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The Monomial Symmetric Functions and the Frobenius Map

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There is a well known isometry between the center $Z(S_n)$ of the group algebra of the symmetric group S_n and the space of homogeneous symmetric functions H^n of degree n. This isometry is defined via the Frobenius map $\mathbf{F}: Z(S_n) \to H^n$, where

$$\mathbf{F}(f) = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) \psi_{\lambda(\sigma)}.$$

Let $M^\lambda = \mathbf{F}^{-1}(m_\lambda)$ be the preimage of the monomial symmetric function m_λ under $\mathbf{F}$. We give an interpretation of M^λ in terms of certain combinatorial structures called λ-domino tabloids. Using this interpretation, a number of properties of M^λ can be derived. The combinatorial interpretation of the preimage of the so called forgotten basis of Doubilet and Rota can also be obtained by similar techniques. © 1990 Academic Press, Inc.

0. INTRODUCTION

It is well known that there is an isometry between the center $Z(S_n)$ of the group algebra of the symmetric group S_n and the space of homogeneous symmetric functions H^n of degree n. Indeed, this isometry forms the basis of an important interplay between the combinatorics of symmetric functions and tableaux and the representation theory of finite groups, which has been so successfully developed in recent years.

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There are five bases of H^n which are normally considered, namely

$\{m_\lambda\}_{\lambda \vdash n}$ (the monomial symmetric functions),

$\{h_\lambda\}_{\lambda \vdash n}$ (the homogeneous symmetric functions),

$\{e_\lambda\}_{\lambda \vdash n}$ (the elementary symmetric functions),

$\{\psi_\lambda\}_{\lambda \vdash n}$ (the power symmetric functions),

$\{S_\lambda\}_{\lambda \vdash n}$ (the Schur symmetric functions).

From the point of view of the symmetric functions, the most natural basis of H^n is the monomial symmetric functions. However it is the one basis for which the corresponding Frobenius preimage in $Z(S_n)$ is not known. Recall that the Frobenius map $\mathbf{F}: Z(S_n) \to H^n$ is defined by

$$\mathbf{F}(f) = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) \psi_{\lambda(\sigma)}, \tag{0.0}$$

where $\lambda(\sigma)$ denotes the partition of n induced by the lengths of the cycles in the cycle decomposition of σ. Then

$$\mathbf{F}^{-1}(\psi_\lambda) = \lambda?\, C^\lambda, \tag{0.1}$$

$$\mathbf{F}^{-1}(h_\lambda) = p^\lambda, \tag{0.2}$$

$$\mathbf{F}^{-1}(e_\lambda) = n^\lambda, \tag{0.3}$$

$$\mathbf{F}^{-1}(S_\lambda) = \chi^\lambda, \tag{0.4}$$

are all known. Here C^λ denotes the function which is 1 on the conjugacy class corresponding to λ and 0 otherwise, and $\lambda?$ is a constant which will be defined shortly. p^λ and n^λ are well known idempotents of $Z(S_n)$ first defined by A. Young [9], and χ^λ is the character of the irreducible representation of S_n corresponding to the partition λ. We shall give precise definitions of all the bases of H^n and the elements of $Z(S_n)$ mentioned above in Section 1.

The main purpose of this paper is to give a combinatorial interpretation of $M^\lambda = \mathbf{F}^{-1}(m_\lambda)$ for every partition λ. Before we can state our results we need to establish some notation. Given a partition $\lambda = (0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k)$ of n, we let $k(\lambda) = k$ denote the number of parts of λ. The Ferrers' diagram F_λ of λ is the diagram which consists of left justified rows of squares or cells of lengths $\lambda_1, \ldots, \lambda_k$ reading from top to bottom. For example, see Fig. 0.1.

Next we introduce a new set of combinatorial objects called λ-domino

$F_{(1,2,3,3)} =$

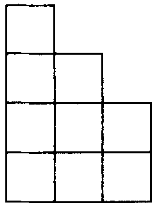


FIG. 0.1

tabloids (or λ-brick tabloids) which can be used to give combinatorial interpretation of the entries for several of the transition matrices between bases of H^n; see [3].

Given partitions $\lambda = (0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k)$ and μ, a *λ-domino tabloid T of shape μ* is a filling of F_μ with dominos $d_1, ..., d_k$ of lengths $\lambda_1, ..., \lambda_k$, respectively, such that

(i) each domino d_i covers exactly λ_i squares of F_μ all of which lie in a single row of F_μ, and

(ii) no two dominos overlap.

For example, if $\lambda = (1, 2, 2, 3)$ and $\mu = (3, 5)$, then we must cover F_μ with the dominos shown in Fig. 0.2.

Here, dominos of the same size are indistinguishable. Then there are seven λ-domino tabloids of shape μ, as shown in Fig. 0.3.

Next we define a weight $w(T)$ for each λ-domino tabloid. Given a domino d in T, let $|d|$ denote the length of d. First, define the T-weight of d, $w_T(d)$, by

$$w_T(d) = \begin{cases} 1 & \text{if } d \text{ is not at the end of a row in } T, \\ |d| & \text{if } d \text{ is at the end of a row in } T. \end{cases} \tag{0.5}$$

Then $w(T)$ is defined by

$$w(T) = \prod_{d \in T} w_T(d). \tag{0.6}$$

Thus $w(T)$ is the product of the lengths of the rightmost dominos in T. For example, for our seven $(1, 2, 2, 3)$-domino tabloids of shape $(3, 5)$ given in Fig. 0.3, $w(T_1) = 6$, $w(T_2) = 6$, $w(T_3) = 3$, $w(T_4) = 6$, $w(T_5) = 3$, $w(T_6) = 4$, and $w(T_7) = 2$.

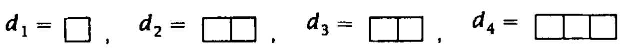


FIG. 0.2

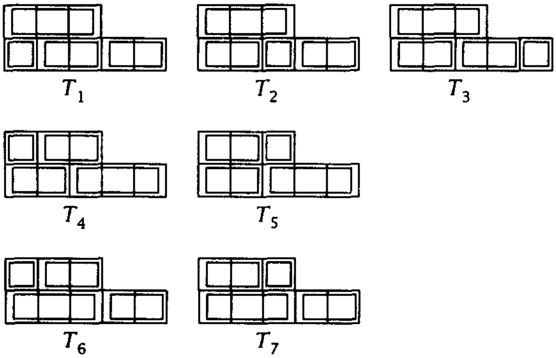


FIG. 0.3

We let $D_{\lambda,\mu}$ denote the set of all λ-domino tabloids of shape μ. This given, we can state our main result.

THEOREM 1. *Let $M^{\lambda}=\mathbf{F}^{-1}(m_{\lambda})$ be the preimage of the monomial symmetric function m_{λ} under the Frobenius map $\mathbf{F}\colon Z(S_n)\to H^n$. Then if $\sigma\in S_n$ and $\lambda(\sigma)=\mu$,*

$$M^{\lambda}(\sigma)=(-1)^{k(\lambda)-k(\mu)}\sum_{T\in D_{\lambda,\mu}} w(T). \tag{0.7}$$

The outline of this paper is as follows. In Section 1, we shall establish some basic notation and briefly review basic facts about the group algebra of S_n and the space of homogeneous symmetric functions H^n. In Section 2, we shall establish the connection between the computation of $M^{\lambda}(\sigma)$ and the transition matrix which transforms $\{\psi_{\lambda}\}_{\lambda\vdash n}$ into $\{h_{\lambda}\}_{\lambda\vdash n}$, and prove our main result.

