SIAM J. DISCRETE MATH.
Vol. 24, No. 4, pp. 1753–1762


# BESSEL POLYNOMIALS AND THE PARTIAL SUMS OF THE EXPONENTIAL SERIES*

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**Abstract.** Let $e_k(x)$ denote the $k$-th partial sum of the Maclaurin series for the exponential function. Define the $(n+1)\times(n+1)$ Hankel determinant by setting $\widetilde{H}_n(x) = \det[e_{i+j}(x)]_{0\le i,j\le n}$. We give a closed form evaluation of this determinant in terms of the Bessel polynomials using the method of recently introduced $\gamma$-operators.

**Key words.** Bessel polynomials, exponential series, Hankel determinants

**AMS subject classifications.** 05A10, 05A15, 05A19, 05E35, 11C20

**DOI.** 10.1137/090760337

## 1. Introduction.

Let

$$e_k(x) = \sum_{m=0}^{k} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} \tag{1}$$

denote the $k$-th partial sum of the Maclaurin series for the exponential function. Consider the $(n+1)\times(n+1)$ Hankel determinant

$$\widetilde{H}_n(x) = \det[e_{i+j}(x)]_{0\le i,j\le n}. \tag{2}$$

The first few of these are as follows:

$$\begin{aligned}
\widetilde{H}_0(x) &= 1,\\
\widetilde{H}_1(x) &= -\frac{x}{2}(x+2),\\
\widetilde{H}_2(x) &= -\frac{x^4}{144}\left(x^2+6x+12\right),\\
\widetilde{H}_3(x) &= \frac{x^9}{1036800}\left(x^3+12x^2+60x+120\right),\\
\widetilde{H}_4(x) &= \frac{x^{16}}{1463132160000}\left(x^4+20x^3+180x^2+840x+1680\right).
\end{aligned}$$

The *Bessel polynomials* $y_n(x)$ were introduced by Krall and Fink in 1948 [13] as the polynomial solutions of the second order differential equation

$$x^2y'' + 2(x+1)y' - n(n+1)y = 0.$$

They are given explicitly by

$$y_n(x) = \sum_{i=0}^{n} \frac{(n+i)!}{i!(n-i)!}\left(\frac{x}{2}\right)^i. \tag{3}$$

*Received by the editors May 28, 2009; accepted for publication (in revised form) November 5, 2010; published electronically December 21, 2010.
http://www.siam.org/journals/sidma/24-4/76033.html

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The Bessel polynomials are known to satisfy two different orthogonality properties. The first one is with respect to an analytic weight function along a simple closed curve encircling the origin in the complex plane [10]. The second is with respect to a signed measure supported on $[0,\infty)$. The reader is referred to [8, 11] for this more recent result. The Bessel polynomials satisfy the recursion

$$y_{n+1}(x) = (2n+1)xy_n(x) + y_{n-1}(x) \tag{4}$$

with $y_{-1}(x) = y_0(x) = 1$.

In this paper we prove that the $\widetilde{H}_n(x)$ has the following evaluation.

THEOREM 1. *Suppose the polynomials $e_k$ and $y_k$ and the $(n+1)\times(n+1)$ Hankel determinant $\widetilde{H}_n(x)$ are defined as in (1), (3), and (2), respectively. Then*

$$\widetilde{H}_n(x) = c_n\, x^{n(n+1)}\, y_n\left(\frac{2}{x}\right), \tag{5}$$

*where*

$$c_n = (-1)^{\frac{n(n+1)}{2}}\, 2^n \prod_{j=1}^{n} \frac{j!^2}{(2j)!^2}. \tag{6}$$

Thus the explicit evaluation of $\widetilde{H}_n(x)$ is

$$\widetilde{H}_n(x) = (-1)^{\frac{n(n+1)}{2}}\, x^{n^2}\, 2^n \prod_{j=1}^{n} \frac{j!^2}{(2j)!^2} \sum_{i=0}^{n} \frac{(n+i)!}{i!(n-i)!} x^{n-i}. \tag{7}$$

