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Boundary enumerator polynomial of hypercubes in Fibonacci cubes

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ABSTRACT

Hypercubes and their special subgraphs, Fibonacci cubes, have been proposed as basic models for interconnection networks. By the recursive nature of Fibonacci cubes, they contain many smaller dimensional hypercubes as subgraphs. In this work, we consider the boundary enumerator polynomial of the k-dimensional hypercubes in Fibonacci cubes of dimension n. We obtain recursive relations satisfied by these polynomials.

1. Introduction

The n-dimensional hypercube graph denoted by Q_n is one of the basic models for interconnection networks. Here vertices denote the processors and edges denote the communication links between processors. Disallowing certain processors, Fibonacci cubes Γ_n of dimension n were introduced as a new model of computation for interconnection networks in [6]. These cubes are subgraphs of Q_n.

There are many interesting properties and applications of the Fibonacci cubes in the literature. In [6,2] their application as an interconnection network and some properties that are important in network design are presented. We refer to [7] for their usage in theoretical chemistry and some results on the structure of Fibonacci cubes, including their representations, recursive construction, hamiltonicity, and the nature of the degree sequence. Moreover, the number of vertex and edge orbits of Fibonacci cubes are determined in [1], the characterization of maximal induced hypercubes in Γ_n is given in [11] and results on disjoint hypercubes in Γ_n are presented in [5,14,12]. The cube enumerator polynomial itself of Γ_n is considered in [8,15,9] and many combinatorial results are obtained.

In this paper, we are interested in the number of edges in the boundary of a hypercube in a Fibonacci cube (see Section 2 for a definition). Our motivation and the starting point is the fact that there are several results only on the degrees of the vertices (in our case, 0-dimensional hypercubes) in Γ_n.

In [6] it is shown that the degrees of vertices in Γ_n are between $\left\lfloor \frac{n+2}{3} \right\rfloor$ and n. Furthermore, in [4] depending on the recursive structure of Γ_n, a recursive formula for computing the degree of any vertex is given. In [13] vertices of degrees n, $n-1$, $n-2$ and $n-3$ are explicitly described and the degrees of vertices are used to investigate the domination number Γ_n. Finally in [10], by deriving and solving a corresponding system of linear recurrences the number of vertices of any degree in Γ_n is obtained (see, Remark 1). Also a direct approach to this problem by considering degrees via the partition of $V(\Gamma_n)$ into strings of any weight is given. In the last section of [10] a method using generating functions is also presented with the

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remark that the presented approach is "somewhat more involved than the one taken in previous Section 5, however it can be further used to obtain several additional properties of the sequence of degrees of the Fibonacci and Lucas cubes". In this paper, we extend this approach to find the number of edges in the boundary of k-dimensional hypercubes in Fibonacci cube Γ_n for any k. The case $k = 0$ reduces to the case of vertex enumeration by degrees and the case $k = 1$ is the enumeration of edges by the number of incident edges. As noted in [10] this method is naturally more involved and complicated as we are considering k-dimensional hypercubes in Fibonacci cubes for arbitrary k.

2. Preliminaries

The vertices of Q_n are represented by all binary strings of length n and two vertices are adjacent if and only if they differ in exactly one position. Formally, we can describe the vertex set $V(Q_n)$ and edge set $E(Q_n)$ of Q_n as follows:

$$V(Q_n) = \{v_1 v_2 \cdots v_n \mid v_i \in \{0, 1\},\ 1 \le i \le n\} \text{ and}$$
$$E(Q_n) = \{(u, v) \mid u, v \in V(Q_n),\ d_H(u, v) = 1\},$$

where $d_H(u, v)$ denotes the Hamming distance between u and v, that is, the number of different positions in u and v. In Γ_n, the vertices correspond to those binary strings without two consecutive 1s in their string representation. In other words, the labels of the vertices of Γ_n ($n \ge 1$) are the Fibonacci strings of length n and therefore the vertex set and the edge set of Γ_n can be written as

$$V(\Gamma_n) = \{v_1 v_2 \cdots v_n \mid v_i \in \{0, 1\},\ 1 \le i \le n,\ v_i v_{i+1} = 0\} \text{ and}$$
$$E(\Gamma_n) = \{(u, v) \mid u, v \in V(\Gamma_n),\ d_H(u, v) = 1\}.$$

For convenience Γ_0 is defined as Q_0, the graph with a single vertex and no edges. Note that the number of vertices of the Γ_n is f_{n+2}, where $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \ge 2$ are the Fibonacci numbers.

One of the useful structures of Fibonacci cubes is their recursive definition, which is called as the *fundamental decomposition* of Γ_n in [7]. The main idea is to decompose $V(\Gamma_n)$ into two disjoint sets, the vertices that start with 0 and 10, respectively. Let $V(\Gamma_n) = A_n \cup B_n$ where

$$A_n = \{1v \mid v \in B_{n-1}\} \quad \text{and} \quad B_n = \{0v \mid v \in A_{n-1} \cup B_{n-1}\}$$

with $A_0 = \emptyset$ and $B_0 = \{\epsilon \mid \epsilon \text{ is the empty string}\}$. Note that for $n \ge 2$ any vertex in A_n must start with 10. Therefore, for $n \ge 2$ the vertices in B_n will constitute a subgraph isomorphic to Γ_{n-1} and the vertices in A_n will constitute a subgraph isomorphic to Γ_{n-2}. In this paper, we will use the following formulation for this decomposition for $n \ge 2$

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}\ . \tag{1}$$

Furthermore, we note that $0\Gamma_{n-1}$ has a subgraph isomorphic to $00\Gamma_{n-2}$, and there is a natural identification (matching) between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$. In the following section we will use the decomposition (1) recursively to obtain our main results.