While in general the scalar product $\langle M^{\lambda}, C^{\mu}\rangle$ is not an integer, we shall also show that this is the case if λ has distinct parts. This will require a second combinatorial interpretation of $\langle M^{\lambda}, C^{\mu}\rangle$. Finally, in Section 3, we show that we can also give an interpretation for the Frobenius preimage of the so called *forgotten* basis $\{f_{\lambda}\}_{\lambda\vdash n}$ of H^n introduced by Doubilet [1].

1. THE SPACES $Z(S_n)$ AND H^n

Let $A(S_n)\to\{f\colon S_n\to\mathbf{C}\}$ denote the group algebra of the symmetric group S_n over the complex numbers $\mathbf{C}$. We shall often write $f\in A(S_n)$ as a formal sum $f=\sum_{\sigma\in S_n} f(\sigma)\sigma$. This given, the sum, product, and scalar product on $A(S_n)$ are defined by

$$f+g=\sum_{\sigma\in S_n}(f(\sigma)+g(\sigma))\sigma,$$

$$f*g=\sum_{\sigma\in S_n}\left(\sum_{\tau\in S_n}f(\tau)\,g(\tau^{-1}\sigma)\right)\sigma,$$

$$\langle f,g\rangle=\frac{1}{n!}\sum_{\sigma\in S_n}f(\sigma)\,\overline{g(\sigma)},$$

where $\overline{g(\sigma)}$ denotes the complex conjugate of $g(\sigma)$. It is easy to show that $f\in A(S_n)$ is in the center of the group algebra of S_n, denoted by $Z(S_n)$, if and only if f is constant on the conjugacy classes. Now given $\sigma\in S_n$, we let $\lambda(\sigma)$ denote the (numerical) partition of n which results from the increasing rearrangement of the cycle lengths of σ. E.g., if $\sigma=(1456)(28)(379)(10\ 11)$, $\lambda(\sigma)=(2,2,3,4)$. It is well known that σ is conjugate to τ in S_n if and only if $\lambda(\sigma)=\lambda(\tau)$ so that the conjugacy classes of S_n are naturally indexed by partitions of n. We write $\lambda\vdash n$ if λ is a partition of n. Given a partition $\lambda=(0<\lambda_1\leqslant\cdots\leqslant\lambda_k)$, we define $\lambda!=\lambda_1!\cdots\lambda_k!$. We shall also write $\lambda=(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})$ to indicate that λ has α_i parts of size i for $i=1,2,...,n$. If $\lambda=(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})$, we let $\lambda?=\alpha_1!\alpha_2!\cdots\alpha_n!1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$. The conjugacy class corresponding to $\lambda\vdash n$ is denoted by C_λ. Thus $C_\lambda=\{\sigma\in S_n\mid\lambda(\sigma)=\lambda\}$. Then it is well known that $|C_\lambda|=n!/\lambda?$. We denote by C^λ, the element of the group algebra corresponding to C_λ. That is,

$$C^\lambda=\sum_{\sigma\in S_n}\chi(\lambda(\sigma)=\lambda)\sigma,\tag{1.1}$$

where for any statement S, the indicator $\chi(S)=1$ if S is true and $\chi(S)=0$ if S is false. Thus $\{C^\lambda\mid\lambda\vdash n\}$ forms a natural basis for $Z(S_n)$.

For completeness, we now give Young's definitions of p^λ, n^λ, and χ^λ mentioned in the introduction.

Let $\lambda=(0<\lambda_1\leqslant\cdots\leqslant\lambda_k)$ be a partition of n. We let λ' denote the conjugate of λ, i.e., λ' is the partition which results by reading the column lengths of F_λ from right to left. A column strict tableau T of shape λ is a filling of F_λ with positive integers in such a way that the numbers weakly increase in each row from left to right and strictly increase in each column from bottom to top. T is called a standard tableau if each $i\in\{1,2,...,n\}$ occurs once and only once in T. For example, T_1 and T_2 in Fig. 1.1 are examples of a column strict tableau and a standard tableau of shape $(1,2,2,3)$, respectively.

Given any subset $\{i_1,...,i_k\}$ of $\{1,...,n\}$, we let

$$[i_1,...,i_k]=\sum_{\sigma\in S_n}\sigma\chi(\sigma(j)=j\text{ if }j\notin\{i_1,...,i_k\})$$

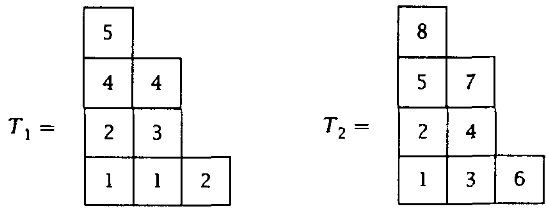


FIG. 1.1

and

$$[i_1, ..., i_k]' = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, \sigma \chi(\sigma(j) = j \text{ if } j \notin \{i_1, ..., i_k\}).$$

Thus $[i_1, ..., i_k]$ is the element of the group algebra which is 1 on those permutations which permute $i_1, ..., i_k$ among themselves, and leave all other elements fixed. The value of $[i_1, ..., i_k]$ is 0 on all permutations which do not fix the complement of $\{i_1, ..., i_k\}$ elementwise. $[i_1, ..., i_k]'$ is defined similarly except that we take the sign of σ instead of 1. Given a standard tableau T of shape λ, we let $[R_i] = [r_1^i, ..., r_{\lambda_i}^i]$, where $r_1^i, ..., r_{\lambda_i}^i$ are the elements in the ith row of T, where we label the rows from top to bottom. $[C_i]' = [c_1^i, ..., c_{t_i}^i]'$, where $c_1^i, ..., c_{t_i}^i$ are the elements in the ith column of T, where we label the columns from left to right. Then we define two elements $P(T)$ and $N(T)$ in the group algebra $A(S_n)$ by

$$P(T) = \frac{1}{\lambda!} [R_1][R_2] \cdots [R_{k(\lambda)}], \tag{1.2}$$

$$N(T) = \frac{1}{\lambda'!} [C_1]' [C_2]' \cdots [C_{k(\lambda')}]'. \tag{1.3}$$

For example, for T_2 pictured above

$$P(T_2) = \frac{1}{2!2!3!} [8][5, 7][2, 4][1, 3, 6],$$

$$N(T_2) = \frac{1}{1!3!4!} [1, 2, 5, 8]' [3, 4, 7]' [6]'.$$

Finally, we define the *hook length* of cell (i, j) in F_λ, denoted by h_{ij}, to be 1 plus the number of cells to the right or above cell (i, j) in F_λ. Put $h_\lambda = \prod_{(i,j) \in F_\lambda} h_{ij}$. For example, the hook lengths of the cells of $F_{(1,2,2,3)}$ are given in Fig. 1.2.