Bessel polynomials have other known determinantal representations [4]. The first representation below is an $n\times n$ determinant via the Rodrigues formula

$$2^n y_n(x) = \det\begin{bmatrix} 2(1+nx) & -1 & 0 & 0 & \cdots & 0 \\ -2x(2+nx) & 2(1+nx) & -2 & 0 & \cdots & 0 \\ 2x^2(3+nx) & -2x(2+nx) & 2(1+nx) & -3 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1-n \\ 2(-x)^{n-1}(n+nx) & 2(-x)^{n-2}(n-1+nx) & \cdots & \cdots & \cdots & 2(1+nx) \end{bmatrix},$$

and the second one is an $(n+1)\times(n+1)$ determinant from the linear recurrence

$$y_n(x) = \det\begin{bmatrix} (2n-1)x & -1 & 0 & 0 & \cdots & 0 \\ 1 & (2n-3)x & -1 & 0 & \cdots & \vdots \\ 0 & 1 & (2n-5)x & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ \vdots & \vdots & \vdots & 1 & x & -1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Instead of working with the $e_k(x)$ directly, it is somewhat easier to work with the polynomials $a_k(x)$ defined as

$$a_k(x) = \sum_{m=0}^{k} \frac{x^m}{(k-m)!} \tag{8}$$

which are related to the $e_k(x)$ by

$$e_k(x) = x^k a_k(x^{-1}).$$

Let

$$H_n(x) = \det[a_{i+j}(x)]_{0\le i,j\le n}. \tag{9}$$

Since, for any Hankel matrix $[h_{i+j}]_{0\le i,j\le n}$, we have

$$\det[x^{i+j}h_{i+j}] = x^{n(n+1)}\det[h_{i+j}]\ , \tag{10}$$

the determinants $\widetilde{H}_n(x)$ and $H_n(x)$ are related by

$$\widetilde{H}_n(x) = x^{n(n+1)}H_n(x^{-1}). \tag{11}$$

Therefore it suffices to evaluate $H_n(x)$. Henceforth we will be working with the polynomials $a_k(x)$ and the determinants $H_n(x)$. We will prove the following evaluation, from which Theorem 1 follows by the relation (11).

THEOREM 2. *Suppose the polynomials $e_k$ and $y_k$ and the $(n+1)\times(n+1)$ Hankel determinant $H_n(x)$ are defined as in* (1), (3), *and* (9), *respectively. Then*

$$H_n(x) = c_n\, y_n(2x),$$

*where $c_n$ is as defined in* (6).

Theorem 2 itself will be proved by establishing that $y = H_n(x)$ satisfies the second order differential equation

$$x^2y'' + (2x+1)y' - n(n+1)y = 0. \tag{12}$$

We remark that the polynomials $a_k(x)$ (and therefore the $e_k(x)$) are closely related to another sequence of polynomials defined by

$$d_k(x) = \sum_{m=0}^{k}(-1)^m\frac{k!}{m!}x^{k-m}.$$

These are the so-called *derangement polynomials* as $d_k(1)$ is the number of derangements of $k$ points. Radoux [17] proved that

$$\det[d_{i+j}(x)]_{0\le i,j\le n} = x^{n(n+1)}\prod_{i=1}^{n} i!^2.$$

An evaluation of shifted Hankel determinants of the derangement polynomials

$$\det[d_{i+j+1}(x)]_{0\le i,j\le n} \quad \text{and} \quad \det[d_{i+j+2}(x)]_{0\le i,j\le n}$$

can be found in [20]. See also [7], [12], [14], and [21]. The polynomials $e_k(x)$ and $d_k(x)$ are trivially related by

$$e_k(x) = \frac{(-1)^k}{k!}x^k d_k(-x^{-1}); \tag{13}$$

however, there does not seem to be a simple analogue of (10) to evaluate $\det[\frac{(-1)^{i+j}}{(i+j)!}h_{i+j}]$ from that of the known evaluation of $\det[h_{i+j}]$.

**2. Preliminaries.** We defined $H_n(x)$ in (9) as the determinant of the $(n+1)\times(n+1)$ Hankel matrix

$$A_n = A_n(x) = [a_{i+j}(x)]_{0\le i,j\le n}.$$

By shifting the column indices of the entries of $A_n$, we define three additional $(n+1)\times(n+1)$ determinants $S_n(x), T_n(x)$, and $U_n(x)$ as follows:

1. $S_n(x)$ is the determinant of the matrix whose first $n$ columns are the same as the columns of $A$, but the subscripts of the $a_i$ that appear in the last column are each shifted up by 1.
2. $T_n(x)$ is the determinant of the matrix whose first $n-1$ columns are the same as the columns of $A$, but the subscripts of the $a_i$ that appear in the last two columns are shifted up by 1.
3. $U_n(x)$ is the determinant of the matrix whose first $n$ columns are the same as the columns of $A$, but the subscripts of the $a_i$ in the last column are shifted up by 2.

