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The number of short cycles in Fibonacci cubes

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ABSTRACT

The Fibonacci cube is the subgraph of the hypercube induced by the vertices whose binary string representations do not contain two consecutive 1s. These cubes were presented as an alternative interconnection network. In this paper, we calculate the number of induced paths and cycles of small length in Fibonacci cubes by using the recursive structure of these graphs.

1. Introduction

The n dimensional hypercube (or Boolean cube) Q_n is one of the most famous interconnection model for parallel computing. It consists of 2^n vertices and each vertex is labeled by a unique n-bit binary string. Two vertices are adjacent if and only if their labels differ in exactly one position, that is, their Hamming distance is one. Fibonacci cubes [10], twisted cube [16,8], locally twisted cube [9,17], augmented cube [3] and many other variants were introduced as alternative interconnection networks.

The n dimensional Fibonacci cube Γ_n is a subgraph of Q_n which is obtained by removing vertices in Q_n that have two consecutive 1s in its binary labeling [10]. In literature many interesting properties of Γ_n are obtained. For a brief survey including the results on representations, hamiltonicity, degree sequence and independence number of Γ_n we refer to [12]. The number of induced hypercubes in Γ_n is considered in [13,15]. In [2,14] some domination type invariants of Γ_n are obtained and the irregularity of Γ_n is presented in [1,6].

It is known that counting cycles and paths in arbitrary graphs is a hard problem [7]. For a general bipartite graph with m vertices and girth g (length of a shortest cycle), a search algorithm to count the short cycles is presented in [4]. Time complexity of the algorithm is $\mathcal{O}(m^2\Delta)$ to count g-cycles and $(g+2)$-cycles, and $\mathcal{O}(m^2\Delta^2)$ to count $(g+4)$-cycles, where Δ is the maximum degree of a vertex in the graph. The numbers of different triangles (cycles of length three) and quadrilaterals (cycles of length four) in augmented cubes are obtained in [5]. We remark that for $n \geq 7$ every edge of Γ_n belongs to cycles of every even length [19]. But as far as we know there are no results on the number of induced cycles and paths in Γ_n, which are directly related with the connectivity and fault-tolerant capability of the network. In this paper, we calculate the number of short induced paths and cycles of small length in Γ_n by using the recursive structure of these

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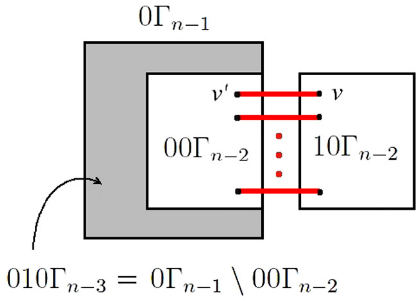

Fig. 1. The fundamental decomposition of Γ_n.

graphs. We present the exact values of induced 4-cycles, 6-cycles and 8-cycles in Γ_n and by considering these graphs as a poset we also count the number of specific type of induced 6-cycles and 8-cycles.

2. Preliminaries

Fibonacci numbers are defined by the recursion $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$, with $f_0 = 0$ and $f_1 = 1$. A subgraph of a graph G having vertex set V and edge set E is a graph H having vertex set contained in V and edge set contained in E. If the edge set of a subgraph of G consists of all edges of G both of whose endpoints lie in G, then it is said to be an induced subgraph of G.

Let B_n denote the set of all binary strings of length n. Then for $n \geq 1$

$$\mathcal{F}_n = \{b_1 b_2 \ldots b_n \in B_n \mid b_i \cdot b_{i+1} = 0,\ 1 \leq i \leq n-1\}$$

is the set of all binary strings of length n that contain no two consecutive 1s, which are called *Fibonacci strings* of length n and it is known that $|\mathcal{F}_n| = f_{n+2}$. For $n \geq 1$ the Fibonacci cube Γ_n has vertex set $\mathcal{F}_n$ and two vertices are adjacent if they differ in exactly one coordinate. Note that $\Gamma_1 = K_2$ and for convenience Γ_0 is assumed to be K_1. Using the properties of Fibonacci strings $\mathcal{F}_n$ we can decompose Γ_n into the subgraphs induced by the vertices that start with 0 and 10 respectively. The vertices that start with 0 constitute a graph isomorphic to Γ_{n-1} and the vertices that start with 10 constitute a graph isomorphic to Γ_{n-2}. This decomposition can be written symbolically as

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}$$

and it is called the *fundamental decomposition* of Γ_n [12]. Note that $0\Gamma_{n-1}$ contains a subgraph isomorphic to Γ_{n-2} which we denote by $00\Gamma_{n-2}$ and there is a perfect matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$. The edges in this perfect matching are called the *link* edges.

In the fundamental decomposition we use primes to denote the mates of the vertices $v \in 10\Gamma_{n-2}$. So for $v = 10\alpha \in 10\Gamma_{n-2}$, $v' = 00\alpha \in 00\Gamma_{n-2} \subset 0\Gamma_{n-1}$ as shown in Fig. 1.