Let $G = (V, E)$ be a simple graph, and X be a subset of V. There are two standard ways to define the boundary of X in G: *the vertex-boundary* or *the edge-boundary*. The vertex boundary ∂X of X is the set of vertices that are not in X but adjacent to some vertex in X. In other words

$$\partial X = \{v \in V \setminus X \mid v \text{ is adjacent to some } u \in X\}.$$

On the other hand, the edge-boundary $\partial_e X$ of X is the set of edges in E which connect vertices in X with vertices in $V \setminus X$, that is,

$$\partial_e X = \{(u, v) \in E \mid u \in X \text{ and } v \in V \setminus X\}.$$

In this paper, we consider $G = \Gamma_n$, $H = Q_k$ for some non-negative integer k and X as the vertex set of H. By the structure of $G\ (= \Gamma_n \subseteq Q_n)$ and $H\ (= Q_k)$ we know that a vertex $v \in \partial X$ is adjacent to the vertex $u \in X$ by a unique edge $(u, v) \in E(\Gamma_n)$. This gives $|\partial X| = |\partial_e X|$. But this is not true in general, since a vertex in the vertex-boundary of X can be joined to more than one vertex in X. Thus the numbers $|\partial X|$ and $|\partial_e X|$ may be different.

Throughout this paper we will use the term *boundary* for edge-boundary.

3. Main results

Let $\mathbb{D}_{n,k}(d)$ be a polynomial that enumerates the boundary of the k-dimensional hypercubes in Fibonacci cube Γ_n of dimension n, where the degree of any monomial in $\mathbb{D}_{n,k}(d)$ shows the number of edges in the boundary of the corresponding k-dimensional hypercube and the coefficient of that monomial shows the number of such hypercubes. Let $\mathbb{C}_{n,k}(d)$ denote the boundary enumerator polynomial of the k-dimensional hypercubes in Γ_n that are obtained during the connection of Γ_{n-1} and Γ_{n-2}. For small values of n and k, by direct inspection on Fig. 1 we obtain the explicit polynomials $\mathbb{D}_{n,k}(d)$ and $\mathbb{C}_{n,k}(d)$ given in Table 1.

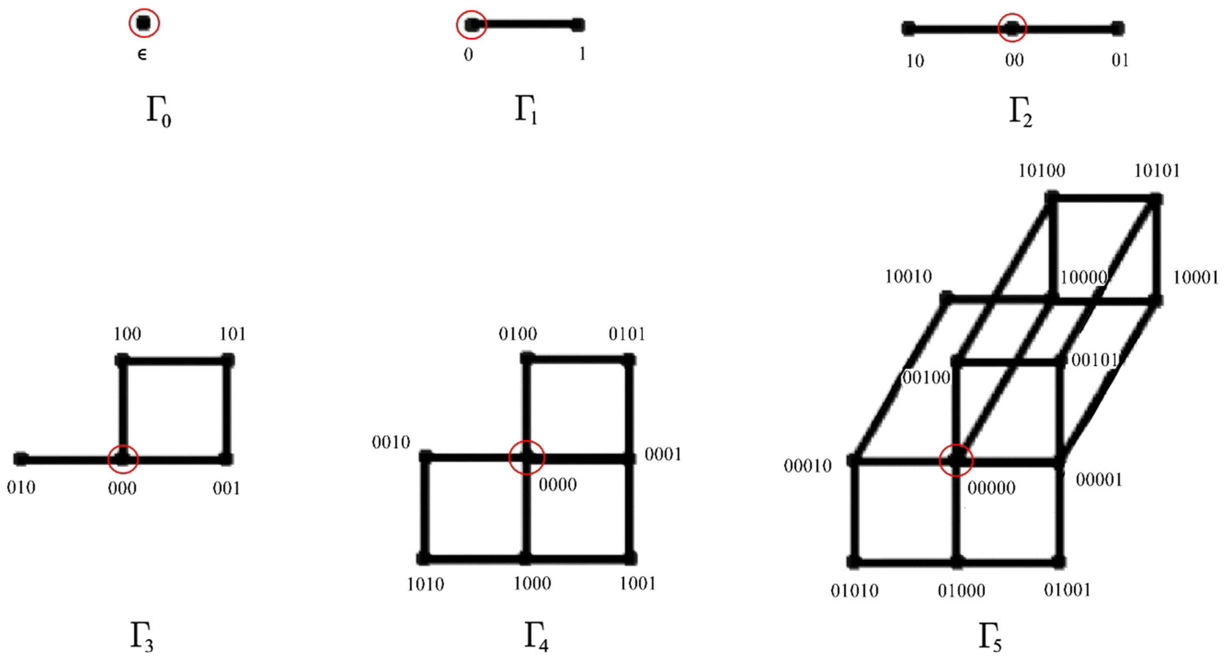

Fig. 1. Fibonacci cubes $\Gamma_0, \Gamma_1, \ldots, \Gamma_5$.

Table 1
The table of boundary enumerator polynomials $\mathbb{D}_{n,k}$ and $\mathbb{C}_{n,k}$ for small values of n and k.

n	$\mathbb{D}_{n,0}$	$\mathbb{C}_{n,1}$	$\mathbb{D}_{n,1}$	$\mathbb{C}_{n,2}$	$\mathbb{D}_{n,2}$
0	1	0	0	0	0
1	$2d$	1	1	0	0
2	$2d+d^2$	d	$2d$	0	0
3	$d+3d^2+d^3$	d^2+d^3	$3d^2+2d^3$	d	d
4	$5d^2+2d^3+d^4$	$d^2+d^3+d^5$	$2d^2+4d^3+2d^4+2d^5$	d^3+d^4	$2d^3+d^4$

As an example, let us consider the Fibonacci cube Γ_5. From Fig. 1 one can see that there are only 0, 1, 2 and 3-dimensional hypercubes in Γ_5. So we will consider only $\mathbb{D}_{5,k}(d)$ for $k \in \{0, 1, 2, 3\}$. As seen from Fig. 1, Γ_5 has 13 vertices, 3 of them have degree 2, 7 of them have degree 3, 2 of them have degree 4 and one of them has degree 5. Similarly, there are only one 3-dimensional cube and there are 4 edges in its boundary. Hence,

$$\mathbb{D}_{5,0}(d) = 3d^2 + 7d^3 + 2d^4 + d^5,$$
$$\mathbb{D}_{5,3}(d) = d^4.$$

