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Note

Counting disjoint hypercubes in Fibonacci cubes

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ABSTRACT

We provide explicit formulas for the maximum number $q_k(n)$ of disjoint subgraphs isomorphic to the k-dimensional hypercube in the n-dimensional Fibonacci cube Γ_n for small k, and prove that the limit of the ratio of such cubes to the number of vertices in Γ_n is $\frac{1}{2^k}$ for arbitrary k. This settles a conjecture of Gravier, Mollard, Špacapan and Zemljič about the limiting behavior of $q_k(n)$.

1. Introduction

One of the basic models for interconnection networks is the hypercube graph Q_n of dimension n. The vertices of Q_n are represented by binary strings of length n and two vertices are adjacent if and only if they differ in exactly one position. In this model vertex set of the graph denotes the processors and edge set denotes the communication links between processors.

In [4] Fibonacci cubes Γ_n were introduced as a new model of computation for interconnection networks. There is extensive literature on the properties and applications of the Fibonacci cubes. A survey of their usage in theoretical chemistry and summary results on the structure of Fibonacci cubes, including representations, recursive construction, hamiltonicity, the nature of the degree sequence and some enumeration results can be found in the survey [5]. Important properties of Fibonacci cubes in network design are given in [4,2]. The characterization of maximal induced hypercubes in Γ_n was presented in [8]. Many interesting results on the cube polynomial of Γ_n were proved in [6]. A refinement of the cube polynomial of Γ_n in [6] is considered in [10]. In the latter combinatorial interpretation, an extra variable acts as the enumerator of the hypercubes in Γ_n by their distance to the all 0 vertex. Recent papers on additional properties of Fibonacci cubes that have appeared in the literature indicate the continuing interest in these graphs, see for example [1,7,9,11].

Let $q_k(n)$ denote the maximum number of disjoint subgraphs isomorphic to k-dimensional hypercube Q_k in Γ_n. In a recent study several recursive relations and a summation formula for $q_k(n)$ in terms of Fibonacci numbers were presented [3]. Let $|V(\Gamma_n)|$ denote the number of vertices of Γ_n. It was conjectured in [3, Question 3.2] that the limit of the ratio $\frac{q_k(n)}{|V(\Gamma_n)|}$ as n increases without bound exists and is equal to $\frac{1}{2^k}$. In this paper we use the expression for $q_k(n)$ given in [3] in terms of Fibonacci numbers and generating function techniques to derive explicit formulas for $q_k(n)$ for small k. Our computation also gives the form of the $q_k(n)$ for general k, from which it follows that $\lim_{n\to\infty} \frac{q_k(n)}{|V(\Gamma_n)|} = \frac{1}{2^k}$ (Theorem 1).

This paper is organized as follows. In Section 2 we present some useful identities on generating functions of certain subsequences of Fibonacci numbers. We derive our main result in Section 3.

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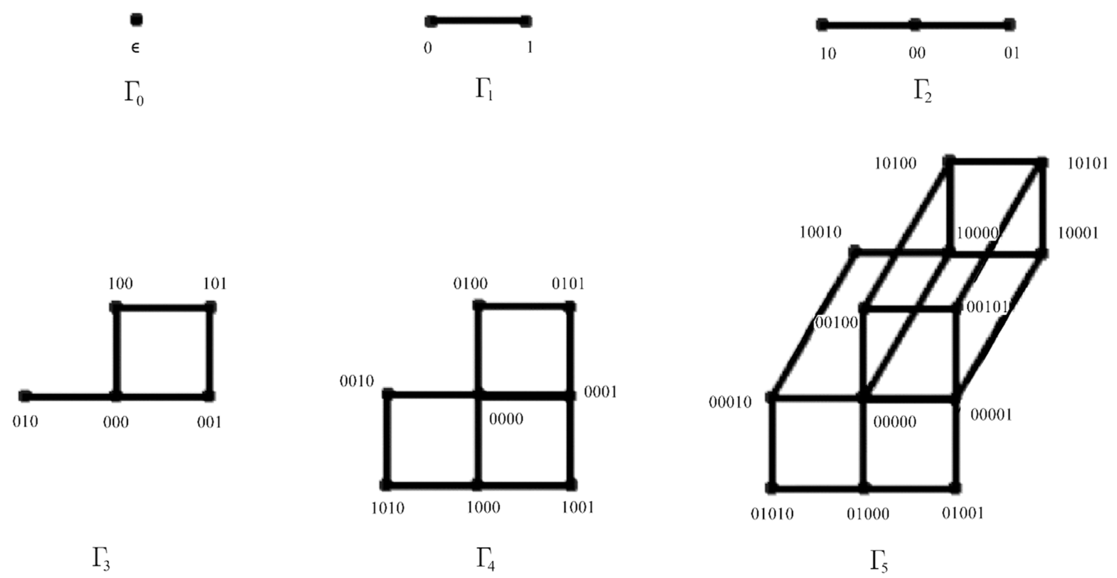

Fig. 1. Fibonacci graphs $\Gamma_0, \Gamma_1, \ldots, \Gamma_5$.