1		
3	1	
4	2	
6	4	1

FIG. 1.2

Thus $h_{(1,2,2,3)} = 6 \cdot 4 \cdot 4 \cdot 3 \cdot 2 = 576$. This given, Young defines one more element of the group algebra based on a standard tableau T of shape λ, namely

$$Y(T) = \frac{\lambda!\lambda'!}{h_\lambda} P(T)\, N(T). \tag{1.4}$$

If we then think of $\sigma \in S_n$ as the element of the group algebra which is 1 at σ and 0 otherwise, we can define the three Young idempotents mentioned in the introduction as follows. Take any standard tableau T of shape λ. Then

$$p^\lambda = \sum_{\sigma \in S_n} \sigma P(T) \sigma^{-1}, \tag{1.5}$$

$$n^{\lambda'} = \sum_{\sigma \in S_n} \sigma N(T) \sigma^{-1}, \tag{1.6}$$

$$\chi^\lambda = \sum_{\sigma \in S_n} \sigma Y(T) \sigma^{-1}. \tag{1.7}$$

Next we consider the space H^n. Let $x_1, ..., x_N$ be any set of variables where $N \geqslant n$. A polynomial $P(x_1, ..., x_N)$ over $\mathbf{C}$ is *symmetric* if and only if

$$P(x_1, ..., x_N) = P(x_{\sigma_1}, ..., x_{\sigma_N}) \tag{1.8}$$

for all $\sigma \in S_N$. We let $H^n = H^n(x_1, ..., x_N)$ denote the space of symmetric polynomials, homogeneous of degree n. The five bases for H^n mentioned in the introduction may be defined as follows. Given a monomial $m = x_1^{i_1} \cdots x_N^{i_N}$ of degree n, we let the type of m, $\tau(m)$, be the partition of n induced by increasing rearrangement of the exponents $(i_1, ..., i_N)$. For example, $T(x_1^2 x_3^3 x_5 x_6^2) = (1, 2, 2, 3)$. Then for any $\lambda \vdash n$, the *monomial symmetric function* m_λ is defined by

$$m_\lambda = \sum_{i_1, ..., i_N} x_1^{i_1} \cdots x_N^{i_N} \chi(\tau(x_1^{i_1} \cdots x_N^{i_N}) = \lambda). \tag{1.9}$$

Since permuting the indices of a monomial preserves its type, it easily follows that $\{m_\lambda\}_{\lambda\vdash n}$ forms a basis for H^n. Thus the dimension of H^n and $Z(S_n)$ are both equal to the number of partitions of n.

Now there are three so called multiplicative bases of H^n. Namely, given $\lambda = (0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k)$, we define the *homogeneous symmetric function* h_λ by

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}, \tag{1.10}$$

where for each integer r,

$$h_r = \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant N} x_{i_1} \cdots x_{i_r}.$$

The *elementary symmetric function* e_λ is defined by

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}, \tag{1.11}$$

where for each integer r,

$$e_r = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant N} x_{i_1} \cdots x_{i_r}.$$

We define the *power symmetric function* ψ_λ by

$$\psi_\lambda = \psi_{\lambda_1} \psi_{\lambda_2} \cdots \psi_{\lambda_k}, \tag{1.12}$$

where for each integer r,

$$\psi_r = \sum_{j=1}^{N} x_j^r.$$

Our final basis is the Schur functions. We can define a Schur function S_λ in two different ways. In terms of determinants we have

$$S_\lambda = \frac{\det \|x_i^{\lambda_j + n - j}\|}{\det \|x_i^{n-j}\|}. \tag{1.13}$$

A more combinatorial definition of S_λ is the following. Given a column strict tableau T with entries from the set $\{1, ..., N\}$, define the monomial weight $m(T)$ of T by $m(T) = x_1^{i_1} \cdots x_N^{i_N}$, where for each j, i_j denotes the number of times the integer j occurs in T. For example, for the column strict tableau T_1 given earlier in Fig. 1.1, $m(T_1) = x_1^2 x_2^2 x_3 x_4^2 x_5$. If $\mathbf{T}_\lambda$ denotes the set of all column strict tableaux T of shape λ with entries from $\{1, ..., N\}$, an alternative definition of S_λ is

$$S_\lambda = \sum_{T \in \mathbf{T}_\lambda} m(T). \tag{1.14}$$

By a vertical bar, we denote the operation of taking coefficients. Thus, for example, $\psi_\lambda|_{h_\mu}$ denotes the coefficient of h_μ in the expansion of ψ_λ in the homogeneous basis.

As mentioned in the introduction, the Frobenius map $\mathbf{F}: Z(S_n) \to H^n$ is defined by

$$\mathbf{F}(f) = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) \psi_{\lambda(\sigma)}. \tag{1.15}$$

Now it is easy to see directly from the definition that

$$F(\lambda ? C^\lambda) = \lambda ? \frac{1}{n!} \psi_\lambda |C_\lambda| = \frac{\lambda ?}{n!} \psi_\lambda \frac{n!}{\lambda ?} = \psi_\lambda,$$

as claimed in (0.1). The other facts stated in (0.2)–(0.4) about $\mathbf{F}$ are considerably more difficult to prove. They were known to many researchers in the field and are certainly implicit in Littlewood [6] and Murnaghan [8]. A complete proof of these facts may be found in [5].

An inner product on H^n, the so called Hall inner product, is defined by declaring $\{h_\lambda\}_{\lambda \vdash n}$ and $\{m_\lambda\}_{\lambda \vdash n}$ to be dual bases. Under this inner product, the Frobenius map $\mathbf{F}: Z(S_n) \to H^n$ is an isometry; see [5] for details. From the fact that F is an isometry, it follows that

$$\langle \psi_\lambda, \psi_\mu \rangle = \lambda ? \chi(\lambda = \mu). \tag{1.16}$$

That is, it is easy to see from our definitions that $\langle \lambda ? C^\lambda, \mu ? C^\mu \rangle = 0$ if $\lambda \neq \mu$ and that

$$\begin{aligned} \langle \lambda ? C^\lambda, \lambda ? C^\lambda \rangle &= \lambda ? \lambda ? \frac{1}{n!} |C_\lambda| \\ &= \lambda ? \lambda ? \frac{1}{n!} \frac{n!}{\lambda ?} = \lambda ?. \end{aligned}$$

Since $\mathbf{F}(\lambda ? C^\lambda) = \psi_\lambda$, we see that (1.16) follows.

2. The Connection Matrices between Monomial, Homogeneous, and Power Symmetric Functions

The main purpose of this section is to give combinatorial interpretations of $\langle M^\lambda, C^\mu \rangle$ and $M^\lambda(\sigma)$, where $\lambda(\sigma) = \mu$. We shall see that these problems reduce to finding the connection matrices between the bases $\{m_\lambda\}_{\lambda \vdash n}$, $\{h_\lambda\}_{\lambda \vdash n}$, and $\{\psi_\lambda\}_{\lambda \vdash n}$. First, we put some total order on the partitions of n; the lexicographic order will do. We then consider the row vectors

$\langle m_\lambda \rangle_{\lambda \vdash n}$, $\langle h_\lambda \rangle_{\lambda \vdash n}$, and $\langle \psi_\lambda \rangle_{\lambda \vdash n}$ and let $A = \|A_{\lambda,\mu}\|$ and $U = \|U_{\lambda,\mu}\|$ be the transition matrices, where

$$\langle \psi_\lambda \rangle_{\lambda \vdash n} = \langle h_\lambda \rangle_{\lambda \vdash n} A, \tag{2.1}$$

$$\langle \psi_\lambda \rangle_{\lambda \vdash n} = \langle m_\lambda \rangle_{\lambda \vdash n} U. \tag{2.2}$$