Similarly, if we consider the edges and squares in Γ_5 we obtain

$$\mathbb{D}_{5,1}(d) = 4d^3 + 7d^4 + 4d^5 + 3d^6 + 2d^7,$$
$$\mathbb{D}_{5,2}(d) = d^4 + 3d^5 + 2d^6 + 2d^7 + d^8.$$

Note that if n and k have bigger values then it is not easy to find $\mathbb{D}_{n,k}(d)$ by direct counting. Now for $k \in \{0, 1\}$ we present a counting method to find $\mathbb{D}_{5,k}(d)$ by using the fundamental decomposition (1) of Fibonacci cubes recursively which can be generalized to obtain a recursion for $\mathbb{D}_{n,k}(d)$ for any nonnegative integer n and k (see, Theorem 1).

Consider $k = 0$. From (1) (see also, Fig. 2) we know that

$$\Gamma_5 = 0\Gamma_4 + 10\Gamma_3 = (00\Gamma_3 + 010\Gamma_2) + 10\Gamma_3. \tag{2}$$

Now let v be any vertex in Γ_5. Then we have two cases:

1. Assume that $v \in 10\Gamma_3$. If we consider all $v \in \Gamma_3$ then the boundary enumerator becomes $\mathbb{D}_{3,0}(d)$. The degrees of these vertices increase by one in Γ_5. Therefore all of these vertices contribute $d \cdot \mathbb{D}_{3,0}(d)$ to $\mathbb{D}_{5,0}(d)$.
2. Assume that $v \in 0\Gamma_4$. Then there are two subcases.

(a) Assume that $v \in 010\Gamma_2$. If we consider all $v \in \Gamma_2$ then the boundary enumerator becomes $\mathbb{D}_{2,0}(d)$. Similarly we see that degrees of these vertices again increase by one in $0\Gamma_4$ and Γ_5. Therefore all of these vertices contribute $d \cdot \mathbb{D}_{2,0}(d)$ to $\mathbb{D}_{4,0}(d)$ and $\mathbb{D}_{5,0}(d)$.

(b) Assume that $v \in 0\Gamma_4 \setminus 010\Gamma_2$. If we consider all such vertices in Γ_4 then we have $\mathbb{D}_{4,0}(d) - d \cdot \mathbb{D}_{2,0}(d)$ from the previous subcase. Also the degrees of these vertices again increase by one in Γ_5. Therefore, all of these vertices contribute $d\left(\mathbb{D}_{4,0}(d) - d \cdot \mathbb{D}_{2,0}(d)\right)$ to $\mathbb{D}_{5,0}(d)$.

Therefore, in total we have

$$\begin{aligned}\mathbb{D}_{5,0}(d) &= d \cdot \mathbb{D}_{3,0}(d) + d \cdot \mathbb{D}_{2,0}(d) + d\left(\mathbb{D}_{4,0}(d) - d \cdot \mathbb{D}_{2,0}(d)\right)\\ &= d\left(\mathbb{D}_{4,0}(d) + \mathbb{D}_{3,0}(d) + (1-d) \cdot \mathbb{D}_{2,0}(d)\right).\end{aligned}$$

Consider $k = 1$. In this case we will use the decomposition of Γ_5 given in (2) again. Now let e be any edge in Γ_5. This time we have three cases:

1. Assume that $e \in 10\Gamma_3$. If we consider all $e \in \Gamma_3$ then the boundary enumerator becomes $\mathbb{D}_{3,1}(d)$. The degrees of vertices of these edges increase by one in Γ_5, that is, the boundary of the edges increase by two. Therefore all of these edges contribute $d^2 \cdot \mathbb{D}_{3,1}(d)$ to $\mathbb{D}_{5,1}(d)$.
2. Assume that $e = (v_1, v_2) \in \Gamma_5$ with $v_1 \in 10\Gamma_3$ and $v_2 \in 0\Gamma_4$ (in particular $v_2 \in 00\Gamma_3$). These edges are not in Γ_m for $m < 5$ and they first occur when we connect Γ_4 and Γ_3 to obtain Γ_5. We denote the contribution of these edges to $\mathbb{D}_{5,1}(d)$ by $\mathbb{C}_{5,1}(d)$ and we will consider this case later in detail.
3. Assume that $e \in 0\Gamma_4$. Since $0\Gamma_4 = 00\Gamma_3 + 010\Gamma_2$ we have three subcases.

(a) Assume that $e \in 010\Gamma_2$. Considering all $e \in \Gamma_2$ we have $\mathbb{D}_{2,1}(d)$ and we see that the boundary of these edges increase by two in Γ_5. Therefore all of these edges contribute $d^2 \cdot \mathbb{D}_{2,1}(d)$ to $\mathbb{D}_{5,1}(d)$.

(b) Assume that $e = (v_1, v_2) \in 0\Gamma_4$ with $v_1 \in 010\Gamma_2$ and $v_2 \in 00\Gamma_3$. These edges are not in Γ_m for $m < 4$ and they first occur when we connect Γ_3 and Γ_2 to obtain Γ_4. Then as we need to consider $e \in \Gamma_5$ we see that the degree of v_2 increase by one when we connect $0\Gamma_4$ and $10\Gamma_3$ to obtain Γ_5. Then as in the above Case 2, the contribution of these edges to $\mathbb{D}_{5,1}(d)$ becomes $d \cdot \mathbb{C}_{4,1}(d)$.