3. Short induced paths in Γ_n

Let $p_k(n)$ denote the number of induced paths of length k in Γ_n. We will refer to such paths as induced k-paths. Clearly $p_1(n)$ is the number of edges of Γ_n. It is given by

$$p_1(n) = \frac{1}{5}\Big(2(n+1)f_n + nf_{n+1}\Big) \tag{1}$$

with generating function

$$\sum_{n\geq 0} p_1(n)t^n = \frac{t}{(1-t-t^2)^2}\,. \tag{2}$$

3.1. Enumerating 2-paths

Next we calculate $p_2(n)$. Since Γ_n is bipartite, every 2-path is induced. Evidently this is given by the expression

$$p_2(n) = \sum_{v\in\Gamma_n} \binom{d_v}{2}\,, \tag{3}$$

where d_v is the degree (number of neighbors) of v in Γ_n.

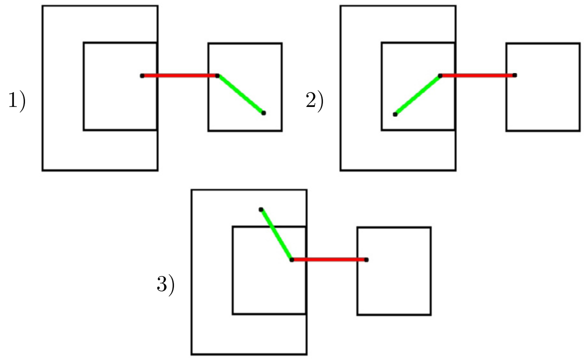

Fig. 2. Possible cases for 2-paths which use link edges.

Proposition 1.

$$p_2(n) = \frac{1}{25}\Big((n-1)(10n+9)f_n - 2nf_{n-1}\Big)\,. \tag{4}$$

Proof. We will obtain $p_2(n)$ by making use of the fundamental decomposition of Γ_n. A 2-path is either completely in $0\Gamma_{n-1}$, or completely in $10\Gamma_{n-2}$, or uses one of the link edges between $00\Gamma_{n-2} \subset 0\Gamma_{n-1}$ and $10\Gamma_{n-2}$. The first two types are enumerated by $p_2(n-1)$ and $p_2(n-2)$, respectively. If one of the edges of the 2-path is a link edge, there are three cases to consider. These can be denoted schematically as in Fig. 2.
Case 1: Such 2-paths are enumerated by

$$\sum_{v\in\Gamma_{n-2}} d_v = 2p_1(n-2)\,,$$

where d_v is the degree of v in Γ_{n-2}.
Case 2: This is similar to Case 1, and the number of such 2-paths is again $2p_1(n-2)$.
Case 3: Any vertex in $0\Gamma_{n-1} \setminus 00\Gamma_{n-2}$ is of the form 010α where α is a Fibonacci string of length $n-3$. Therefore the number of 2-paths of this type is f_{n-1}.

As a consequence we have the recurrence relation

$$p_2(n) = p_2(n-1) + p_2(n-2) + 4p_1(n-2) + f_{n-1} \tag{5}$$

for $n \geq 2$ with $p_2(0) = p_2(1) = 0$. We multiply (5) by t^n and sum for $n \geq 2$. Using the expression (2) for the generating function of the $p_1(n)$ and the generating function of the Fibonacci numbers themselves, we obtain

$$\sum_{n\geq 0} p_2(n)t^n = \frac{t^2(1+3t-t^2)}{(1-t-t^2)^3}\,. \tag{6}$$

The formula (4) can be obtained from this generating function by standard calculations. □

The sequence $p_2(n)$ for $n \geq 1$ starts as

$$0, 1, 6, 17, 46, 108, 242, 515, 1062, 2131, 4188, 8088, \ldots$$

Proposition 1, in conjunction with (1) and the observation (3) yield the following expression for the second moment of the degrees in Γ_n.

Corollary 1.

$$\sum_{v\in\Gamma_n} d_v^2 = 2p_2(n) + 2p_1(n) = \frac{2}{25}\Big((10n^2+14n+1)f_n + 3nf_{n-1}\Big)\,.$$

The sequence of the second moments of the degrees for $n \geq 0$ start as

$$0, 2, 6, 22, 54, 132, 292, 626, 1290, 2594, 5102, 9864, \ldots$$

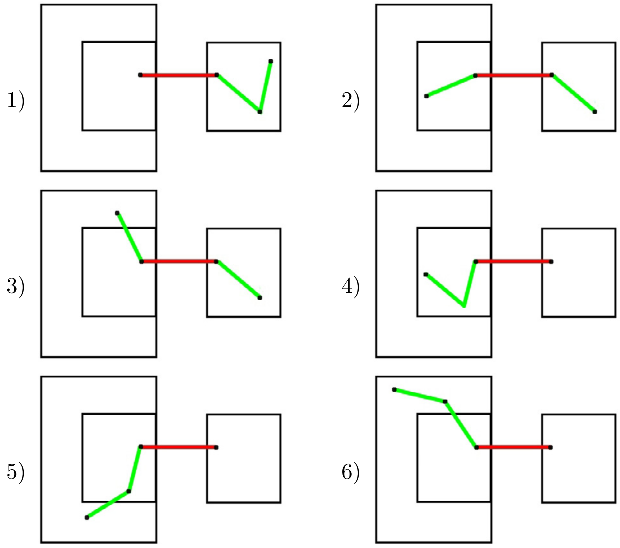

Fig. 3. Possible cases for induced 3-paths that use link edges.

3.2. Enumerating induced 3-paths

In the enumeration of the induced 3-paths in Γ_n we need to keep in mind that these are induced paths, and they cannot form a square in Γ_n.