For example, when $n=3$, we have

$$H_n(x) = \det\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{bmatrix},\ S_n(x) = \det\begin{bmatrix} a_0 & a_1 & a_2 & a_4 \\ a_1 & a_2 & a_3 & a_5 \\ a_2 & a_3 & a_4 & a_6 \\ a_3 & a_4 & a_5 & a_7 \end{bmatrix},$$

$$T_n(x) = \det\begin{bmatrix} a_0 & a_1 & a_3 & a_4 \\ a_1 & a_2 & a_4 & a_5 \\ a_2 & a_3 & a_5 & a_6 \\ a_3 & a_4 & a_6 & a_7 \end{bmatrix},\ U_n(x) = \det\begin{bmatrix} a_0 & a_1 & a_2 & a_5 \\ a_1 & a_2 & a_3 & a_6 \\ a_2 & a_3 & a_4 & a_7 \\ a_3 & a_4 & a_5 & a_8 \end{bmatrix}.$$

**2.1. The three identities.** The bulk of the work for the derivation of the differential equation (12) for $H_n(x)$ relies on three essential identities, which are characteristic of the method of $\gamma$-operators. The first is a differential equation. The second is a linear equation involving the $a_k$ but no derivatives. The third is a relation between shifted columns of the matrix $A_n = [a_{i+j}]_{0\le i,j\le n}$.

LEMMA 1. *(First Identity (FI)) Suppose $a_k$ is as defined in* (8). *Then*

$$x^2\frac{d}{dx}a_n = (n+1)a_{n+1} - (x+1)a_n. \tag{14}$$

LEMMA 2. *(Second Identity (SI)) Suppose $a_k$ is as defined in* (8). *Then*

$$xa_n - (1+2x+nx)a_{n+1} + (n+2)a_{n+2} = 0. \tag{15}$$

LEMMA 3. *(Third Identity (TI)) Suppose $a_k$ is as defined in* (8). *Then*

$$\sum_{j=0}^{n+2} w_{n,j}(x)a_{i+j}(x) = 0 \tag{16}$$

*for $i = 0,1,\dots,n$, where*

$$w_{n,j} = (-1)^{n-j}\left[\frac{2(n+j+1)!}{(n+1)(2n)!}\binom{n+1}{j}x + \frac{2(n+j)!}{(n+1)(2n)!}\binom{n+1}{j-1}\right]. \tag{17}$$

*Proofs of Lemmas* 1, 2*, and* 3. The generating function $f$ of the sequence $a_k(x)$ is given by

$$f(x,y) = \sum_{n=0}^{\infty} a_n(x) y^n = \frac{e^y}{1-xy}.$$

These identities are easily verified by comparing coefficients.

We will use Lemma 3 in the following form. Let $v_j = [a_j, a_{j+1}, \ldots, a_{j+n}]^T$ . The TI (16) says that the vectors $v_0, v_1, \ldots, v_{n+2}$ are linearly dependent with coefficients $w_{n,j}$. Consider the determinant of the $(n+1) \times (n+1)$ matrix whose first $n$ columns are the columns of $[a_{i+j}]$ and whose last column is the zero vector. Writing the zero vector in the form (16) and expanding the determinant by linearity, we find

$$w_{n,n+2} U_n(x) + w_{n,n+1} S_n(x) + w_{n,n} H_n(x) = 0. \tag{18}$$

From the weights (17), we have that, in particular,

$$\begin{aligned} w_{n,n+2} &= 4(2n+1), \\ w_{n,n+1} &= -2(2n+1)(2x+1), \\ w_{n,n} &= n + 2x + 4nx. \end{aligned} \tag{19}$$

Using (18) and (19), we have the following lemma.