2. Preliminaries

In this section we present some notations and preliminary results related to Fibonacci cubes and Fibonacci numbers.

An n-dimensional hypercube Q_n is the simple graph with vertex set

$$V(Q_n) = \{b_1 b_2 \cdots b_n \mid b_i \in \{0, 1\},\ 1 \le i \le n\}$$

where the edges are between vertices differing in a single bit. An n-dimensional Fibonacci cube Γ_n is a subgraph of Q_n with vertex set $V(\Gamma_n)$ corresponding to those in Q_n without two consecutive 1s in their string representation. Therefore the vertices of Γ_n have the property that $b_i b_{i+1} = 0$ for all $i \in \{1, 2, \ldots, n-1\}$. For convenience Γ_0 is defined as the graph with a single vertex and no edges.

$V(\Gamma_n)$ is enumerated by Fibonacci number f_{n+2}, where $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \ge 2$. In Fig. 1 we present the first six Fibonacci cubes with their vertices labeled with the corresponding binary strings in the hypercube graph.

Next we consider some special generating functions that we use in the proof of our main result.

If $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ is the generating function of the sequence a_n, then

$$a_0 + a_3 x^3 + a_6 x^6 + \cdots = \frac{1}{3}\left(g(x) + g(\omega x) + g(\omega^2 x)\right)$$

where ω is a primitive cube root of unity. It follows that for the Fibonacci numbers we have

$$\sum_{i \ge 0} f_{3i} x^{3i} = \frac{1}{3}\left(\frac{x}{1-x-x^2} + \frac{\omega x}{1-\omega x-\omega^2 x^2} + \frac{\omega^2 x}{1-\omega^2 x-\omega x^2}\right) = \frac{2x^3}{1-4x^3-x^6},$$

and therefore

$$\sum_{i \ge 0} f_{3i} x^i = \frac{2x}{1-4x-x^2} \quad \text{and} \quad \sum_{i \ge 0} f_{3i+3} x^i = \frac{2}{1-4x-x^2}. \tag{1}$$

Note that (1) is a consequence of the general result that the generating function of the every rth Fibonacci number for $r > 0$ is given by

$$\sum_{i \ge 0} f_{ri} x^i = \frac{f_r x}{1-L_r x-x^2}$$

where L_r is the rth Lucas number defined by $L_0 = 2, L_1 = 1$ and $L_r = L_{r-1} + L_{r-2}$ for $r > 1$. However we do not need this expansion in its full generality.

By a similar manipulation using the cube roots of unity, we also obtain the following generating functions for subsequences of the Fibonacci sequence:

$$\sum_{i\geq 0} f_{3i+1}x^i = \frac{1-x}{1-4x-x^2}, \qquad \sum_{i\geq 0} f_{3i+2}x^i = \frac{1+x}{1-4x-x^2}. \tag{2}$$

Also note that by Newton's expansion we have

$$\sum_{i\geq 0} \binom{i}{k-1} x^i = \frac{x^{k-1}}{(1-x)^k}. \tag{3}$$

We will use the generating functions in (1)–(3) in the following section to prove our main results.

3. Main results

To show that $\lim_{n\to\infty} \frac{q_k(n)}{|V(\Gamma_n)|} = \frac{1}{2^k}$, our starting point is the closed formula for $q_k(n)$ in terms of Fibonacci numbers given in [3].