Because the Frobenius map is an isometry, we have

$$\begin{aligned}\langle m_\lambda, \psi_\mu \rangle &= \langle M^\lambda, \mu?\, C^\mu \rangle = \mu? \langle M^\lambda, C^\mu \rangle \\ &= \frac{\mu?}{n!} \sum_{\sigma \in S_n} M^\lambda(\sigma) \chi(\lambda(\sigma) = \mu) \\ &= \frac{\mu?}{n!} |C_\mu|\, M^\lambda(\sigma) = M^\lambda(\sigma),\end{aligned} \tag{2.3}$$

where $\lambda(\sigma) = \mu$. We can then compute $\langle m_\lambda, \psi_\mu \rangle$ in two different ways. First, we can expand ψ_μ in terms of the homogeneous basis and use the fact that $\langle m_\lambda \rangle_{\lambda \vdash n}$ and $\langle h_\lambda \rangle_{\lambda \vdash n}$ are dual bases. Thus

$$\langle m_\lambda, \psi_\mu \rangle = \left\langle m_\lambda, \sum_{\rho \vdash n} h_\rho A_{\rho,\mu} \right\rangle = A_{\lambda,\mu}. \tag{2.4}$$

Expanding m_λ in terms of the power basis and using (1.16) and (2.3), we obtain

$$\begin{aligned}\langle m_\lambda, \psi_\mu \rangle &= \left\langle \sum_{\rho \vdash n} \psi_\rho U^{-1}_{\rho,\lambda}, \psi_\mu \right\rangle \\ &= U^{-1}_{\mu,\lambda} \langle \psi_\mu, \psi_\mu \rangle \\ &= \mu?\, U^{-1}_{\mu,\lambda} = \langle M^\lambda, \mu?\, C^\mu \rangle.\end{aligned} \tag{2.5}$$

Thus we see that

$$M^\lambda(\sigma) = A_{\lambda,\mu}, \tag{2.6}$$

where $\lambda(\sigma) = \mu$, and

$$\langle M^\lambda, C^\mu \rangle = U^{-1}_{\mu,\lambda} = \frac{A_{\lambda,\mu}}{\mu?}. \tag{2.7}$$

Note that by (2.6), Theorem 1 is equivalent to

THEOREM 2. *Let A be the transition matrix from the homogeneous basis to the power basis in H^n, i.e.,*

$$\langle \psi_\lambda \rangle_{\lambda \vdash n} = \langle h_\lambda \rangle_{\lambda \vdash n} A.$$

Then

$$A_{\lambda,\mu} = (-1)^{k(\lambda)-k(\mu)} \sum_{T \in D_{\lambda,\mu}} w(T).$$

Proof. Our proof of Theorem 2 will require two steps. First we shall define a new weight on λ-domino tabloids. Given a λ-domino tabloid T and a domino $d \in T$, let

$$\bar{w}_T(d) = \begin{cases} -1 & \text{if } d \text{ is not at the end of a row in } T, \\ |d| & \text{if } d \text{ is at the end of a row in } T, \end{cases} \tag{2.8}$$

and put

$$\bar{w}(T) = \prod_{d \in T} \bar{w}_T(d). \tag{2.9}$$

Now if $T \in D_{\lambda,\mu}$, then it is easy to see that

$$\bar{w}(T) = (-1)^{k(\lambda)-k(\mu)} w(T). \tag{2.10}$$

Thus to prove Theorem 2, it suffices to show that $A_{\lambda,\mu} = \bar{A}_{\lambda,\mu}$, where

$$\bar{A}_{\lambda,\mu} = \sum_{T \in D_{\lambda,\mu}} \bar{w}(T). \tag{2.11}$$

Next we observe that our combinatorial definition of $\bar{A}_{\lambda,\mu}$ via (2.11) leads to some very simple recursions for the $\bar{A}_{\lambda,\mu}$. First, given two partitions $\alpha = (\alpha_1, ..., \alpha_p)$ and $\beta = (\beta_1, ..., \beta_q)$, let $\alpha + \beta$ denote the partition which is the increasing rearrangement of $(\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q)$. For example, $(1, 1, 2, 3) + (1, 2, 4, 5, 5) = (1, 1, 1, 2, 2, 3, 4, 5, 5)$. Given a partition λ, we let $\lambda/(j)$ be the partition of $|\lambda| - j$ that results by removing a part of size j from λ if λ has a part of size j. $\lambda/(j)$ is left undefined if λ has no part of size j. We make the convention in this latter case that $\bar{A}_{\lambda/(j),\mu} = 0$ for any partition μ.

LEMMA 3. *Let $\lambda = (\lambda_1, ..., \lambda_k)$ and $\mu = (\mu_1, ..., \mu_l)$ be partitions of n, where $k, l \geqslant 2$. Then*

$$\text{(a)} \quad \bar{A}_{(n),(n)} = n, \tag{2.12}$$

$$\text{(b)} \quad \bar{A}_{\lambda,(n)} = -\sum_{j=1}^{n-1} \bar{A}_{\lambda/(j),(n-j)}, \tag{2.13}$$

$$\text{(c)} \quad \bar{A}_{\lambda,\mu} = \sum_{\substack{\alpha \vdash \mu_1 \\ \beta \vdash \mu_2 + \cdots + \mu_l}} \bar{A}_{\alpha,(\mu_1)} \bar{A}_{\beta,(\mu_2,...,\mu_l)} \chi(\alpha + \beta = \lambda). \tag{2.14}$$

Proof. For (a), note that there is only one domino tabloid T in $D_{(n),(n)}$, namely the domino of size n covers the row of size n, and $\bar{w}(T)=n$. The right hand side of (b) just organizes the sum $\sum_{T\in D_{\lambda,\mu}}\bar{w}(T)$ according to the size of the first domino in $F_{(n)}$. Similarly, the right hand side of (c) organizes the sum $\sum_{T\in D_{\lambda,\mu}}\bar{w}(T)$ according to the dominos that lie in the top row of F_μ. ∎

Next we shall show that the coefficient $A_{\lambda,\mu}$ defined by (2.1) satisfies the same recursions as those given in Lemma 3. To prove this fact, we shall use the following well known relations between ψ_n and $\{h_k\}_{k=0,1,\ldots,n}$.

LEMMA 4. *We have*

$$nh_n=\sum_{k=1}^{n}\psi_k h_{n-k}. \tag{2.15}$$

Proof. We shall give a simple combinatorial proof of (2.15) which first appeared in the first author's thesis [2]. We define a *marked column strict tableau* M of shape (n) as a column strict tableau where one of the cells of $F_{(n)}$ is marked. The monomial weight $m(M)$ of M is defined as the monomial weight of M when it is regarded as just a column strict tableau. Since each column strict tableau T of shape (n) gives rise to n marked column strict tableaux of shape (n), it is easy to see that

$$nh_n=\sum_{M\in\mathcal{MT}_{(n)}} m(M), \tag{2.16}$$

where $\mathcal{MT}_{(n)}$ denotes the set of marked column strict tableaux of shape (n). Given $M\in\mathcal{MT}_{(n)}$, we associate to M a pair of column strict tableaux (C_M, T_M), where C_M is the tableau which results from taking the marked cell c of M plus all the cells following c whose entry is identical to the entry in c. T_M is the column strict tableau which is obtained from M by removing the cells of C_M. For example, if we denote the marked cell of M by placing an X over it, then an example of our correspondence is given in Fig. 2.1.