Lemma 4. *Suppose* $a_k$ *is as defined in* (8) *and* $S_n(x)$*,* $T_n(x)$*, and* $U_n(x)$ *are as defined in the beginning of this section. Then*

$$4(2n+1)U_n(x) - 2(2n+1)(2x+1)S_n(x) + (n+2x+4nx)H_n(x) = 0. \tag{20}$$

**2.2. The $\gamma$-operator.** The $\gamma$-operator is a multilinear operator defined on $m$-tuples of matrices.

Definition 1. *Given* $(n+1)\times(n+1)$ *matrices* $A$ *and* $X_1, X_2, \ldots, X_m$ *with* $m \geq 1$, *define* $\gamma_A(\,) = \det(A)$ *and*

$$\gamma_A(X_1, \ldots, X_m) = \\ \partial_{t_1}\partial_{t_2}\cdots\partial_{t_m} \det(A + t_1X_1 + t_2X_2 + \cdots + t_mX_m)|_{t_1=\cdots=t_m=0},$$

*where* $t_1, t_2, \ldots, t_m$ *are variables that do not appear in* $A$ *or* $X_1, X_2, \ldots, X_m$.

The $\gamma$-operators behave well with respect to differentiation; the derivative of a $\gamma$ is a sum of $\gamma$'s.

Proposition 1. *For* $m \leq n$,

$$\frac{d}{dx}\gamma_A(X_1, \ldots, X_m) = \gamma_A\left(\frac{d}{dx}A, X_1, \ldots, X_m\right) \\ + \sum_{j=1}^{m} \gamma_A\left(X_1, \ldots, X_{j-1}, \frac{d}{dx}X_j, X_{j+1}, \ldots, X_m\right).$$

The reader is referred to [6] for the proofs of various properties of $\gamma$-operators. The values of the $\gamma$-operators need not be calculated from scratch for different Hankel determinant evaluations. Let $A_n = [a_{i+j}]_{0 \leq i,j \leq n}$ be a Hankel matrix in the generic symbols $a_k$. Extensive tables of values of $\gamma$-operators on various kinds of matrices as well as a computationally feasible combinatorial interpretation of $\gamma_A(X_1, \ldots, X_m)$ for small $m$ can be found in [6]. As examples,

$$\begin{aligned} \gamma_A([a_{i+j+1}]) &= S_n, \\ \gamma_A([a_{i+j+2}]) &= U_n - T_n, \\ \gamma_A([(i+j)a_{i+j}]) &= n(n+1)H_n, \\ \gamma_A([a_{i+j+1}],[a_{i+j}]) &= nS_n, \\ \gamma_A([a_{i+j+1}],[(i+j)a_{i+j+1}]) &= 2(2n-1)T_n. \end{aligned} \tag{21}$$

**3. The differential equation.** We obtain the differential equation (12) for $H_n(x)$ from a linear system of differential equations for $\frac{d}{dx}H_n(x)$ and $\frac{d}{dx}S_n(x)$ in terms of of $H_n(x)$ and $S_n(x)$. To construct this system, we first obtain $U_n(x)$ and $T_n(x)$ in terms of $H_n(x)$ and $S_n(x)$, and then we compute the derivatives of $H_n(x)$ and $S_n(x)$.

**3.1. Equation from the SI.** Apply $\gamma_A(*)$ to the $(n+1)\times(n+1)$ matrix whose $(i,j)$-th entry is obtained from the SI (15) evaluated at $i+j$, and expand using linearity. Making use of the entries in the $\gamma_A(*)$ computations from Table 2 of [6], we get

$$\begin{aligned} 0 = x(n+1)H_n(x) &- (2x+1)S_n(x) - x(2nS_n(x)) \\ &+ (2nU_n(x) - 2(n-1)T_n(x)) + 2(U_n(x) - T_n(x)). \end{aligned}$$

Simplifying gives the linear relation

$$(n+1)xH_n(x) - (2x+2nx+1)S_n(x) + 2(n+1)U_n(x) - 2nT_n(x) = 0. \tag{22}$$

**3.2. Equation from the TI.** We use (20). Solving the linear system (22) and (20), we obtain $U_n(x)$ and $T_n(x)$ in terms of $H_n(x)$ and $S_n(x)$ as follows:

$$\begin{aligned} 4(2n+1)U_n(x) &= 2(2n+1)(2x+1)S_n(x) - (n+2x+4nx)H_n(x), \\ 4(2n+1)T_n(x) &= 2(2n+1)S_n(x) - (n+1)H_n(x). \end{aligned} \tag{23}$$

We now proceed with the calculation of the derivatives of $H_n(x)$ and $S_n(x)$.