(c) Assume that $e \in 00\Gamma_3$. Then one can observe that these edges are the ones in $0\Gamma_4$ that are not in $010\Gamma_2$ and that does not occur as connection edges between $00\Gamma_3$ and $010\Gamma_2$. Also note that the boundary of these edges increase by two during the connection of $0\Gamma_4$ and $10\Gamma_3$ to obtain Γ_5. Therefore, the contribution of these edges to $\mathbb{D}_{5,1}(d)$ becomes $d^2\left(\mathbb{D}_{4,1}(d) - d^2 \cdot \mathbb{D}_{2,1}(d) - \mathbb{C}_{4,1}(d)\right)$.

Therefore, in total we have

$$\begin{aligned}\mathbb{D}_{5,1}(d) &= d^2 \cdot \mathbb{D}_{3,1}(d) + \mathbb{C}_{5,1}(d) + d^2 \cdot \mathbb{D}_{2,1}(d) + d \cdot \mathbb{C}_{4,1}(d)\\ &\quad + d^2\left(\mathbb{D}_{4,1}(d) - d^2 \cdot \mathbb{D}_{2,1}(d) - \mathbb{C}_{4,1}(d)\right)\\ &= d^2\left(\mathbb{D}_{4,1}(d) + \mathbb{D}_{3,1}(d) + (1-d^2) \cdot \mathbb{D}_{2,1}(d)\right) + \mathbb{C}_{5,1}(d)\\ &\quad + (d - d^2) \cdot \mathbb{C}_{4,1}(d).\end{aligned}$$

Now let us consider $\mathbb{C}_{5,1}(d)$ in detail. This number enumerates the boundary of the edges e that first occur during the connection of $0\Gamma_4$ and $10\Gamma_3$ to obtain Γ_5. From the fundamental decomposition (1) we have

$$\begin{aligned}\Gamma_5 = 0\Gamma_4 + 10\Gamma_3 &= (00\Gamma_3 + 010\Gamma_2) + 10\Gamma_3\\ &= ((000\Gamma_2 + 0010\Gamma_1) + 010\Gamma_2) + (100\Gamma_2 + 1010\Gamma_1).\end{aligned}$$

Assume that $e = (v_1, v_2)$ with $v_1 \in 0\Gamma_4$ and $v_2 \in 10\Gamma_3$. We have two different cases:

1. Assume $v_2 \in 1010\Gamma_1$. Then $v_1 \in 0010\Gamma_1$ and the degrees of all such vertices in Γ_1 are $\mathbb{D}_{1,0}(d)$ and therefore they become $d \cdot \mathbb{D}_{1,0}(d)$ in both $10\Gamma_3$ and $0\Gamma_4$. So, the boundary of all such edges becomes $d^2 \cdot \mathbb{D}_{1,0}(d^2)$ in Γ_5.
2. Assume $v_2 \in 100\Gamma_2$. Then $v_1 \in 000\Gamma_2$ and the degrees of all such vertices in $00\Gamma_3$ are $\mathbb{D}_{3,0}(d) - d \cdot \mathbb{D}_{1,0}(d)$ as $00\Gamma_3 = 000\Gamma_2 + 0010\Gamma_1$. But note that the degree of v_1 becomes $d \cdot \left(\mathbb{D}_{3,0}(d) - d \cdot \mathbb{D}_{1,0}(d)\right)$ in $0\Gamma_4$. So, the boundary of all such edges becomes $d \cdot \left(\mathbb{D}_{3,0}(d^2) - d^2 \cdot \mathbb{D}_{1,0}(d^2)\right)$ in Γ_5.

If we sum up we get

$$\begin{aligned}\mathbb{C}_{5,1}(d) &= d^2 \cdot \mathbb{D}_{1,0}(d^2) + d \cdot \left(\mathbb{D}_{3,0}(d^2) - d^2 \cdot \mathbb{D}_{1,0}(d^2)\right)\\ &= d \cdot \mathbb{D}_{3,0}(d^2) + \left(d^2 - d^3\right)\mathbb{D}_{1,0}(d^2).\end{aligned}$$

By the same argument we have $\mathbb{C}_{4,1}(d) = d \cdot \mathbb{D}_{2,0}(d^2) + \left(d^2 - d^3\right)\mathbb{D}_{0,0}(d^2)$.

Before giving our main result we give the following remark which states the connection between our results and the results given in [10, Section 6].

Remark 1. Let $f_{n,m}$ denote the number of vertices of Γ_n having degree m. In [10, Section 6] the generating function $f(x,y)=\sum_{n,m\geq 0} f_{n,m}x^n y^m$ is obtained. From the denominator of $f(x,y)$ it is concluded that $f_{n,m}$ satisfies the recursive relation

$$f_{n,m}=f_{n-1,m-1}+f_{n-2,m-1}+f_{n-3,m-1}-f_{n-3,m-2} \tag{3}$$

for all large enough n and m. By exploiting the geometric meaning of this recursion and using the fundamental decomposition (1) of Γ_n we obtain a general recursion for $\mathbb{D}_{n,k}$ and $\mathbb{C}_{n,k}$ in Theorem 1. Note that in our notation (3) corresponds to our $k=0$ case, that is,

$$\mathbb{D}_{n,0}(d)=d\left(\mathbb{D}_{n-1,0}(d)+\mathbb{D}_{n-2,0}(d)+\mathbb{D}_{n-3,0}(d)\right)-d^2D_{n-3,0}(d).$$

Now using the fundamental decomposition (1) of Fibonacci cubes recursively and by generalizing the above arguments, we obtain the following results on $\mathbb{D}_{n,k}(d)$ and $\mathbb{C}_{n,k}(d)$.