Proposition 1 ([3, Corollary 2.4]). *For every $n \geq 0$ and $k \geq 1$*

$$q_k(n) = \sum_{i=0}^{\lfloor \frac{n+k-2}{3} \rfloor} \binom{i}{k-1} f_{n+k-3i-1}. \tag{4}$$

The right hand side of (4) is the convolution of the sequence in (3) with a shifted Fibonacci sequence. Our aim is to find an explicit formula for $q_k(n)$ using this expression. The main difficulty is the nature of the upper limit of the summation in (4). For this purpose we will consider the cases $n+k-2 \bmod 3$ separately in the following subsections. Assume that $n+k-2 = 3m+j$ with $j \in \{0, 1, 2\}$. Then from (4) we need to find

$$s_k(m) := q_k(n) = \sum_{i=0}^{m} \binom{i}{k-1} f_{3(m-i)+j+1}.$$

Note that for $m \leq k-1$ obviously we have $s_k(m) = 0$. Furthermore, the generating function of the convolution $s_k(m)$ with the shifted Fibonacci numbers depending the value of j is given by the product of the one of the generating functions in (1) or (2) with the one in (3) as given below.

$$\sum_{m\geq 0} s_k(m)x^m = \begin{cases} \dfrac{x^{k-1}}{(1-x)^{k-1}(1-4x-x^2)} & \text{if } n+k-2 = 3m, \\ \dfrac{(1+x)x^{k-1}}{(1-x)^k(1-4x-x^2)} & \text{if } n+k-2 = 3m+1, \\ \dfrac{2x^{k-1}}{(1-x)^k(1-4x-x^2)} & \text{if } n+k-2 = 3m+2. \end{cases} \tag{5}$$

Using (5) and taking also the value of k mod 3 into account there would actually be nine different cases to consider, though they are all similar computations in nature. Combining the results obtained in each of these cases (see subsections below) we obtain the following result. Also note that since $|V(\Gamma_n)| = f_{n+2}$, for any fixed exponent $c > 0$, $\lim_{n\to\infty} \frac{n^c}{|V(\Gamma_n)|} = 0$.

Theorem 1. *For every $n \geq 0$ and $k \geq 1$*

$$q_k(n) = \frac{1}{2^k} f_{n+2} + p(n)$$

where $p(n)$ is a polynomial with rational coefficients of degree at most $k-1$. Consequently,

$$\lim_{n\to\infty} \frac{q_k(n)}{|V(\Gamma_n)|} = \frac{1}{2^k}.$$

Before presenting the details of the proof we provide the exact expressions for $q_k(n)$ for $k = 1, 2, \dots, 5$. The $k = 1$ case is given in [3].

$$q_1(n) = \begin{cases} \frac{1}{2}f_{n+2} - \frac{1}{2} & \text{if } n \equiv 0 \bmod 3, \\ \frac{1}{2}f_{n+2} & \text{if } n \equiv 1 \bmod 3, \\ \frac{1}{2}f_{n+2} - \frac{1}{2} & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

For $1 < k \le 5$ we obtain the following exact formulas for $q_k(n)$.

$$q_2(n) = \begin{cases} \frac{1}{4}f_{n+2} - \frac{1}{4} & \text{if } n \equiv 0 \bmod 3, \\ \frac{1}{4}f_{n+2} - \frac{n}{6} - \frac{1}{3} & \text{if } n \equiv 1 \bmod 3, \\ \frac{1}{4}f_{n+2} - \frac{n}{6} - \frac{5}{12} & \text{if } n \equiv 2 \bmod 3, \end{cases}$$

$$q_3(n) = \begin{cases} \frac{1}{8}f_{n+2} - \frac{1}{8} - \frac{1}{12}n - \frac{1}{36}n^2 & \text{if } n \equiv 0 \bmod 3, \\ \frac{1}{8}f_{n+2} - \frac{1}{9} - \frac{1}{9}n - \frac{1}{36}n^2 & \text{if } n \equiv 1 \bmod 3, \\ \frac{1}{8}f_{n+2} - \frac{5}{24} - \frac{1}{12}n & \text{if } n \equiv 2 \bmod 3, \end{cases}$$

$$q_4(n) = \begin{cases} \frac{1}{16}f_{n+2} - \frac{1}{16} - \frac{1}{72}n - \frac{1}{72}n^2 - \frac{1}{324}n^3 & \text{if } n \equiv 0 \bmod 3, \\ \frac{1}{16}f_{n+2} - \frac{1}{18} - \frac{1}{18}n - \frac{1}{72}n^2 & \text{if } n \equiv 1 \bmod 3, \\ \frac{1}{16}f_{n+2} - \frac{103}{1296} - \frac{5}{216}n - \frac{1}{108}n^2 - \frac{1}{324}n^3 & \text{if } n \equiv 2 \bmod 3, \end{cases}$$