By making use of the fundamental decomposition of Γ_n, we have the contribution of the induced 3-paths in $0\Gamma_{n-1}$ and the ones in $10\Gamma_{n-2}$ to $p_3(n)$, which are $p_3(n-1)$ and $p_3(n-2)$, respectively. All other induced 3-paths must include a single link edge. We consider the contribution arising from this last possibility, which can be denoted graphically as shown in Fig. 3.

Case 1: Here we make use of the calculation we have already made for 2-paths. Since each extreme vertex of the 2-path in Γ_{n-2} can be connected to its mate by a link edge, the contribution from this case is $2p_2(n-2)$.

Case 2: In this case for any vertex $v \in 10\Gamma_{n-2}$ and its mate $v' \in 00\Gamma_{n-2}$, we can pick a neighbor of v in d_v ways, and then a neighbor of v' for each such selection in $d_v - 1$ ways, avoiding the neighbor picked for v in order not to form a square. Therefore the total number of induced 3-paths contributed in this case is

$$\sum_{v\in\Gamma_{n-2}} d_v(d_v-1) = 2p_2(n-2) + 2p_1(n-2) - 2p_1(n-2) = 2p_2(n-2)$$

by Corollary 1.

Case 3: A vertex $v' = 00\alpha \in 00\Gamma_{n-2}$ has no neighbors in $0\Gamma_{n-1} \setminus 00\Gamma_{n-2}$ if α starts with a 1. If α starts with a 0, then v' has a unique neighbor $01\alpha \in 0\Gamma_{n-1} \setminus 00\Gamma_{n-2}$. Therefore the contribution to the induced 3-paths in this case is

$$\sum_{0\alpha\in\Gamma_{n-2}} d_{0\alpha} \ . \tag{7}$$

We can write

$$\begin{aligned}\sum_{v\in\Gamma_{n-2}} d_v &= \sum_{0\alpha\in\Gamma_{n-2}} d_{0\alpha} + \sum_{1\alpha\in\Gamma_{n-2}} d_{1\alpha}\\ &= \sum_{0\alpha\in\Gamma_{n-2}} d_{0\alpha} + \sum_{10\beta\in\Gamma_{n-2}} d_{10\beta}\\ &= \sum_{0\alpha\in\Gamma_{n-2}} d_{0\alpha} + \sum_{\beta\in\Gamma_{n-4}} d_{\beta} + f_{n-2} \ .\end{aligned}$$

Therefore

$$2p_1(n-2) = \sum_{0\alpha\in\Gamma_{n-2}} d_{0\alpha} + 2p_1(n-4) + f_{n-2}$$

and the contribution from this case is

$$2p_1(n-2) - 2p_1(n-4) - f_{n-2}\ .$$

Case 4: This is the number of 2-paths in $00\Gamma_{n-2}$ with an endpoint selected to link to its mate in $10\Gamma_{n-2}$. Therefore the contribution is $2p_2(n-2)$.
Case 5: In order for a vertex $u' \in 00\Gamma_{n-2}$ to have a neighbor in $0\Gamma_{n-1} \setminus 00\Gamma_{n-2}$, it must be of the form $u' = 000\alpha$, in which case it has exactly one such neighbor. We can then pick a neighbor v' on u' in $00\Gamma_{n-2}$ and connect it to its mate $v \in 10\Gamma_{n-2}$ by a link edge. Thus the contribution in this case is also

$$\sum_{0\alpha \in \Gamma_{n-2}} d_{0\alpha}$$

which is

$$2p_1(n-2) - 2p_1(n-4) - f_{n-2}$$

as found in Case 3.
Case 6: There are $p_1(n-3)$ edges whose both endpoints lie in $0\Gamma_{n-1} \setminus 00\Gamma_{n-2}$. Each endpoint of such an edge is of the form 010α. Each endpoint has a unique neighbor $000\alpha \in 00\Gamma_{n-2}$, which we can then connect by a link edge to its mate in $10\Gamma_{n-2}$. It follows that the number of induced 3-paths contributed in this case is $2p_1(n-3)$.

The total contribution coming from the Cases 1-6 is then

$$4p_1(n-2) + 2p_1(n-3) - 4p_1(n-4) + 6p_2(n-2) - 2f_{n-2}\ .$$

Now from the recursion

$$p_3(n) = p_3(n-1) + p_3(n-2) + 4p_1(n-2) + 2p_1(n-3) - 4p_1(n-4) + 6p_2(n-2) - 2f_{n-2}$$

for $n \geq 2$ we obtain the generating function of the $p_3(n)$ by algebraic manipulations using Mathematica as

$$\frac{2t^3(1+4t+5t^2-4t^3+t^4)}{(1-t-t^2)^4}\ . \tag{8}$$

First few values of $p_3(n)$ for $n \geq 2$ are

$$0, 2, 16, 70, 224, 640, 1648, 3994, 9200, 20414, 43920, 92160, \ldots$$

From the power series expansion of the generating function (8), it is possible to obtain a formula for the coefficient of the term t^n in the expansion. We omit the proof of the following result obtained by using Mathematica.