Theorem 1. *Let n,k be nonnegative integers and let $\mathbb{D}_{n,k}(d)$ be the boundary enumerator polynomial of the k-dimensional hypercubes in Γ_n and $\mathbb{C}_{n,k}(d)$ be the boundary enumerator polynomial of the k-dimensional hypercubes in Γ_n that are obtained during the connection of Γ_{n-1} and Γ_{n-2}. Then for $n\geq 1$ and $k\geq 0$ we have*

$$\begin{aligned}\mathbb{D}_{n,k}(d) &= d^{2^k}\left(\mathbb{D}_{n-1,k}(d)+\mathbb{D}_{n-2,k}(d)+\left(1-d^{2^k}\right)\mathbb{D}_{n-3,k}(d)\right)+\mathbb{C}_{n,k}(d)\\ &\quad+\left(d^{2^{k-1}}-d^{2^k}\right)\mathbb{C}_{n-1,k}(d)\end{aligned}$$

and for $n\geq 2$ and $k\geq 1$,

$$\begin{aligned}\mathbb{C}_{n,k}(d) &= d^{2^k}\left(1-d^{2^{k-1}}\right)\mathbb{D}_{n-4,k-1}(d^2)+d^{2^{k-2}}\left(1-d^{2^{k-2}}\right)\mathbb{C}_{n-2,k-1}(d^2)\\ &\quad+d^{2^{k-1}}\mathbb{D}_{n-2,k-1}(d^2),\end{aligned}$$

where $\mathbb{D}_{-1,0}(d)=\mathbb{D}_{0,0}(d)=\mathbb{C}_{1,1}(d)=1$, $\mathbb{C}_{n,0}(d)=0$ and for $m<2k-1$ $\mathbb{D}_{m,k}(d)=\mathbb{C}_{m,k}(d)=0$.

Proof. The proof for the cases $k=0$ and $k=1$ can be done by following the above arguments. For this reason we only consider the proof of the general case where we assume $k\geq 2$. Note that $Q_k\in\Gamma_n$ if and only if $n\geq 2k-1$. We need to consider the following three cases:

1. Assume that $Q_k\in 10\Gamma_{n-2}$. The boundary enumerator in Γ_{n-2} for all such hypercubes is $\mathbb{D}_{n-2,k}(d)$. The degrees of vertices of all Q_k increase by one in Γ_n, that is, the boundary of the Q_k increases by 2^k. Therefore all of these hypercubes contribute $d^{2^k}\cdot\mathbb{D}_{n-2,k}(d)$ to $\mathbb{D}_{n,k}(d)$.
2. Assume that $Q_k\in\Gamma_n$ and it is obtained from $Q_{k-1}\in 10\Gamma_{n-2}$ and $Q_{k-1}\in 0\Gamma_{n-1}$ (in particular, $Q_{k-1}\in 00\Gamma_{n-2}$) during the connection of $10\Gamma_{n-2}$ and $0\Gamma_{n-1}$ to obtain Γ_n. Note that these hypercubes are not in Γ_m for $m<n$. From the definition of $\mathbb{C}_{n,k}(d)$, the contribution of these hypercubes to $\mathbb{D}_{n,k}(d)$ is $\mathbb{C}_{n,k}(d)$.
3. Assume that $Q_k\in 0\Gamma_{n-1}$. Since $0\Gamma_{n-1}=00\Gamma_{n-2}+010\Gamma_{n-3}$ we have three subcases.
   (a) Assume that $Q_k\in 010\Gamma_{n-3}$. Considering all such hypercubes we see that the boundary of each of them increase by 2^k in Γ_n and therefore all of them contribute $d^{2^k}\cdot\mathbb{D}_{n-3,k}(d)$ to $\mathbb{D}_{n,k}(d)$.
   (b) Assume that $Q_k\in 0\Gamma_{n-1}$ and it is obtained from $Q_{k-1}\in 010\Gamma_{n-3}$ and $Q_{k-1}\in 00\Gamma_{n-2}$ during the connection of $010\Gamma_{n-3}$ and $00\Gamma_{n-2}$ to obtain $0\Gamma_{n-1}$. As in the above second case the contribution of these hypercubes to $\mathbb{D}_{n,k}(d)$ is $d^{2^{k-1}}\mathbb{C}_{n-1,k}(d)$ since in the last connection step only the boundary of $Q_{k-1}\in 00\Gamma_{n-2}$ increases by 2^{k-1}.
   (c) Assume that $Q_k\in 00\Gamma_{n-2}$. These hypercubes are the ones in $0\Gamma_{n-1}$ that are not in $010\Gamma_{n-3}$ and that does not occur during the connection of $00\Gamma_{n-2}$ and $010\Gamma_{n-3}$. Furthermore the boundary of each of these hypercubes increases by 2^k during the last connection step. Therefore, the contribution of these hypercubes to $\mathbb{D}_{n,k}(d)$ is $d^{2^k}\left(\mathbb{D}_{n-1,k}(d)-d^{2^k}\cdot\mathbb{D}_{n-3,k}(d)-\mathbb{C}_{n-1,k}(d)\right)$.

If we sum up all the above contributions we obtain the desired result for $\mathbb{D}_{n,k}(d)$. Now we can deal with the details of $\mathbb{C}_{n,k}(d)$. From the definition this number enumerates the boundary of the hypercubes that first occur during the connection of $0\Gamma_{n-1}$ and $10\Gamma_{n-2}$ to obtain Γ_n. Such hypercubes are obtained by the connection of $Q_{k-1}\in 00\Gamma_{n-2}$ (say H_{00}) and $Q_{k-1}\in 10\Gamma_{n-2}$ (say H_{10}). For $n\geq 4$ from the fundamental decomposition (1) we have (see also, Fig. 2)

$$\begin{aligned}\Gamma_n &= 0\Gamma_{n-1}+10\Gamma_{n-2}\\ &= (00\Gamma_{n-2}+010\Gamma_{n-3})+10\Gamma_{n-2}\\ &= ((000\Gamma_{n-3}+0010\Gamma_{n-4})+010\Gamma_{n-3})+(100\Gamma_{n-3}+1010\Gamma_{n-4}).\end{aligned}$$

From this decomposition we have the following three different cases:

1. Assume that $H_{00}\in 0010\Gamma_{n-4}\subset 00\Gamma_{n-2}$ and $H_{10}\in 1010\Gamma_{n-4}\subset 10\Gamma_{n-2}$. Then the boundary of all such Q_{k-1}'s in Γ_{n-4} are $\mathbb{D}_{n-4,k-1}(d)$ and therefore they become $d^{2^{k-1}}\cdot\mathbb{D}_{n-4,k-1}(d)$ in $10\Gamma_{n-2}$ and $00\Gamma_{n-2}(\subset 0\Gamma_{n-1})$. If we connect these subcubes H_{00} and H_{10}'s they constitutes Q_k's and these hypercubes contribute $d^{2^k}\cdot\mathbb{D}_{n-4,k-1}(d^2)$ in Γ_n to $\mathbb{C}_{n,k}(d)$.