It is then easy to see that this correspondence $M\to(C_M, T_M)$ can be reserved. Thus

$$\sum_{M\in\mathcal{MT}_{(n)}} m(M)=\sum_{(C,T)} m(C)\,m(T), \tag{2.17}$$

where the second summation runs over all pairs (C, T) in which C is a column strict tableau of shape (k) with $1\leqslant k\leqslant n$ and identical entries, and T is a column strict tableau of shape $(n-k)$. It is clear that the right hand side of (2.17) is a combinatorial interpretation of $\sum_{k=1}^{n}\psi_k h_{n-k}$. Thus (2.17) is equivalent to (2.15). ∎

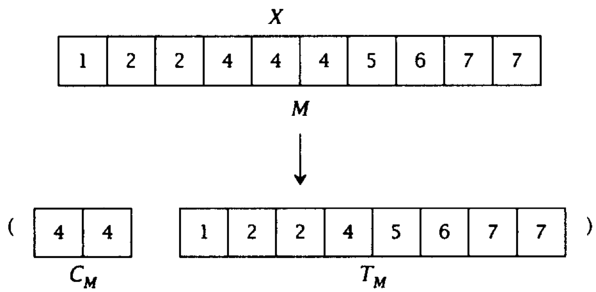


FIG. 2.1

Now (2.15) implies

$$\psi_n = nh_n - \sum_{k=1}^{n-1} \psi_k h_{n-k}. \tag{2.18}$$

We claim that (2.18), together with the fact that $\{\psi_\lambda\}_{\lambda \vdash n}$ and $\{h_\lambda\}_{\lambda \vdash n}$ are multiplicative bases, implies that the $A_{\lambda,\mu}$'s satisfy the same recursions as the $\bar{A}_{\lambda,\mu}$'s. That is, first observe that from (2.18) we have

$$\sum_{\lambda \vdash n} h_\lambda A_{\lambda,(n)} = \psi_n = nh_n - \sum_{k=1}^{n-1} \psi_k h_{n-k}. \tag{2.19}$$

$$= nh_n - \sum_{k=1}^{n-1} \left(\sum_{\alpha \vdash k} A_{\alpha,(k)} \right) h_{n-k}$$

$$= nh_n + \sum_{\substack{\lambda \vdash n \\ k(\lambda) \geqslant 2}} h_\lambda \left(\sum_{j=1}^{n-1} - A_{\lambda/(j),(n-j)} \right). \tag{2.20}$$

Then (2.19) shows that for every n,

$$A_{(n),(n)} = n \tag{2.21}$$

and if $k(\lambda) \geqslant 2$, from (2.20) we have

$$A_{\lambda,(n)} = \sum_{j=1}^{n-1} - A_{\lambda/(j),(n-j)}. \tag{2.22}$$

Finally, observe that if $\mu = (\mu_1, ..., \mu_l)$, where $l \geqslant 2$, then

$$\sum_{\lambda \vdash n} h_\lambda A_{\lambda,\mu} = \psi_\mu = \psi_{\mu_1} \psi_{(\mu_2, ..., \mu_l)} \tag{2.23}$$

$$= \left(\sum_{\alpha \vdash \mu_1} A_{\alpha,(\mu_1)} \right) \left(\sum_{\beta \vdash n - \mu_1} A_{\beta,(\mu_2, ..., \mu_l)} h_\beta \right)$$

$$= \sum_{\lambda \vdash n} h_\lambda \left(\sum_{\substack{\alpha \vdash \mu_1 \\ \beta \vdash n - \mu_1}} A_{\alpha,(\mu_1)} A_{\beta,(\mu_2, ..., \mu_l)} \chi(\alpha + \beta = \lambda) \right). \tag{2.24}$$

Thus (2.24) shows that if μ is as above, then

$$A_{\lambda,\mu}=\sum_{\substack{\alpha\vdash\mu_1\\ \beta\vdash n-\mu_1}} A_{\alpha,(\mu_1)}A_{\beta,(\mu_2,\ldots,\mu_l)}\chi(\alpha+\beta=\lambda). \tag{2.25}$$

It is easy to see that (2.21), (2.22), and (2.25) completely determine the $A_{\lambda,\mu}$'s. Now it follows from Lemma 3 that $A_{\lambda,\mu}=\bar{A}_{\lambda,\mu}$ for all λ and μ. ∎

We remark that it is possible to replace the recursive type proof of Theorem 2 by a direct combinatorial proof of the fact that

$$\psi_\mu=\sum_{\lambda\vdash n} h_\lambda \bar{A}_{\lambda,\mu},$$

where $\bar{A}_{\lambda,\mu}$ is defined via (2.11). See [3] for such a proof.

Observe that $A_{\lambda,\mu}\neq 0$ if and only if there is a λ-domino tabloid of shape μ. But this means that λ is a refinement of μ. Conversely, if λ is a refinement of μ, then it is easy to see that there is a λ-domino tabloid of shape μ. Thus we can conclude the following:

COROLLARY 3. *Let λ and μ be partitions of n. If $\sigma\in S_n$ with $\lambda(\sigma)=\mu$ then*

$$M^\lambda(\sigma)=M^\lambda|_{C^\mu}\neq 0$$

if and only if λ is a refinement of μ.

For example, if $\lambda=(1,2,3)$, $M^\lambda|_{C^\mu}\neq 0$ only when μ is one of the partitions (1, 2, 3), (3, 3), (2, 4), (1, 5), (6). Then it is easy to check that the λ-domino tabloids can be listed as in Fig. 2.2. Thus

$$A_{\lambda,(1,2,3)}=6,\ A_{\lambda,(3,3)}=-18,\ A_{\lambda,(2,4)}=-8,\ A_{\lambda,(1,5)}=-5,\ A_{\lambda,(6)}=12$$

and therefore

$$M^{(1,2,3)}=6C^{(1,2,3)}-18C^{(3,3)}-8C^{(2,4)}-5C^{(1,5)}+12C^{(1,2,3)}.$$

For certain class of partitions λ, we can obtain explicit formulas for $M^\lambda(\sigma)$ as a direct consequence of Theorem 2.