Proposition 2.

$$p_3(n) = \frac{2}{25}\Big((n-2)(2n^2+n-4)f_n + n(2n^2-9n+6)f_{n-2}\Big)\ . \tag{9}$$

4. Short induced cycles in Γ_n

We denote by $c_k(n)$ the number of induced k-cycles in Γ_n. Since Γ_n is bipartite, this number is zero unless k is even. As special cases, we let $c_0(n)$ and $c_2(n)$ denote the number of vertices and the number of edges of Γ_n. Then $c_0(n) = f_{n+2}$ and $c_2(n) = p_1(n)$ as given in (1). We can also use the alternate expression

$$c_2(n) = \frac{1}{5}\Big(2(2n+1)f_{n-1} + (3n+2)f_{n-2}\Big) \tag{10}$$

with generating function

$$\sum_{n\geq 0} c_2(n)t^n = \frac{t}{(1-t-t^2)^2}\ . \tag{11}$$

The number of induced 4-cycles in Γ_n is a special case of counting the number of hypercubes Q_k in Γ_n for $k=2$ (see, [11,13,15]). We have

$$c_4(n) = \frac{1}{50}\Big((n-2)(5n+1)f_n + 6nf_{n-2}\Big) \tag{12}$$

with generating function

$$\sum_{n\geq 0} c_4(n)t^n = \frac{t^3}{(1-t-t^2)^3}\ . \tag{13}$$

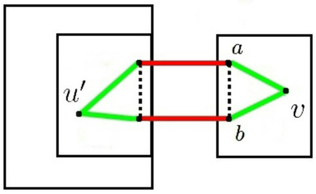

Fig. 4. An induced 6-cycle that uses link edges.

4.1. Enumerating induced 6-cycles

Using the fundamental decomposition of Γ_n, an induced 6-cycle is either completely contained in $0\Gamma_{n-1}$, completely contained in $10\Gamma_{n-2}$, or includes two link edges as shown in Fig. 4. The first two types are counted by $c_6(n-1)$ and $c_6(n-2)$, respectively. It remains to calculate the third type of induced 6-cycles to arrive at a recurrence relation for $c_6(n)$. In Fig. 4, let us call the rightmost vertex v, the leftmost vertex u', and refer to the two neighbors of v in Γ_{n-2} as a (top) and b (bottom). Then the neighbors of u' are a' and b', the mates of a and b, respectively. In the figure the vertical dotted lines indicate that the corresponding edge is not there. We make two observations:

1. $u' \notin 0\Gamma_{n-1} \setminus 00\Gamma_{n-2}$, for the vertices in the set difference have exactly one neighbor in $00\Gamma_{n-2}$.
2. u' is not the vertex v', the mate of v, for otherwise the 6-cycle would not be induced because of the existence of the link edge vv'.

Using these observations we deduce that u, a, v, b is a 4-cycle in Γ_{n-2}. Furthermore, a 4-cycle formed by four vertices u, a, v, b in Γ_{n-2} is responsible for a total of four induced 6-cycles in this way. This is because the pair a, b, a dotted diagonal, contributes two induced 6-cycles, one with using u' and v for the two extreme points, and the other by using u and v'. Similarly, the other dotted diagonal formed by u and v contribute two induced 6-cycles.

Therefore we have the recursion

$$c_6(n) = c_6(n-1) + c_6(n-2) + 4c_4(n-2) \tag{14}$$

for $n \geq 2$ with $c_6(0) = c_6(1) = 0$. We already have the generating function (13). Using the recurrence relation (14) in conjunction with this, followed by generating function manipulations and partial fractions expansion, we obtain the following.

Proposition 3. *For $n \geq 0$, let $c_6(n)$ denote the number of induced 6-cycles in the Fibonacci cube Γ_n. Then*

$$\sum_{n\geq 0} c_6(n)t^n = \frac{4t^5}{(1-t-t^2)^4} \tag{15}$$

and $c_6(n)$ is explicitly given by

$$c_6(n) = \frac{1}{75}n(n-2)\Big((n-7)f_{n+1} + 3(n+1)f_{n-2}\Big)\,. \tag{16}$$

First few values of the sequence of numbers $c_6(n)$ for $n \geq 4$ are

$$0, 4, 16, 56, 160, 420, 1024, 2376, 5296, 11440, 24080, 49608, \ldots$$

It is easy to show by using induction on n and the fundamental decomposition of Γ_n that the 6-cycles in Γ_n are either induced, or have exactly one diagonal edge. This second type constitutes noninduced 6-cycles in Γ_n. These are pairs of 4-cycles in Γ_n sharing an edge. Let us denote by $s_6(n)$ this latter type of 6-cycles in Γ_n.

Proposition 4. *The generating function of $s_6(n)$ is given by*

$$\sum_{n\geq 0} s_6(n)t^n = \frac{2t^4(1+57-t^2)}{(1-t-t^2)^4} \tag{17}$$

with

$$s_6(n) = \frac{1}{50}\Big((n-2)(2n^2-9n-3)f_{n+1} + (n+1)(6n^2-17n+6)f_{n-2}\Big)\,. \tag{18}$$