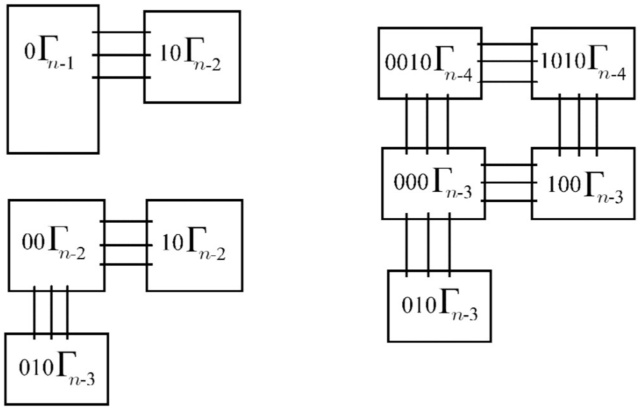

Fig. 2. Fundamental decomposition and identifications in the Fibonacci cube Γ_n, $n \geq 4$.

2. Assume that $H_{00} \in 00\Gamma_{n-2}$ has two $(k-2)$-dimensional hypercubes $Q_{k-2} \in 0010\Gamma_{n-4}$ and $Q_{k-2} \in 000\Gamma_{n-3}$. Since we are looking for a hypercube Q_k and in the connection step there is a matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$, the other $(k-1)$-dimensional hypercube $H_{10} \in 10\Gamma_{n-2}$ must have two parts $Q_{k-2} \in 1010\Gamma_{n-4}$ and $Q_{k-2} \in 100\Gamma_{n-3}$. By the definition of $\mathbb{C}_{n,k}(d)$ we know that the $(k-1)$-dimensional hypercube H_{00} has boundary enumerator $\mathbb{C}_{n-2,k-1}(d)$ in $00\Gamma_{n-2}$ and H_{10} has boundary enumerator $\mathbb{C}_{n-2,k-1}(d)$ in $10\Gamma_{n-2}$. Furthermore, in the connection of $00\Gamma_{n-2}$ and $010\Gamma_{n-3}$ to obtain $0\Gamma_{n-1}$ the boundary of $Q_{k-2} \in 000\Gamma_{n-3}$ increases by 2^{k-2} (since there is a matching between $000\Gamma_{n-3} \subset 00\Gamma_{n-2}$ and $010\Gamma_{n-3}$), that is, the boundary enumerator of H_{00} becomes $d^{2^{k-2}} \cdot \mathbb{C}_{n-2,k-1}(d)$ in $0\Gamma_{n-1}$. As our Q_k is obtained by the matching of H_{00} and H_{10}, these Q_k's contribute $d^{2^{k-2}} \cdot \mathbb{C}_{n-2,k-1}(d^2)$ to $\mathbb{C}_{n,k}(d)$ in Γ_n.
3. Assume that $H_{00} \in 000\Gamma_{n-3} \subset 00\Gamma_{n-2}$ and $H_{10} \in 100\Gamma_{n-3} \subset 10\Gamma_{n-2}$. These $(k-1)$-dimensional hypercubes are the ones in $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$ that are not in $0010\Gamma_{n-4}$ and $1010\Gamma_{n-4}$ and that does not occur during the connection of $(0010\Gamma_{n-4}$ and $000\Gamma_{n-3})$ or $(1010\Gamma_{n-4}$ and $100\Gamma_{n-3})$. Note that the boundary of each of these hypercubes are $\mathbb{D}_{n-2,k-1}(d) - d^{2^{k-1}} \cdot \mathbb{D}_{n-4,k-1}(d) - \mathbb{C}_{n-2,k-1}(d)$ in $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$. Furthermore, the boundary of each $H_{00} \in 000\Gamma_{n-3}$ increases by 2^{k-1} while the connection step to obtain $0\Gamma_{n-1}$. Therefore, the contribution of all Q_k's having such two Q_{k-1}'s to $\mathbb{C}_{n,k}(d)$ is $d^{2^{k-1}}\left(\mathbb{D}_{n-2,k-1}(d^2) - d^{2^k} \cdot \mathbb{D}_{n-4,k-1}(d^2) - \mathbb{C}_{n-2,k-1}(d^2)\right)$.

If we add all the above contributions we get the desired result for $\mathbb{C}_{n,k}(d)$. □

Using Theorem 1 one can obtain the generating functions for $\mathbb{D}_{n,k}(d)$ and $\mathbb{C}_{n,k}(d)$.

Theorem 2. *Let $D_k(d,t) = \sum_{n\geq 0} \mathbb{D}_{n,k}(d)t^n$ and $C_k(d,t) = \sum_{n\geq 0} \mathbb{C}_{n,k}(d)t^n$ be the generating functions of $\mathbb{D}_{n,k}(d)$ and $\mathbb{C}_{n,k}(d)$. Then we have*

$$D_0(d,t) = \frac{1+dt+(d-d^2)t^2}{1-dt-dt^2-(d-d^2)t^3},$$

$$C_k(d,t) = \frac{d^{(k-1)2^{k-2}} t^{2k-1}\left(1+\left(d^{2^{k-1}}-d^{2^k}\right)t\right)\left(f(d^{2^k},t)\right)^{k-1}}{\left(g(d^{2^k},t)\right)^k},$$

$$D_k(d,t) = \frac{d^{(k-1)2^{k-2}} t^{2k-1}\left(1+\left(d^{2^{k-1}}-d^{2^k}\right)t\right)^2\left(f(d^{2^k},t)\right)^{k-1}}{\left(g(d^{2^k},t)\right)^{k+1}},$$

where $f(d,t) = 1+\left(d^3-d^4\right)\left(t+t^2\right) - d^4\left(1-2d+2d^3-d^4\right)t^3$ and $g(d,t) = 1-d\left(t+t^2+t^3\right)+d^2t^3$.