**3.3. The derivative of $H_n(x)$.** From Definition 1, $H_n(x) = \gamma_A(\ )$. Therefore, by Proposition 1,

$$\frac{d}{dx}H_n(x) = \gamma_A\left(\left[\frac{d}{dx}a_{i+j}\right]\right).$$

Let $FI(i+j)$ denote the $(n+1)\times(n+1)$ matrix whose $(i,j)$-th entry is obtained from the FI (14) evaluated at $i+j$ and is expanded using linearity. Using $FI(i+j)$,

$$x^2\frac{d}{dx}H_n(x) = \gamma_A([(i+j)a_{i+j+1}]) + \gamma_A([a_{i+j+1}]) - (x+1)\gamma_A([a_{i+j}]).$$

The values for $\gamma_A(*)$ from Table 2 of [6] give

$$x^2\frac{d}{dx}H_n(x) = 2nS_n(x) + S_n(x) - (x+1)(n+1)H_n(x). \tag{24}$$

Therefore

$$x^2\frac{d}{dx}H_n(x) = (2n+1)S_n(x) - (n+1)(x+1)H_n(x). \tag{25}$$

**3.4. The derivative of $S_n(x)$.** To differentiate $S_n(x)$, we use the expression $S_n(x) = \gamma_A([a_{i+j+1}])$ from Table 2 of [6]. From Proposition 1 we have

$$\frac{d}{dx}S_n(x) = \gamma_A\left([a_{i+j+1}], \left[\frac{d}{dx}a_{i+j}\right]\right) + \gamma_A\left(\left[\frac{d}{dx}a_{i+j+1}\right]\right).$$

Therefore, to compute $\frac{d}{dx}S_n(x)$,

$$\gamma_A([a_{i+j+1}], FI(i+j)) \quad \text{and} \quad \gamma_A(FI(i+j+1))$$

are needed. Using the entries in Table 3 of [6] for the $\gamma_A([a_{i+j+1}], *)$ computations for the first one of these, we get

$$x^2\gamma_A([a_{i+j+1}], FI(i+j)) = 2(2n-1)T_n(x) + 2T_n(x) - (x+1)nS_n(x), \tag{26}$$

and for the second one, we get

$$\begin{aligned} x^2\gamma_A(FI(i+j+1)) = 2nU_n(x) - 2(n-1)T_n(x) \\ + 2(U_n(x) - T_n(x)) - (x+1)S_n(x) \end{aligned}$$

by using Table 2 of [6]. Adding the two expressions,

$$x^2\frac{d}{dx}S_n(x) = (2n+2)U_n(x) + 2nT_n(x) - (n+1)(x+1)S_n(x). \tag{27}$$

Substituting the expressions for $U_n(x)$ and $T_n(x)$ in terms of $S_n(x)$ and $H_n(x)$ from (23), we obtain the derivative of $S_n(x)$ as

$$(2n+1)x^2\frac{d}{dx}S_n(x) = (2n+1)(n+x+nx)S_n(x) - (n+1)(n+x+2nx)H_n(x). \tag{28}$$

Next we eliminate $S_n(x)$ and $\frac{d}{dx}S_n(x)$ from the linear system of differential equations (25) and (28) to find that

$$y = H_n(x) = \det[a_{i+j}(x)]_{0\le i,j\le n}$$

satisfies the differential equation (12). Next a straightforward application of the Frobenius method gives

$$H_n(x) = H_n(0)\sum_{i=0}^{n}\frac{(n+i)!}{i!(n-i)!}x^i = H_n(0)y_n(2x). \tag{29}$$

**3.5. Evaluation of $H_n(0)$.** The fact that the constant of integration

$$H_n(0) = \det\left[\frac{1}{(i+j)!}\right]_{0\le i,j\le n} \tag{30}$$

evaluates to $c_n$ as defined in (6) is already known. A proof appears in Lavioe [15], and even earlier, Muir [16] refers to an 1893 paper of Segar [18] which gives the evaluation of

$$\det\left[\frac{1}{(i+j+1)!}\right]_{0\le i,j\le n}$$

whose general techniques can most likely evaluate (30) as well. Similar evaluations can be found in [19]. With the identities we have, we can provide an alternate derivation of the evaluation of (30) as follows. Specialize (20), (22), and (25) at $x = 0$ to obtain the linear system

$$\begin{aligned} 4(2n+1)U_n(0) - 2(2n+1)S_n(0) + nH_n(0) &= 0, \\ 2(n+1)U_n(0) - 2nT_n(0) - S_n(0) &= 0, \\ (2n+1)S_n(0) - (n+1)H_n(0) &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} S_n(0) &= \frac{n+1}{2n+1} H_n(0), \\ U_n(0) &= \frac{n+2}{4(2n+1)} H_n(0), \\ T_n(0) &= \frac{n+1}{4(2n+1)} H_n(0). \end{aligned}$$