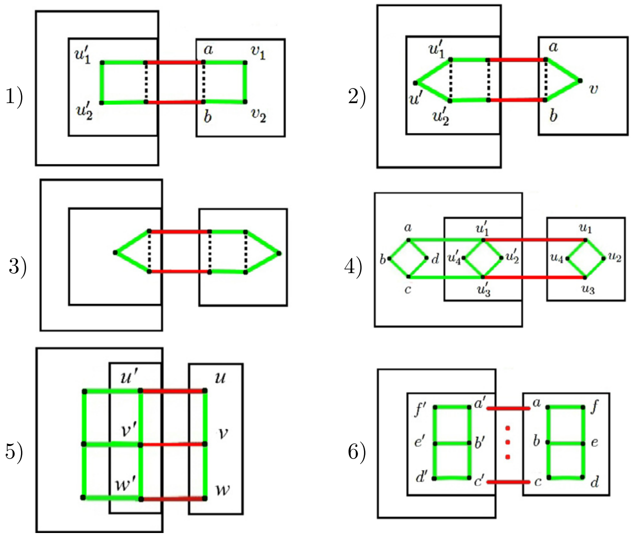

Fig. 5. Possible cases for induced 8-cycles that use link edges.

Proof. First we prove the recurrence relation

$$s_6(n) = s_6(n-1) + s_6(n-2) + 8c_4(n-2) + p_2(n-2) + p_1(n-3) \tag{19}$$

with $s_6(0) = s_6(1) = s_6(2) = 0$. First of all noninduced 6-cycles that are contained completely in $0\Gamma_{n-1}$ and $10\Gamma_{n-2}$ account for the first two terms in (19). Next we look at those that involve link edges. Given any 4-cycle in $10\Gamma_{n-2}$, any edge uv of the 4-cycle together with $u'v'$ gives a noninduced 6-cycle which uses two link edges. This contributes $4c_4(n-2)$ to the count. The symmetric case where the 4-cycle is picked in $00\Gamma_{n-2} \subset 0\Gamma_{n-1}$ contributes another $4c_4(n-2)$. We can also pick an edge in $100\Gamma_{n-3}$, connect the endpoints uv to $u'v'$, and finally connect both u' and v' to the unique edge in $0\Gamma_{n-1} \setminus 00\Gamma_{n-2}$. This last edge is opposite uv in the noninduced 6-cycle formed. This accounts for the term $p_1(n-3)$ in (19). Finally, there are $p_2(n-2)$ noninduced 6-cycles which use three link edges, each corresponding to a 2-path in $00\Gamma_{n-2}$.

Using the generating functions (13), (6) and (2) together with (19) we derive the generating function of the proposition. The formula (18) can be obtained using generating function methods and partial fractions expansion. □

Adding (16) and (18), we have

Corollary 2. *The number of 6-cycles (induced or noninduced) in Γ_n is given by*

$$\frac{1}{150}\Big((n-2)(8n^2-9n-3)f_{n+1} + 3(n+1)(8n^2-21n+6)f_{n-2}\Big) . \tag{20}$$

The sequence of numbers in Corollary 2 is [18, A291915]. The sequence starts for $n \geq 3$ as

$$0, 2, 22, 82, 268, 742, 1902, 4562, 10452, 23068, 49432, \ldots$$

4.2. Enumeration of induced 8-cycles in Γ_n

In the calculation of $c_8(n)$, we again make use of the fundamental decomposition of Γ_n. There are $c_8(n-1)$ induced 8-cycles that are contributed by $0\Gamma_{n-1}$ and $c_8(n-2)$ that are contributed by $10\Gamma_{n-2}$. The remaining induced 8-cycles must involve two of the link edges. The possible cases for this last family are shown in Fig. 5.

Case 1: Since the 8-cycle is induced, there are no additional edges than the ones shown in Fig. 5. Therefore $u_1' \neq v_1'$, $u_2' \neq v_2'$, $u_1' \neq v_2'$, $u_2' \neq v_1'$. This means that the vertices v_1, a, u_1, u_2, b, v_2 form an induced 6-cycle in $10\Gamma_{n-2}$. Furthermore, given any induced 6-cycle in Γ_{n-2}, picking a polar opposite diagonal pair a, b in one of three ways, then picking which side of the 6-cycle is to be in $10\Gamma_{n-2}$ gives a total of six choices for each induced 6-cycle in Γ_{n-2}. Therefore the contribution of this case is $6c_6(n-2)$.

Case 2: The vertices v, a, u_1, u, u_2, b form an induced 6-cycle in $10\Gamma_{n-2}$. Furthermore, any vertex v of an induced 6-cycle and its neighbors a and b on the cycle can be used as the part in $10\Gamma_{n-1}$ of an induced 8-cycle with link edges aa' and bb'. Therefore induced 6-cycles in Γ_{n-2} contribute a total of $6c_6(n-2)$ induced 8-cycles to the count $c_8(n)$.

Case 3: The contribution of this case is identical to the one in Case 2.