Proof. For $k=0$ and $n \geq 1$ using Theorem 1 we have

$$\mathbb{D}_{n,0}(d) = d\left(\mathbb{D}_{n-1,k}(d) + \mathbb{D}_{n-2,0}(d) + (1-d)\,\mathbb{D}_{n-3,0}(d)\right).$$

Then since $\mathbb{D}_{-1,0}(d) = 1$ and $\mathbb{D}_{m,k}(d) = 0$ for $m < 2k-1$ (see, Theorem 1) we get

$$D_0(d,t) - 1 = dtD_0(d,t) + dt^2(D_0(d,t) - t^{-1}) + (d-d^2)t^3(D_0(d,t) - t^{-1})$$

which gives the desired result for $D_0(d,t)$. Similarly, using Theorem 1 for $k=1$ and $n \geq 2$ we have

$$\mathbb{C}_{n,1}(d) = d^2\,(1-d)\,\mathbb{D}_{n-4,0}(d^2) + d \cdot \mathbb{D}_{n-2,0}(d^2)$$

which gives

$$C_1(d,t)-t=d^2\,(1-d)\left(D_0(d^2,t)-t^{-1}\right)+d\cdot\mathbb{D}_0(d^2,t). \tag{4}$$

Furthermore, for $n\geq 2$ and $k\geq 2$ we have

$$\mathbb{C}_{n,k}(d)=d^{2^k}\left(1-d^{2^{k-1}}\right)\mathbb{D}_{n-4,k-1}(d^2)+d^{2^{k-2}}\left(1-d^{2^{k-2}}\right)\mathbb{C}_{n-2,k-1}(d^2)$$
$$+d^{2^{k-1}}\mathbb{D}_{n-2,k-1}(d^2)$$

which gives

$$C_k(d,t)=\left(d^{2^k}\left(1-d^{2^{k-1}}\right)t^4+d^{2^{k-1}}t^2\right)D_{k-1}(d^2,t)$$
$$+d^{2^{k-2}}\left(1-d^{2^{k-2}}\right)t^2C_{k-1}(d^2,t), \tag{5}$$

since $\mathbb{D}_{m,k}(d)=\mathbb{C}_{m,k}(d)=0$ for $m<2k-1$.

On the other hand, for $n\geq 1$ and $k\geq 1$ we have

$$\mathbb{D}_{n,k}(d)=d^{2^k}\left(\mathbb{D}_{n-1,k}(d)+\mathbb{D}_{n-2,k}(d)+\left(1-d^{2^k}\right)\mathbb{D}_{n-3,k}(d)\right)+\mathbb{C}_{n,k}(d)$$
$$+\left(d^{2^{k-1}}-d^{2^k}\right)\mathbb{C}_{n-1,k}(d)$$

which gives

$$D_k(d,t)=d^{2^k}\left(tD_k(d,t)+t^2D_k(d,t)+\left(1-d^{2^k}\right)t^3D_k(d,t)\right)+C_k(d,t)$$
$$+\left(d^{2^{k-1}}-d^{2^k}\right)tC_k(d,t)$$
$$=\frac{1+\left(d^{2^{k-1}}-d^{2^k}\right)t}{1-d^{2^k}(t+t^2+t^3)+d^{2^{k+1}}t^3}C_k(d,t). \tag{6}$$

Finally, using $D_0(d,t)$, (4), (5) and (6) we complete the proof. □

Remark 2. Note that $D_0(d,t)$ in Theorem 2 is the generating function $f(x,y)$ given in [10, Section 6]. In our notation d and t corresponds to y and x, respectively.

Remark 3. For $k=1$, we know from Theorem 2 that

$$D_1(d,t)=\frac{t\left(1+\left(d-d^2\right)t\right)^2}{\left(1-d^2(t+t^2)-(d^2-d^4)t^3\right)^2}$$

and let

$$s(d)=\frac{dt^2\left(1+(d^3-d^4)(t+t^2)-\left(d^4-2d^5+2d^7-d^8\right)t^3\right)}{1-d^4(t+t^2)-\left(d^4-d^8\right)t^3}. \tag{7}$$

Then, for $k\geq 1$ we have

$$D_{k+1}(d,t)=D_1\left(d^{2^k},t\right)s\left(d^{2^{k-1}}\right)^k.$$

In closed form, for $k\geq 1$, $D_k(d,t)$ can also be written as

$$D_k(d,t)=\frac{d^{(k-1)2^{k-2}}t^{2k-1}(1+a_1t)^2\left(1+b_1(t+t^2)+b_2t^3\right)^{k-1}}{\left(1-c_1(t+t^2)-c_2t^3\right)^{k+1}}$$

with

$$\begin{aligned} a_1&=d^{2^{k-1}}-d^{2^k},\\ b_1&=d^{3\cdot 2^{k-2}}-d^{2^k},\;\; b_2=-d^{2^k}+2d^{5\cdot 2^{k-2}}-2d^{7\cdot 2^{k-2}}+d^{2^{k+1}},\\ c_1&=d^{2^k},\;\; c_2=d^{2^k}-d^{2^{k+1}} \end{aligned} \tag{8}$$

where we omitted the dependence on k for notational convenience.