COROLLARY 4. *Suppose $M^\lambda=\mathbf{F}^{-1}(m_\lambda)$ and $\sigma\in S_n$ with $\lambda(\sigma)=\mu$.*

(a) *If $\lambda=(1^rs)$ is a hook shape, then*

$$M^\lambda(\sigma)=(-1)^{r+1-k(\mu)}\sum_{i=1}^{k(\mu)}\mu_i\chi(s\leqslant\mu_i);$$

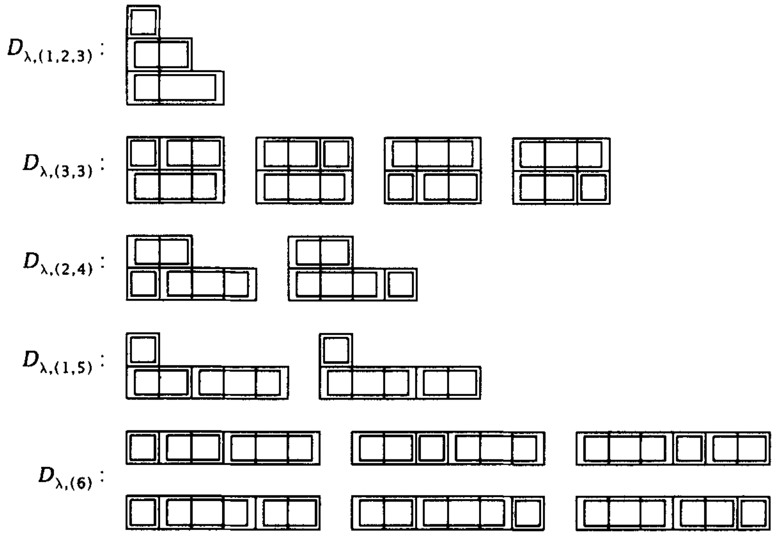


FIG. 2.2

(b) *if* $\lambda = (s^r)$ *then*

$$M^\lambda(\sigma) = (-1)^{r-k(\mu)} s^{k(\mu)} \chi \text{ (} s \text{ divides each } \mu_i\text{)}.$$

Next we turn to the problem of giving a combinatorial interpretation of $\langle M^\lambda, C^\mu \rangle = A_{\lambda,\mu}/\mu?$. The first thing to observe is that $\langle M^\lambda, C^\mu \rangle$ is not always an integer. For example, if $\lambda = (1^n)$, so that all λ-dominos are of size 1, then for each $\mu \vdash n$, there is precisely one domino tabloid T of shape μ and the weight of this domino tabloid T is 1. Thus $\langle M^{(1^n)}, C^\mu \rangle = (-1)^{n-k(\mu)} (1/\mu?)$ is never an integer for $n > 1$. However, we can give another combinatorial interpretation to $\langle M^\lambda, C^\mu \rangle$ which is of interest in its own right and which will show that if λ has distinct parts, then $\langle M^\lambda, C^\mu \rangle$ is always an integer. For this second interpretation, we need another new class of combinatorial objects, which is that of *λ-domino permutations*. Let $\lambda = (0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k)$ be a partition of n. Again we consider dominos $d_1, d_2, ..., d_k$ of sizes $\lambda_1, \lambda_2, ..., \lambda_k$, respectively, except this time we want to distinguish between dominos of the same size. Thus we put subscripts on the dominos. For example, if $\lambda = (1, 1, 2, 3, 3)$, then our set of dominos is as shown in Fig. 2.3. We then let $S(\lambda)$ denote the set of all permutations of these dominos. We shall write a λ-domino permutation in its

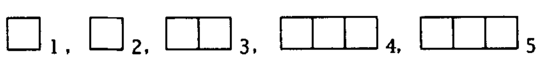


FIG. 2.3

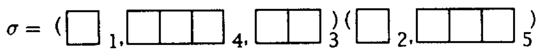


FIG. 2.4

cycle structure. For example, if we have the permutation in Fig. 2.4, then σ corresponds to the λ-domino permutation whose cycle diagram is given in Fig. 2.5. In one line notation, σ would be written as in Fig. 2.6.

Given a cycle $c = (d_{i_1}, ..., d_{i_l})$, we define the *length* of c, $l(c) = l$, the *sign* of c, $\operatorname{sign}(c) = (-1)^{l(c)-1}$, and the *shape* of c, $\operatorname{sh}(c) = \sum_{j=1}^{l} |d_{i_j}|$, where $|d_{i_j}|$ denotes the size of the domino d_{i_j}. For example, if c is the first cycle of the permutation in Figure 2.4, then $l(c) = 3$, $\operatorname{sign}(c) = (-1)^{3-1} = 1$, and $\operatorname{sh}(c) = 6$. Then given a permutation σ which consists of cycles $c_1, ..., c_l$, we have $\operatorname{sign}(\sigma) = \prod_{i=1}^{l} \operatorname{sign}(c_i)$. The *shape* of σ, $\operatorname{sh}(\sigma)$, is defined to be the partition which is the increasing rearrangement of the sequence $\operatorname{sh}(c_1), ..., \operatorname{sh}(c_l)$. Thus for our $(1^2 2 3^2)$-domino permutation σ given in Figure 2.5, $\operatorname{sign}(\sigma) = (-1)^{3-1}(-1)^{2-1} = -1$, and $\operatorname{sh}(\sigma) = (4, 6)$. We let $S(\lambda)^\mu$ denote the set of all λ-domino permutations of shape μ. Note that if $\sigma \in S(\lambda)^\mu$, then $\operatorname{sign}(\sigma) = (-1)^{k(\lambda)-k(\mu)}$. Finally, if $\lambda = (1^{\alpha_1} \cdots n^{\alpha_n})$, we let $\alpha(\lambda)! = \alpha_1! \cdots \alpha_n!$.

THEOREM 5. *Let λ and μ be partitions of n. Then*

$$\langle M^\lambda, C^\mu \rangle = \frac{1}{\alpha(\lambda)!} \sum_{\sigma \in S(\lambda)^\mu} \operatorname{sign}(\sigma) = \frac{(-1)^{k(\lambda)-k(\mu)}}{\alpha(\lambda)!} |S(\lambda)^\mu|.$$

Proof. There are several ways which we could proceed to prove Theorem 5. We shall use the fact that we have already proved that

$$\langle M^\lambda, C^\mu \rangle = \frac{A_{\lambda,\mu}}{\mu?} = \frac{(-1)^{k(\lambda)-k(\mu)}}{\mu?} \sum_{T \in D_{\lambda,\mu}} w(T). \tag{2.26}$$

Since

$$\sum_{\sigma \in S(\lambda)^\mu} \operatorname{sign}(\sigma) = (-1)^{k(\lambda)-k(\mu)} |S(\lambda)^\mu|, \tag{2.27}$$

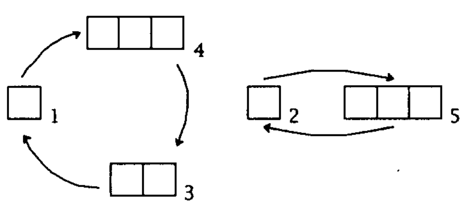


FIG. 2.5

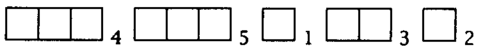


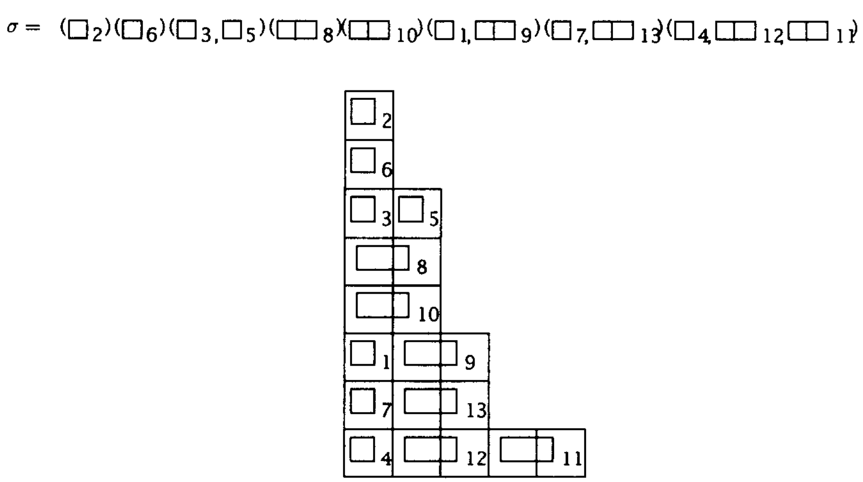
FIG. 2.6

in the light of (2.26) our result will follow if we can show

$$\sum_{T \in D_{\lambda,\mu}} w(T) = \frac{\mu?}{\alpha(\lambda)!} |S(\lambda)^{\mu}|. \tag{2.28}$$

