $P^*_{r,s}$. This is shown in Figure 8.

Then, inequality (3.4) reduces to proving

$$B'(P_{p,q}) + B'(P_{r,s}) \geq B'(Y_1)$$

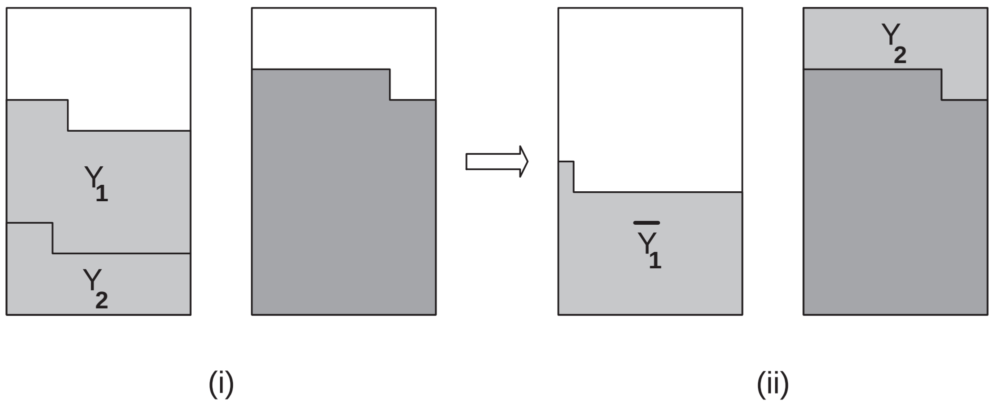

FIG. 8. *Combining two incomplete subgraphs where the elements cannot fit into one.* (i) *Subgraphs before, and* (ii) *after.*

as $B'(Y_1) \geq B'(\overline{Y_1})$ holds by the induction hypothesis. Now note that $B'(P_{r,s}) = B'(Y_1) + B'(Y_2) - 2B'(Y_1, Y_2)$ and $B'(Y_2) = B'(P_{p,q})$ since Y_2 and $P_{p,q}$ are complementary in $P^*_{p,q}$. Thus the above inequality is equivalent to

$$B'(Y_2) \geq B'(Y_1, Y_2),$$

which obviously holds. Thus $B(P_{p,q}) + B(P_{r,s})$ does not increase as claimed. By applying this process to all $(d-2)$-dimensional incomplete subgraphs, we can assume that X has only one incomplete $(d-2)$-dimensional Hamming subgraph.

Finally we treat the $(d-2)$-dimensional Hamming subgraphs as single vertices and use the two-dimensional case to minimize $B(X)$ by putting them in reverse-lexicographic order with the only incomplete one highest in the order, as shown by phase (iv) in Figure 6. This completes the proof of Theorem 1.2. ∎

4. Conclusions. We proved that the set of first k vertices of the Hamming graph $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ $(k_1 \leq k_2 \leq \cdots \leq k_d)$ in reverse-lexicographic order constitutes an extremal set minimizing the dimension-normalized edge-boundary over all k-vertex subsets of the graph. The boundary edges $\partial_i X$ along the ith dimension of $X \subset V(H^d)$ are normalized by a weight

$$c_i = \begin{cases} k_i^2 & \text{if } k_i \text{ is even,} \\ k_i^2 - 1 & \text{if } k_i \text{ is odd,} \end{cases}$$

which naturally arises in the isoperimetric number problem for d-dimensional arrays. The weighted boundary to be minimized is then

$$B(X) = \frac{|\partial_1 X|}{c_1} + \frac{|\partial_2 X|}{c_2} + \cdots + \frac{|\partial_d X|}{c_d}$$

over $X \subset V(H^d)$. Interestingly, when all factors of H^d have equal size, the lexicographic and reverse-lexicographic orders both result in structurally symmetric subsets and hence are equivalent with respect to extremal sets minimizing the boundary (dimension-normalized or otherwise). Thus our result is identical to Lindsey's for $k_1 = k_2 = \cdots = k_d$.

We formulated the problem for the two-dimensional case as the maximization of the function f defined on partitions $\lambda \in \mathbb{P}(m,n)$ $(m \le n)$ by

$$f(\lambda) = \gamma_n \sum_{i=1}^{n} \lambda_i^2 + \gamma_m \sum_{j=1}^{m} \lambda_j'^2$$

and proved that f is maximized for $N \le nm$, by the reverse-lexicographically smallest partition of N in $\mathbb{P}(m,n)$, where

$$\gamma_n = \begin{cases} n^2 & \text{if } n \text{ is even,} \\ n^2 - 1 & \text{if } n \text{ is odd.} \end{cases}$$

This result for $d = 2$ forms the base step of the higher-dimensional case.

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