$$q_5(n) = \begin{cases} \frac{1}{32}f_{n+2} - \frac{1}{32} - \frac{1}{144}n - \frac{1}{144}n^2 - \frac{1}{648}n^3 & \text{if } n \equiv 0 \bmod 3, \\ \frac{1}{32}f_{n+2} - \frac{37}{972} - \frac{43}{1944}n - \frac{1}{648}n^2 - \frac{1}{1944}n^3 - \frac{1}{3888}n^4 & \text{if } n \equiv 1 \bmod 3, \\ \frac{1}{32}f_{n+2} - \frac{389}{7776} - \frac{67}{3888}n + \frac{1}{1296}n^2 - \frac{1}{972}n^3 - \frac{1}{3888}n^4 & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

Remark 1. Even though the coefficients of the polynomials $p(n)$ appear to be all negative, this is not always the case. The polynomial in $q_5(n)$ for $n \equiv 2 \bmod 3$ contains a term with positive sign.

We begin the proof of Theorem 1 for the case $n+k-2 \equiv 2 \bmod 3$, which is illustrative of how the computation proceeds in general.

3.1. Case $n + k - 2 \equiv 2 \bmod 3$

Let $n + k - 2 = 3m + 2$. From (5), we need to expand the right hand side of

$$\sum_{m\ge 0} s_k(m)x^m = \frac{2x^{k-1}}{(1-x)^k(1-4x-x^2)}. \tag{6}$$

This in turn depends on the value of k mod 3. First assume that $k = 3t - 1$ for some integer $t > 1$, that is, $n + 2 = 3(m - t + 2) + 1$. Then by using the equation on the left side of (2) we obtain

$$\sum_{m\ge t-2} f_{n+2}x^m = x^{t-2} \sum_{m\ge t-2} f_{3(m-t+2)+1}x^{m-t+2} = \frac{(1-x)x^{t-2}}{1-4x-x^2}. \tag{7}$$

Now using the numerator of (7), the right hand side of (6) can be written as

$$\frac{2x^{k-1}}{(1-x)^k(1-4x-x^2)} = (1-x)x^{t-2}\left[\frac{2x^{2t}}{(1-x)^{k+1}(1-4x-x^2)}\right].$$

Using partial fractions we can write,

$$\frac{2x^{2t}}{(1-x)^{k+1}(1-4x-x^2)} = \frac{Ax+B}{1-4x-x^2} + \sum_{i=1}^{k+1} \frac{C_i}{(1-x)^i}. \tag{8}$$

Now let z_1 and z_2 be the roots of $1-4x-x^2=0$. By equating the numerators in (8) and putting $x=z_1$, $x=z_2$ and $k=3t-1$ we obtain the two equations

$$\begin{aligned} 2z_1^{2t} &= (Az_1+B)(1-z_1)^{3t} \\ 2z_2^{2t} &= (Az_2+B)(1-z_2)^{3t}. \end{aligned}$$

This gives $A=0$, $B=\frac{1}{2^{3t-1}}=\frac{1}{2^k}$. The C_i's in (8) are certain rational numbers. Consequently,

$$\sum_{m\geq 0} s_k(m)x^m = \frac{1}{2^k}\left(\frac{(1-x)x^{t-2}}{1-4x-x^2}\right) + x^{t-2}\sum_{i=0}^{k} \frac{C_{i+1}}{(1-x)^i}.$$

Then by using (7) we get

$$q_k(n) = s_k(m) = \frac{1}{2^k} f_{n+2} + p(n)$$

where $p(n)$ is polynomial with rational coefficients of degree at most $k-1$ which comes from the $x^{t-2}\sum_{i=0}^{k}\frac{C_{i+1}}{(1-x)^i}$ part.

Example 1. We consider the $k=5$ $(t=2)$ case in detail. We assume that $n+k-2=n+3=3m+2$. From (6) we have

$$\begin{aligned} \sum_{m\geq 0} s_k(m)x^m &= \frac{2x^4}{(1-x)^5(1-4x-x^2)} \\ &= (1-x)\left[\frac{2x^4}{(1-x)^6(1-4x-x^2)}\right]. \end{aligned}$$

By using partial fractions,

$$\frac{2x^4}{(1-x)^6(1-4x-x^2)} = \frac{Ax+B}{1-4x-x^2} + \sum_{i=1}^{6}\frac{C_i}{(1-x)^i}.$$