We shall give a combinatorial interpretation of $(\mu?/\alpha(\lambda)!)\,|S(\lambda)^{\mu}|$ which will show that it is equal to the left hand side of (2.28). We start by associating to each $\sigma \in S(\lambda)^{\mu}$ a type of λ-domino tabloid. Given $\sigma \in S(\lambda)^{\mu}$, we arrange the cycles of σ so that the smallest indexed domino comes first in each cycle, and then we order the cycles first by the increasing size of their shapes and then we order cycles of the same shape among themselves by increasing indices of the first domino in each cycle. Once we have arranged the cycles of σ in this way, we then place the ordered cycles into F_{μ} from top to bottom. For example, if $\lambda = (1^7 2^6)$ and $\mu = (1^2 2^3 3^2 5)$, then $\sigma \in S(\lambda)^{\mu}$, pictured in Fig. 2.7, has the proper arrangement of its cycles and corresponds to the filling of F_{μ} below it.

It is easy to see that the type of λ-domino tabloids that arise from $\sigma \in S(\lambda)^{\mu}$ via this process can be characterized as fillings of F_{μ} with indexed λ-dominos satisfying the following four conditions:



Filling of F_{μ} corresponding to σ

FIG. 2.7

(a) each λ-domino lies in a single row of F_μ,

(b) no two dominos overlap,

(c) within each row of F_μ, the domino with the smallest index comes first,

(d) among rows of the same size, the indices of the first dominos in the rows increase from top to bottom.

Next assume that $\mu = (1^{\beta_1} \cdots n^{\beta_n})$ and consider the interpretation of $\mu?\,|S(\lambda)^\mu| = 1^{\beta_1} \cdots n^{\beta_n} \beta_1! \cdots \beta_n!\,|S(\lambda)^\mu|$. Now because β_i is the number of rows of size i in F_μ, multiplying $\beta_i!$ times $|S(\lambda)^\mu|$ has the effect of counting all fillings of F_μ as above except that we are allowed to permute the rows of size i in such a filling among themselves in any manner we choose. It follows that $\beta_1! \cdots \beta_n!\,|S(\lambda)^\mu|$ counts all fillings of F_μ with indexed λ-dominos satisfying conditions (a), (b), and (c) above. We can then interpret $\mu?\,|S(\lambda)^\mu| = 1^{\beta_1} \cdots n^{\beta_n} \beta_1! \cdots \beta_n!\,|S(\lambda)^\mu|$ as all fillings of F_μ with indexed λ-dominos satisfying conditions (a), (b), and (c) where in addition one square in each row of F_μ is marked. We then modify our marked λ-domino tabloid so that instead of marking a square of F_μ, we mark the square of the λ-domino that covers that square and then cyclically rearrange the λ-dominos in each row so that the marked λ-domino is at the end of the row. For example, Fig. 2.8 gives such a marked filling of F_μ which might arise from the domino tabloid in Figure 2.7 on the left and its corresponding modification on the right. Here the marked squares of F_μ are partially shaded.

It is now not difficult to see that the fillings of F_μ with indexed λ-dominos which we obtain via this procedure can be characterized by the following three conditions:

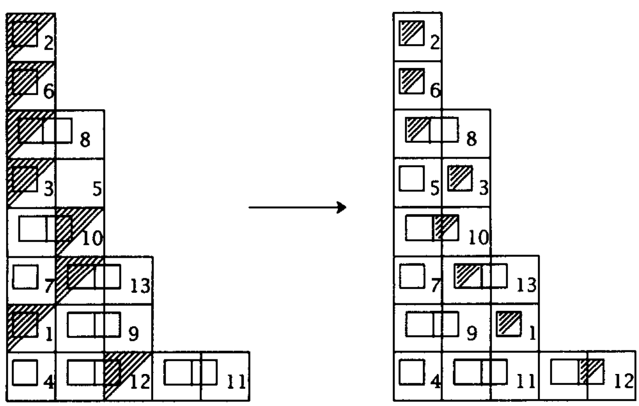


FIG. 2.8

(i) each λ-domino lies in a single row of F_μ,

(ii) no two dominos overlap,

(iii) the last domino in each row of F_μ is marked.

Note that the effect of multiplying $\mu?$ times $|S(\lambda)^\mu|$ has been to remove the conditions on special orders required of the indexed λ-dominos embodied in conditions (c) and (d) at the expense of marking one cell in the last domino in each row.

Next suppose $\lambda = (1^{\alpha_1} \cdots n^{\alpha_n})$. Then the effect of dividing $\mu? |S(\lambda)^\mu|$ by $\alpha_i!$ is to collapse the class of all fillings of F_μ by indexed λ-dominos satisfying (i), (ii), and (iii) which differ by only a permutation of the indices of the λ-dominos of size i into a single element. One way to represent this class is to simply take any such filling of F_μ and erase the indices on the λ-dominos of size i. It thus follows that we can interpret $\mu? |S(\lambda)^\mu|/\alpha(\lambda)!$ as counting the set of all λ-domino tabloids $T \in D_{\lambda,\mu}$, where one cell in the last domino is marked. But then each $T \in D_{\lambda,\mu}$ gives rise to $|d_{i_1}| \cdots |d_{i_l}| = w(T)$ such marked λ-domino tabloids where $d_{i_1}, ..., d_{i_l}$ are the dominos which lie at the end of the rows of T. Thus $\mu? |S(\lambda)^\mu|/\alpha(\lambda)! = \sum_{T \in D_{\lambda,\mu}} w(T)$ as desired. ∎

Note that if λ has distinct parts, $\alpha(\lambda)! = 1$ so that we have

COROLLARY 6. *If λ has distinct parts, then*

$$\langle M^\lambda, C^\mu \rangle = U^{-1}_{\lambda,\mu} = (-1)^{k(\lambda)-k(\mu)} |S(\lambda)^\mu|.$$

As an example of Theorem 5, note that we have computed $A_{\lambda,\mu}$ for $\lambda = (1, 2, 3)$. That is, we found $A_{\lambda,\mu} = 0$ unless μ is one of the partitions (1, 2, 3), (3, 3), (2, 4), (1, 5), (6) in which case $A_{\lambda,(1,2,3)} = 6$, $A_{\lambda,(3,3)} = -18$, $A_{\lambda,(2,4)} = -8$, $A_{\lambda,(1,5)} = -5$, $A_{\lambda,(6)} = 12$. Note that $\alpha(\lambda)! = 1$ in this case so that we know $A_{\lambda,\mu}/\mu? = (-1)^{k(\lambda)-k(\mu)} |S(\lambda)^\mu|$.