The generating functions for a few small particular values of k are given as

$$D_0(d) = \frac{1+dt+\left(d-d^2\right)t^2}{1-d(t+t^2)-\left(d-d^2\right)t^3} = 1+2dt+\left(d^2+2d\right)t^2+\left(d^3+3d^2+d\right)t^3+\cdots$$

$$D_1(d) = \frac{t\left(1+\left(d-d^2\right)t\right)^2}{\left(1-d^2(t+t^2)-(d^2-d^4)t^3\right)^2} = t+2dt^2+\left(2d^3+3d^2\right)t^3+2\left(d^5+d^4+2d^3+d^2\right)t^4+\cdots$$

$$D_2(d) = \frac{dt^3\left(1+(d^2-d^4)t\right)^2\left(1+(d^3-d^4)(t+t^2)+(-d^4+2d^5-2d^7+d^8)t^3\right)}{\left(1-d^4(t+t^2)-(d^4-d^8)t^3\right)^3} = dt^3+\left(d^4+2d^3\right)t^4+\left(d^8+2d^7+2d^6+3d^5+d^4\right)t^5+\cdots$$

$$D_3(d) = \frac{d^4t^5\left(1+(d^4-d^8)t\right)^2\left(1+(d^6-d^8)(t+t^2)+(-d^8+2d^{10}-2d^{14}+d^{16})t^3\right)^2}{\left(1-d^8(t+t^2)-(d^8-d^{16})t^3\right)^4} = d^4t^5+2\left(d^{10}+d^8\right)t^6+\left(2d^{18}+3d^{16}+4d^{14}+3d^{12}+2d^{10}\right)t^7+\cdots$$

$$D_4(d) = \frac{d^{12}t^7\left(1+(d^8-d^{16})t\right)^2\left(1+(d^{12}-d^{16})(t+t^2)+(-d^{16}+2d^{20}-2d^{28}+d^{32})t^3\right)^3}{\left(1-d^{16}(t+t^2)-(d^{16}-d^{32})t^3\right)^5} = d^{12}t^7+\left(3d^{24}+2d^{20}\right)t^8+\left(3d^{40}+5d^{36}+6d^{32}+3d^{28}+3d^{24}\right)t^9+\cdots.$$

We do a similar computation for the $C_k(d)$. We have

$$C_1(d) = \frac{t\left(1+(d-d^2)t\right)}{1+d^4t^3-d^2t\left(1+t+t^2\right)} = t+dt^2+\left(d^3+d^2\right)t^3+\left(d^5+d^3+d^2\right)t^4+\cdots \tag{9}$$

and for $k \geq 1$,

$$C_{k+1}(d) = C_k\left(d^2\right)s\left(d^{2^{k-1}}\right)$$

where $s(d)$ is the polynomial given in (7). A few of these connection polynomials for small k are as follows:

$$C_2(d) = \frac{dt^3\left(1+(d^2-d^4)t\right)\left(1+(d^3-d^4)(t+t^2)+(-d^4+2d^5-2d^7+d^8)t^3\right)}{\left(1-d^4(t+t^2)-(d^4-d^8)t^3\right)^2} = dt^3+\left(d^3+d^4\right)t^4+\left(d^4+d^5+d^6+d^7+d^8\right)t^5+\cdots$$

$$C_3(d) = \frac{d^4t^5\left(1+(d^4-d^8)t\right)\left(1+(d^6-d^8)(t+t^2)+(-d^8+2d^{10}-2d^{14}+d^{16})t^3\right)^2}{\left(1-d^8(t+t^2)-(d^8-d^{16})t^3\right)^3} = d^4t^5+\left(d^8+2d^{10}\right)t^6+\left(2d^{10}+d^{12}+2d^{14}+2d^{16}+2d^{18}\right)t^7+\cdots$$

$$C_4(d) = \frac{d^{12}t^7\left(1+(d^8-d^{16})t\right)\left(1+(d^{12}-d^{16})(t+t^2)+(-d^{16}+2d^{20}-2d^{28}+d^{32})t^3\right)^3}{\left(1-d^{16}(t+t^2)-(d^{16}-d^{32})t^3\right)^4} = d^{12}t^7+\left(d^{20}+3d^{24}\right)t^8+\left(3d^{24}+d^{28}+3d^{32}+4d^{36}+3d^{40}\right)t^9+\cdots.$$

So $C_1(d)$ is as given in (9) and for $k \geq 2$, the explicit expression for the connection polynomials are given by

$$C_k(d) = \frac{d^{(k-1)2^{k-2}}t^{2k-1}(1+a_1t)\left(1+b_1(t+t^2)+b_2t^3\right)^{k-1}}{\left(1-c_1(t+t^2)-c_2t^3\right)^k}$$

where the coefficients are as given in (8).

4. Applications

Consider the boundary enumerator polynomial of the k-dimensional hypercubes in hypercubes Q_n of dimension n. It is known that (see, for example [3]) the number of Q_k contained in an Q_n can be found from the coefficient of x^k in $(2+x)^n$ which is $\binom{n}{k}2^{n-k}$. Hence, for nonnegative integer n and k the boundary enumerator polynomial of Q_k in Q_n becomes

$$\binom{n}{k}2^{n-k}\cdot d^{(n-k)2^k}$$

since each vertex in Q_k (in total 2^k vertices) has degree k and these vertices have degrees n in Q_n.

Remark 4. The cube polynomial $c_n(x)$ of Γ_n is studied in [8] and many interesting related results are obtained. In [15] the q-analogue of the cube polynomial $c_n(x; q)$ of the Fibonacci cubes are considered and many of the results in [8] are extended. Note that $c_n(x; q)$ is a refinement of the cube polynomial $c_n(x)$ and $c_n(x; 1) = c_n(x)$. In this paper, we have a similar refinement of $c_n(x)$. In particular, writing

$$c_n(x) = \sum_{k\geq 0} a_{n,k}x^k \quad \text{and} \quad c_n(x; q) = \sum_{k\geq 0} b_{n,k}(q)x^k$$

we obtain that

$$a_{n,k} = b_{n,k}(1) = \mathbb{D}_{n,k}(1).$$

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