Case 4: For any 4-cycle $abcd$ in $010\Gamma_{n-3}$, we know that there are corresponding 4-cycles in $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$ uniquely determined by $abcd$. Then for any 2-path on $abcd$, we obtain 2 different induced 8-cycle in Γ_n. For instance, if we fix the 2-path abc, then the vertices $a, b, c, u_3', u_3, u_2, u_1, u_1'$ and $a, b, c, u_3', u_3, u_4, u_1, u_1'$ form two induced 8-cycles in Γ_n. Therefore the total contribution of this case is $8c_4(n-3)$.
Case 5: We can use 2-paths in $010\Gamma_{n-3}$ in another way. Any 2-path abc in $010\Gamma_{n-3}$ which is not a part of a 4-cycle in $010\Gamma_{n-3}$, determines a unique 2-path $u'v'w'$ in $00\Gamma_{n-2}$ as shown in Fig. 5. The 2-path $u'v'w'$ in turn has its mate uvw in $100\Gamma_{n-3}$ to which it is connected by link edges. The 8 vertices on the outside boundary of this subgraph form an induced 8-cycle. Since abc is not on a 4-cycle, the contribution of this case is $p_2(n-3) - 4c_4(n-3)$.
Case 6: Consider a noninduced 6-cycle formed by the vertices a, b, c, d, e, f in $10\Gamma_{n-2}$. Their mates a', b', c', d', e', f' form a noninduced 6-cycle in $00\Gamma_{n-2}$. In this situation, we obtain eight induced 8-cycles, namely $abcc'd'e'f'a'$, $fedd'c'b'a'f'$, $a'b'c'cdefa$, $f'e'd'dcbaf$, $abcdd'e'f'a'$, $a'b'c'd'defa$, $fedcc'b'a'f'$ and $f'e'd'c'cbaf$. Therefore the total contribution of this case is $8s_6(n-2)$.

Adding up the contributions, we find that $c_8(n)$ satisfies the recurrence relation

$$c_8(n) = c_8(n-1) + c_8(n-2) + 18c_6(n-2) + 4c_4(n-3) + p_2(n-3) + 8s_6(n-2) \tag{21}$$

for $n \geq 2$ with $c_8(3) = c_8(4) = 0$. Using the generating functions (15), (13), (6) and (17) in conjunction with this recurrence relation, we the generating function for the number of induced 8-cycles in Γ_n as given below.

Proposition 5. *The generating function of $c_8(n)$ is given by*

$$\sum_{n\geq 0} c_8(n)t^n = \frac{t^5(1+22t+143t^2-22t^3+t^4)}{(1-t-t^2)^5}\,. \tag{22}$$

A calculation with Mathematica using (22) gives a closed form expression for the number of induced 8-cycles in Γ_n as

$$c_8(n) = \frac{1}{250}\Big((n-2)(100n^2-400n-21)f_{n+1} + (70n^4-360n^3+195n^2+458n-42)f_{n-2}\Big)\,. \tag{23}$$

First few values of the sequence of numbers $c_8(n)$ for $n \geq 4$ are

$$0, 1, 27, 273, 1198, 4371, 13551, 38297, 100578, 250278, 596316, \ldots$$

5. Poset approach to enumerating short induced cycles

Γ_n is a ranked poset in which the covering relation is flipping a 0 to a 1 inherited from the Boolean algebra of all binary strings of length n. The unique minimal element is the all zero string, which has rank 0. The maximal rank is $\lceil n/2 \rceil$. There is a unique maximal element for n odd, and $\frac{n}{2}+1$ maximal elements for n even. For Γ_n the rank of an element is simply its Hamming weight. The number of elements of Γ_n having rank r is denoted by W_r for $0 \leq r \leq \lceil n/2 \rceil$. These are called the rank numbers (or Whitney numbers). For Γ_n, it is known that

$$W_r = \binom{n-r+1}{r}\,. \tag{24}$$

Making use of the poset interpretation of Γ_n provides an alternate approach to the calculation of the number of induced k-cycles. In this approach cycles are classified according to the pattern of the cardinality of their intersection with the ranks. This has the effect of localizing the study of the induced cycles, since the top vertices and the bottom vertices of a cycle in the Hasse diagram cannot be too far apart in terms of the ranks the cycle spans. More precisely, any induced k-cycle C in Γ_n determines an ordered partition of k. Parts of this partition are given by the number of vertices k_i that C contains from each rank i of the Hasse diagram of Γ_n. Let r and r' denote the maximum and minimum rank of the vertices of C. If we disregard the parts that are zeros, we can define the *type* of C by a string $k_r k_{r-1} \cdots k_{r'}$ corresponding to the ordered partition $k = k_r + k_{r-1} + \cdots + k_{r'}$. Since C is an induced cycle and Γ_n is bipartite, the following lemma is immediate.

Lemma 1. *For $k \geq 4$, the type $k_r k_{r-1} \cdots k_{r'}$ of any induced k-cycle satisfies*

1. *$k_i > 1$ for $r' < i < r$,*
2. *$k_r = k_{r-1} = 2$ or $k_{r'+1} = k_{r'} = 2$ is not possible,*
3. *$k_r + k_{r-2} + k_{r-4} \cdots = k_{r-1} + k_{r-3} + k_{r-5} + \cdots$.*

For the two 4-cycles shown by dark lines in Fig. 6, we have $k_3 = 1, k_2 = 2, k_1 = 1$ and $k_2 = 1, k_1 = 2, k_0 = 1$. In fact for $k = 4$, it is easy to show that 121 is the only possible type. In this case the reason is not because of the formation of a 4-cycle as implied by part 2 above, but it follows from a simple calculation.