Equating the numerators,

$$2x^2 = (Ax+B)(1-x)^6 + \left(\sum_{i=1}^{6} C_i(1-x)^{6-i}\right)(1-4x-x^2).$$

Therefore,

$$A=0, \qquad B=\frac{1}{32},$$

and

$$C_1=0, \qquad C_2=\frac{1}{32}, \qquad C_3=\frac{3}{16}, \qquad C_4=-1, \qquad C_5=\frac{5}{4}, \qquad C_6=-\frac{1}{2}.$$

Then we have

$$\sum_{m\geq 0} s_k(m)x^m = \frac{1}{32}\left(\frac{1-x}{1-4x-x^2}\right) + \sum_{i=1}^{6}\frac{C_i}{(1-x)^{i-1}}$$

and by using (7) we get

$$q_5(n) = s_5(m) = \frac{1}{32} f_{n+2} + p(n).$$

Using the identity $\frac{1}{1-x}=\sum_{m\geq 0} x^m$ and its derivatives we obtain the error term $p(n)$ explicitly as

$$p(n) = -\frac{389}{7776} - \frac{67}{3888}n + \frac{1}{1296}n^2 - \frac{1}{972}n^3 - \frac{1}{3888}n^4.$$

Example 2. In this example, we consider the $k = 2$ ($t = 1$) case separately as we assumed $t > 1$ in the above calculations. For this case we use partial fractions directly for (6) and find

$$\sum_{m\geq 0} s_k(m)x^m = \frac{2x}{(1-x)^2(1-4x-x^2)}$$
$$= \frac{3}{8}\left(\frac{2}{1-4x-x^2}\right) + \frac{1}{8}\left(\frac{2x}{1-4x-x^2}\right) - \frac{1}{2}\left(\frac{1}{(1-x)^2}\right) - \frac{1}{4}\left(\frac{1}{1-x}\right)$$

which gives

$$s_k(m) = \frac{3}{8}f_{3m+3} + \frac{1}{8}f_{3m} - \frac{1}{2}(m+1) - \frac{1}{4}.$$

Recalling that $n = 3m + 2$, we can write this expression in terms of n as

$$q_2(n) = \frac{3}{8}f_{n+1} + \frac{1}{8}f_{n-2} - \frac{n}{6} - \frac{5}{12},$$

which further simplifies to

$$q_2(n) = \frac{1}{4}f_{n+2} - \frac{n}{6} - \frac{5}{12}.$$

Continuing with the proof, now we assume that $k = 3t$, giving $n + 2 = 3(m - t + 2)$. Using the equation on the left side of (1) we obtain

$$\sum_{m\geq t-2} f_{n+2}x^m = x^{t-2}\sum_{m\geq t-2} f_{3(m-t+2)}x^{m-t+2} = \frac{2x^{t-1}}{1-4x-x^2}. \tag{9}$$

Using similar calculations as above we get

$$\sum_{m\geq 0} s_k(m)x^m = \frac{1}{2^k}\left(\frac{2x^{t-1}}{1-4x-x^2}\right) + 2x^{t-1}\sum_{i=1}^{k}\frac{C_i}{(1-x)^i}.$$

Therefore by using (9) we have

$$q_k(n) = s_k(m) = \frac{1}{2^k}f_{n+2} + p(n)$$

as before. Here $p(n)$ is polynomial with rational coefficients of degree at most $k-1$ which comes from the $2x^{t-1}\sum_{i=1}^{k}\frac{C_i}{(1-x)^i}$ part.

Similarly, for the last subcase with $k = 3t - 2$ we obtain

$$\sum_{m\geq t-2} f_{n+2}x^m = \frac{(1+x)x^{t-2}}{1-4x-x^2}$$

and

$$\sum_{m\geq 0} s_k(m)x^m = \frac{1}{2^k}\left(\frac{(1+x)x^{t-2}}{1-4x-x^2}\right) + (1+x)x^{t-2}\sum_{i=1}^{k}\frac{C_i}{(1-x)^i} + C_{k+1}x^{t-2},$$

which again gives

$$q_k(n) = s_k(m) = \frac{1}{2^k}f_{n+2} + p(n)$$

where $p(n)$ is polynomial with rational coefficients of degree at most $k-1$.