By a general result on Hankel determinants (see [5], section 3, Proposition 1), the following identity holds for any $x$:

$$H_{n-1}(x)H_{n+1}(x) = H_n(x)U_n(x) + H_n(x)T_n(x) - S_n(x)^2. \tag{31}$$

Substituting the expressions from (31) at $x = 0$ and simplifying gives

$$H_{n-1}(0)H_{n+1}(0) = -\frac{H_n(0)^2}{4(2n+1)^2}.$$

This is a recurrence relation for the quotient $H_{n+1}(0)/H_n(0)$ with initial value $H_1(0)/H_0(0) = -\frac{1}{2}$. Solving this recurrence we obtain

$$H_{n+1}(0) = 2(-1)^{n+1} \frac{(n+1)!^2}{(2n+2)!^2} H_n(0)$$

from which the formula $c_n$ for the determinant (30) follows.

This completes the proof of Theorem 2. In view of (11), Theorem 1 and the expression for $\widetilde{H}_n(x)$ in (7) follow.

**Remarks.** The recursion (4) for the Bessel polynomials together with a change of variable in the expression (7) for $\widetilde{H}_n(x)$ give a linear recursion involving $\widetilde{H}_{n+1}(x)$, $\widetilde{H}_n(x)$, and $\widetilde{H}_{n-1}(x)$.

THEOREM 3. *Suppose* $\widetilde{H}_n(x)$ *is as defined in* (2). *Then*

$$r_n(x)^2 \widetilde{H}_{n+1}(x) + (-1)^n r_n(x) \widetilde{H}_n(x) + \widetilde{H}_{n-1}(x) = 0,$$

*where*

$$r_n(x) = \binom{2n}{n} (2n+1)!\, x^{-(2n+1)}.$$

There are numerous results involving Bessel polynomials and polynomials related to them [10]. These can be used to obtain additional properties of the $\widetilde{H}_n(x)$ such as its asymptotic behavior. As examples, Carlitz [2] showed that the polynomials $f_n(x) = x^n y_{n-1}(\frac{1}{x})$ form a Sheffer sequence and that

$$e^z = \sum_{n=0}^{\infty} y_{n-1}(x) \frac{(2z - xz^2)^n}{2^n n!}.$$

Interestingly, the right-hand side of this expansion for $e^z$ is independent of $x$. Grosswald [9, 10] gives that, for fixed $x \neq 0$ and $n \to \infty$,

$$y_n(x) \sim \frac{(2n)!}{2^n n!} x^n e^{1/x} \ ,$$

and for fixed $n$ and $|x| \to 0$, $y_n(x) \sim e^{\frac{1}{2}n(n+1)x}$. By Theorem 1, results on Bessel polynomials translate directly into properties of the determinant $\widetilde{H}_n(x)$ and vice versa.

A variant of the numbers $c_n$ of (6) appears as the determinant of another Hankel matrix of moments computed by Al-Salam and Carlitz in [1]. Define $C_n$ by

$$\frac{2}{e^x + 1} = \sum_{n=0}^{\infty} \frac{C_n}{2^n} \frac{x^n}{n!}.$$

Thus $C_{2n} = 0$ for $n \geq 1$, and $|C_{2n+1}|$ are the tangent numbers. Then

$$\det \left[ \frac{|C_{i+j}|}{(i+j)!} \right]_{0 \leq i,j \leq n} = 2^{(n+1)^2} \frac{(n+1)!}{(2n+2)!} \prod_{j=1}^{n} \frac{j!^2}{(2j)!^2}.$$

Finally, as the referee points out, the results of the present paper can be used in conjunction with available techniques [3] to find a recurrence for a family of polynomials orthogonal with respect to a measure $d\mu(x,t)$ supported on $t \in [0, \infty)$ whose moments are the partial sums of the exponential series. The coefficients of this recurrence are rational functions of Bessel polynomials.

**Acknowledgments.** I would like to thank the anonymous referee whose insightful comments and suggestions, including the concluding paragraph above, greatly improved the presentation of this paper. I would also like to thank Christian Krattenthaler for [16].

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