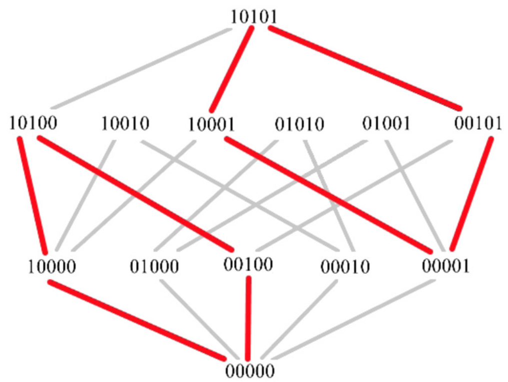

Fig. 6. The Hasse diagram of Γ_n for $n = 5$. The rank numbers $W_r = \binom{n-r+1}{r}$ are $W_0 = 1$, $W_1 = 5$, $W_2 = 6$, $W_3 = 1$.

Table 1
Induced 6-cycles by type in the Fibonacci cube Γ_n, computed by a Mathematica program.

Type	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9
33	0	1	4	14	40	105
132	0	0	0	0	0	0
231	0	0	0	0	0	0
1221	0	3	12	42	120	315
Total	**0**	**4**	**16**	**56**	**160**	**420**

Table 2
Induced 8-cycles by type in the Fibonacci cube Γ_n, computed by a Mathematica program.

Type	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9
44	0	1	6	27	94	291
143	0	0	6	36	144	480
242	0	0	3	42	192	720
341	0	0	0	12	60	240
1232	0	0	0	24	120	480
1331	0	0	0	0	0	0
2222	0	0	0	0	0	0
2321	0	0	12	72	288	960
12221	0	0	0	60	300	1200
Total	**0**	**1**	**27**	**273**	**1198**	**4371**

For $k = 6$ the possible types are 33, 132, 231, 1221; and for $k = 8$, the possible types are 44, 143, 242, 341, 1232, 1331, 2321, 12221.

If we denote by $c_6^{33}(n)$ the number of induced 6-cycles in Γ_n of type 33, and use a similar notation for the other possible types of induced 6-cycles, we have the decomposition

$$c_6(n) = c_6^{33}(n) + c_6^{132}(n) + c_6^{231}(n) + c_6^{1221}(n)$$

and similarly

$$c_8(n) = c_8^{44}(n) + c_8^{143}(n) + c_8^{242}(n) + c_8^{341}(n) + c_8^{1232}(n) + c_8^{1331}(n) + c_8^{2321}(n) + c_8^{12221}(n)\,.$$

These numbers are shown in Table 1 and 2 for $4 \le n \le 9$.

If we use this approach to count induced 6-cycles in Γ_n this way, we note that any 33 type induced 6-cycle must have three vertices of the form $w_0 1 w_1 1 w_2 0 w_3$, $w_0 1 w_1 0 w_2 1 w_3$, $w_0 0 w_1 1 w_2 1 w_3$ of some rank $r-1$, together with $w_0 1 w_1 0 w_2 0 w_3$, $w_0 0 w_1 1 w_2 0 w_3$, $w_0 0 w_1 0 w_2 1 w_3$ of rank $r-2$. These induced 6-cycles are in one to one correspondence with a selection of three 1s from a Fibonacci word of rank r. Therefore by (24), the number of induced 6-cycles of type 33 in Γ_n is then

$$\sum_{r=3}^{n} \binom{n-r+1}{r}\binom{r}{3}\,. \tag{25}$$

Fig. 7. The three induced 6-cycles of type 1221.

Similarly, induced 6-cycles of type 1221 arise from the selection of three 1s from a Fibonacci word of rank r. For each such selection, the replacement of these 1s by 0s all possible ways is isomorphic to a 3-dimensional hypercube, and each of these contribute three induced 6-cycles as shown in Fig. 7.

Therefore the number of induced 6-cycles of type 1221 in Γ_n is three times the sum in (25). It can be shown that Γ_n does not have induced 6-cycles of types 132 or 231. Therefore it remains to evaluate (25) to find $c_6(n)$.

Let

$$a_n(x) = \sum_{r=0}^{n} \binom{n-r+1}{r} x^r .$$

Then

$$A(x,t) = \sum_{n\geq 0} a_n(x) t^n = \frac{1+xt}{1-t-xt^2} . \tag{26}$$

This means that the generating function of the sequence

$$\sum_{r=3}^{n} \binom{n-r+1}{r}\binom{r}{3}$$

can be obtained by evaluating at $x=1$ the expression

$$\frac{1}{6}\frac{\partial^3 A(x,t)}{\partial x^3} \tag{27}$$

where $A(x,t)$ is as in (26). The expression in (27) is found to be

$$\frac{t^5}{(1-t-xt^2)^4} .$$

Therefore with this approach, we find the generating function of the number of induced 6-cycles in Γ_n as

$$\frac{4t^5}{(1-t-t^2)^4} .$$

The generating functions for the number of 1-cycles (vertices), 2-cycles (edges), 4-cycles, induced 6-cycles and induced 8-cycles in Γ_n are respectively

$$\begin{aligned} &\frac{1+t}{1-t-t^2} ,\\ &\frac{t}{(1-t-t^2)^2} ,\\ &\frac{t^3}{(1-t-t^2)^3} ,\\ &\frac{4t^5}{(1-t-t^2)^4} ,\\ &\frac{t^5(1+22t+143t^2-22t^3+t^4)}{(1-t-t^2)^5} . \end{aligned} \tag{28}$$

Here the denominators give a discernable pattern as the denominators of the generating functions of multiple convolutions of Fibonacci numbers.