3.2. *Case* $n + k - 2 \equiv 1 \bmod 3$ *and Case* $n + k - 2 \equiv 0 \bmod 3$

We omit the calculations in these cases, but only indicate what is involved. First assume that $n + k - 2 = 3m + 1$. By using (5) and an argument similar to the previous case with the subcases $k = 3t$, $k = 3t + 1$ and $k = 3t + 2$ separately we need to evaluate the following:

$$\sum_{m\geq 0} s_k(m)x^m = \frac{(1+x)x^{k-1}}{(1-x)^k(1-4x-x^2)} = \begin{cases} \frac{1}{2^k}\left(\frac{(1+x)x^{t-1}}{1-4x-x^2}\right) + (1+x)x^{t-1}\sum_{i=1}^{k}\frac{C_i}{(1-x)^i} & \text{if } k=3t, \\ \frac{1}{2^k}\left(\frac{2x^{t-1}}{1-4x-x^2}\right) + 2x^{t-1}\sum_{i=1}^{k}\frac{C_i}{(1-x)^i} & \text{if } k=3t-1, \\ \frac{1}{2^k}\left(\frac{(1-x)x^{t-2}}{1-4x-x^2}\right) + x^{t-1}\sum_{i=0}^{k-1}\frac{C_i}{(1-x)^i} & \text{if } k=3t-2. \end{cases}$$

Similarly if $n+k-2=3m$ then we need to expand the following:

$$\begin{aligned} \sum_{m\geq 0} s_k(m)x^m &= \frac{x^{k-1}}{(1-x)^{k-1}(1-4x-x^2)} \\ &= \begin{cases} \frac{1}{2^k}\left(\frac{(1-x)x^{t-1}}{1-4x-x^2}\right) + x^{t-1}\sum_{i=0}^{k-1}\frac{C_i}{(1-x)^i} & \text{if } k=3t, \\ \frac{1}{2^k}\left(\frac{(1+x)x^{t-1}}{1-4x-x^2}\right) + (1+x)x^{t-1}\sum_{i=1}^{k-1}\frac{C_i}{(1-x)^i} + C_k x^{t-1} & \text{if } k=3t-1, \\ \frac{1}{2^k}\left(\frac{2x^{t-1}}{1-4x-x^2}\right) + 2x^{t-1}\sum_{i=1}^{k-1}\frac{C_i}{(1-x)^i} & \text{if } k=3t-2. \end{cases} \end{aligned}$$

In all of the above six cases the idea is similar to the one in the previous section and using the appropriate formulas from either (1) or (2) for $\sum_{m\geq t-2} f_{n+2}x^m$ we obtain

$$q_k(n) = s_k(m) = \frac{1}{2^k}f_{n+2} + p(n)$$

where $p(n)$ is polynomial with rational coefficients of degree at most $k-1$.

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References

[1] A.R. Ashrafi, J. Azarija, K. Fathalikhani, S. Klavžar, M. Petkovšek, Vertex and edge orbits of Fibonacci and Lucas cubes, Ann. Comb. 20 (2) (2016) 209–229.
[2] B. Cong, S. Zheng, S. Sharma, On simulations of linear arrays, rings and 2D meshes on Fibonacci cube networks, in: Proceedings of the 7th International Parallel Processing Symposium, 1993, pp. 747–751.
[3] S. Gravier, M. Mollard, S. Špacapan, S.S. Zemljič, On disjoint hypercubes in Fibonacci cubes, Discrete Appl. Math. 190–191 (2015) 50–55.
[4] W.-J. Hsu, Fibonacci cubes–a new interconnection technology, IEEE Trans. Parallel Distrib. Syst. 4 (1) (1993) 3–12.
[5] S. Klavžar, Structure of Fibonacci cubes: a survey, J. Comb. Optim. 25 (2013) 505–522.
[6] S. Klavžar, M. Mollard, Cube polynomial of Fibonacci and Lucas cube, Acta Appl. Math. 117 (2012) 93–105.
[7] S. Klavžar, M. Mollard, Asymptotic properties of Fibonacci cubes and Lucas cubes, Ann. Comb. 18 (3) (2014) 447–457.
[8] M. Mollard, Maximal hypercubes in Fibonacci and Lucas cubes, Discrete Appl. Math. 160 (2012) 2479–2483.
[9] M. Ramras, Congestion-free routing of linear permutations on Fibonacci and Lucas cubes, Australas. J. Combin. 60 (1) (2014) 1–10.
[10] E. Saygı, Ö. Eğecioğlu, q-cube enumerator polynomial of Fibonacci graphs, UCSB Technical Report, ID 2016-01, 2016.
[11] A. Vesel, Linear recognition and embedding of Fibonacci cubes, Algorithmica 71 (4) (2015) 1021–1034.