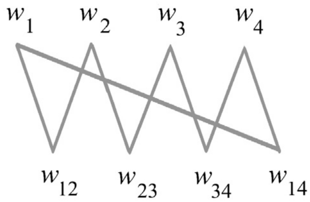

Fig. 8. An induced 8-cycle of type 44 in Γ_n.

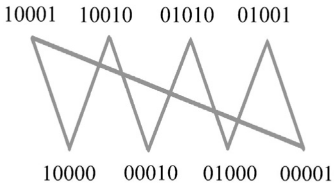

Fig. 9. Induced 8-cycle of type 44 in Γ_n having small ranked vertices.

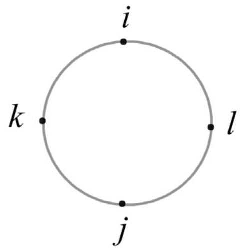

Fig. 10. Four Fibonacci words of rank $r+1$ obtained by replacing the 0 in the positions i, j, k, l to a 1 in a specific Fibonacci word of rank r.

5.1. The calculation of a specific type of induced 8-cycles in Γ_n

Consider a Fibonacci string w of length n and weight $r \geq 4$. Then we necessarily have $n \geq 7$. Pick four 1s in w and assume that these are in positions i_1, i_2, i_3, i_4 in w. Let w_j be Fibonacci word of rank $r-1$ obtained by setting the 1 in position i_j of w to 0 for $1 \leq j \leq 4$. Then two words w_s and w_t meet in rank $r-2$ at the word obtained from w by setting the bits in i_s and i_t to 0. Let us denote this word by w_{st}. The words in Fig. 8 then form an induced 8-cycle of type 44 in Γ_n in ranks $r-1$ and $r-2$. We easily calculate that there are 3 different induced 8-cycles arising from a fixed selection i_1, i_2, i_3, i_4. Therefore the total number of induced 8-cycles of this type is obtained by summing the contributions from each rank, giving

$$\sum_{r\geq 0} 3\binom{n-r+1}{r}\binom{r}{4}.$$

The generating function of this sequence is easily found (by Mathematica) as

$$\frac{3t^7}{(1-t-t^2)^5}. \tag{29}$$

However not all induced 8-cycles are obtained this way. For example the induced 8-cycle in Fig. 9 does not come from the positions of four 1s of a Fibonacci word of rank 3.

This cycle is constructed from the all 0 Fibonacci word in Γ_5 by using the positions 1, 2, 4, 5 for 1s. In this case the induced 8-cycle of type 44 in ranks $r+1$ and $r+2$ is constructed from a Fibonacci word of rank r (for this example $r=0$) by making use of the positions that are 0s in rank r, rather than 1s. To be more precise, this category of induced 8-cycles of type 44 is constructed from a Fibonacci word of rank r and four 0s in positions say i, j, k, l, such that each 0 can be flipped to a 1 and still remain a Fibonacci string over all possible r.

The situation can be succinctly represented by the Fig. 10 which represents the four Fibonacci words of rank $r+1$ obtained by replacing the 0 in each position by 1.

The actual induced 8-cycle of type 44 in the ranks $r+1$ and $r+2$ is then represented by Fig. 11.

There are a number of conditions that the four indices of 0s need to satisfy: none of the pairs il, jl, jk, ik be a pair of consecutive numbers. Since we are looking at undirected cycles, we can also assume that i is the smallest of the four

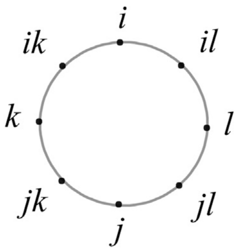

Fig. 11. Completion of four Fibonacci words of rank $r+1$ to an induced 8-cycle of type 44.

indices, and $k<l$. The relative orders of these four indices can then be represented in 3 distinct cases the letter x indicating that position cannot be one of the indices picked.

1. $i \ldots jx \ldots k \ldots l$
2. $ix \ldots k \ldots l \ldots xj$
3. $ix \ldots k \ldots xjx \ldots l.$

Consider the generating function $F(t)$ of Fibonacci strings by length, including the empty word of length 0. Then

$$F = F(t) = \frac{1+t}{1-t-t^2} \, .$$

The generating function of the first case above is the product of $1+tF$, $1+t+t^2F$, $t+t^2F$, $1+t+t^2F$ and $1+tF$; corresponding to the parts of the word before i, between i and j, between j and k, between k and l, and following l, respectively. The generating function of this case is then

$$t^5(1+tF)^3(1+t+t^2F)^2 \, . \tag{30}$$

By a similar argument, the generating functions of the cases 2. and 3. are found to be

$$t^6(1+tF)^4(1+t+t^2F) \text{ and } t^7(1+tF)^5 \, , \tag{31}$$

respectively. Adding (29), (30), (31), we obtain the generating function of the number of induced 8-cycles of type 44 in Γ_n as

$$\frac{t^5(1+t+2t^2-t^3+t^4)}{(1-t-t^2)^5} = t^5 + 6t^6 + 27t^7 + 94t^8 + 291t^9 + 816t^{10} + 2141t^{11} + \cdots \tag{32}$$

From the expansion of the generating function (32), the explicit formula for the number of induced 8-cycles of type 44 in Γ_n is found to be

$$\frac{1}{500}\Big(6(2-n)f_{n+1} + (12-88n+105n^2-40n^3+5n^4)f_{n-2}\Big) \, . \tag{33}$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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