Thus from Fig. 2.9

$$\begin{aligned} m_{(1,2,3)} &= \sum_{\mu \vdash 6} A_{(1,2,3),\mu} \psi_\mu \\ &= \psi_{(1,2,3)} - \psi_{(3,3)} - \psi_{(2,4)} - \psi_{(1,5)} + 2\psi_{(6)}. \end{aligned}$$

We end this section with some remarks on how our two interpretations of

$$U^{-1}_{\lambda,\mu} = \frac{A_{\lambda,\mu}}{\mu?} = \frac{(-1)^{k(\lambda)-k(\mu)}}{\alpha(\lambda)!} |S(\lambda)^\mu|$$

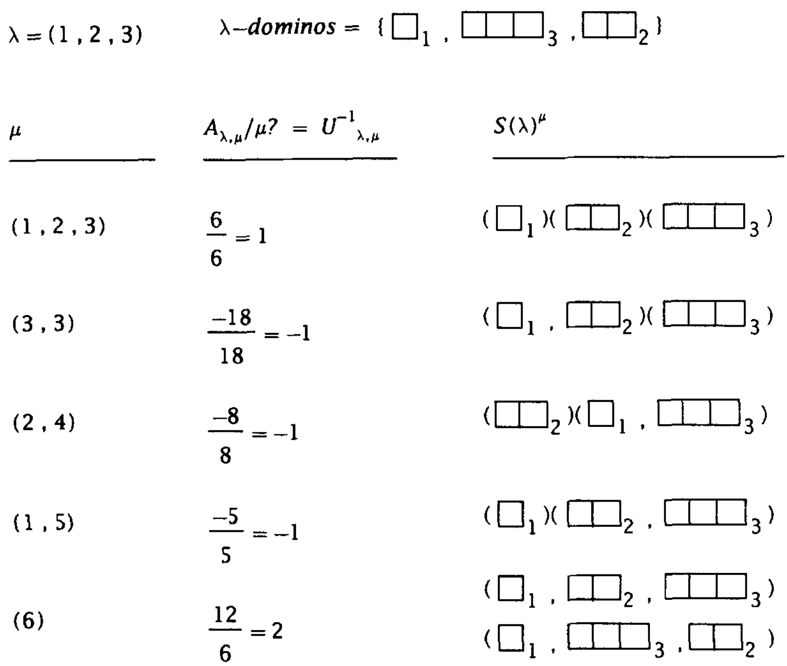


FIG. 2.9

allows us to give combinatorial proofs of the facts that $UU^{-1} = I$ and $U^{-1}U = I$. Now recall that U is the matrix which transforms the monomial basis $\langle m_\lambda \rangle_{\lambda \vdash n}$ into the power symmetric function basis $\langle \psi_\lambda \rangle_{\lambda \vdash n}$. Now in [3], a combinatorial interpretation of $U_{\lambda,\mu}$ was provided. To give this combinatorial definition of $U_{\lambda,\mu}$ here, we need to define the concept of an *ordered λ-domino tabloid of shape μ*. Suppose $\lambda = (0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k)$ and we have a set of dominos $d_1, ..., d_k$, where for each i, $|d_i| = \lambda_i$. Then an ordered λ-domino tabloid T of shape μ is a covering of F_λ with the dominos $d_1, ..., d_k$ in such a way that

(a) a domino d of size i covers i squares of F_λ and these squares must all lie in the same row,

(b) no two dominos overlap,

(c) for any given row of F_λ, the indices of the dominos increase from left to right.

For example, if $\lambda = (1, 1, 2, 3, 3)$, then our set of λ-dominos is as in Fig. 2.3, and there are five ordered λ-dominos of shape (4, 6), shown in Fig. 2.10.

We let $OD_{\lambda,\mu}$ denote the set of all ordered λ-domino tabloids of shape μ. Then the following is proved in [3]:

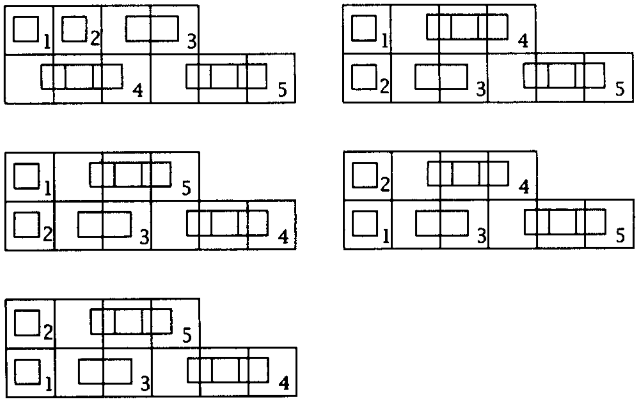


FIG. 2.10

THEOREM 7. *We have*

$$U_{\lambda,\mu} = \psi_\lambda|_{m_\mu} = |OD_{\lambda,\mu}|.$$

In the light of Theorem 7, to give a combinatorial proof of the fact that $I = UU^{-1}$, we must show

$$\chi(\lambda = \mu) = \sum_{\nu \vdash n} U_{\lambda,\nu} U^{-1}_{\nu,\mu}$$

$$= \sum_{\nu \vdash n} |OD_{\lambda,\nu}| \frac{A_{\nu,\mu}}{\mu?}, \tag{2.29}$$

or equivalently,

$$\mu?\chi(\lambda = \mu) = \sum_{\nu \vdash n} \sum_{(T,S) \in OD_{\lambda,\nu} \times D_{\nu,\mu}} \bar{w}(S). \tag{2.30}$$

Now (2.30) suggests that there should be a sign-reversing involution γ on $\bigcup_{\nu \vdash n} OD_{\lambda,\nu} \times D_{\nu,\mu}$ such that the fixed point set of γ is counted by $\mu?\,\chi(\lambda = \mu)$, i.e., that

$$\mu?\,\chi(\lambda = \mu) = \sum_{\nu \vdash n} \sum_{\substack{(T,S) \in OD_{\lambda,\nu} \times D_{\nu,\mu} \\ \gamma(T,S) = (T,S)}} \bar{w}(S). \tag{2.31}$$

Indeed, such a sign-reversing involution is provided by the authors in [3]. Now to give a combinatorial proof that $I = U^{-1}U$, it is advantageous to use the second interpretation of $U^{-1}_{\lambda,\mu}$. That is, we must show that

$$\chi(\lambda=\mu)=\sum_{\nu\vdash n} U^{-1}_{\lambda,\nu} U_{\nu,\mu}$$

$$=\sum_{\nu\vdash n} \frac{(-1)^{k(\lambda)-k(\nu)}}{\alpha(\lambda)!} |S(\lambda)^{\nu}|\ |OD_{\nu,\mu}|, \tag{2.32}$$

or equivalently,

$$\alpha(\lambda)!\ \chi(\lambda=\mu)=\sum_{\nu\vdash n}\ \sum_{(\sigma,S)\in S(\lambda)^{\nu}\times OD_{\nu,\mu}} \operatorname{sign}(\sigma). \tag{2.33}$$

Again (2.33) suggest that there should be a sign-reversing involution on $\bigcup_{\nu\vdash n} S(\lambda)^{\nu}\times OD_{\nu,\mu}$ whose fixed point set is counted by $\alpha(\lambda)!\ \chi(\lambda=\mu)$. Again, such a sign-reversing involution is provided by the authors in [3]. We note that the key to both combinatorial proofs is the fact that we can multiply (2.29) and (2.32) by $\mu?$ and $\alpha(\lambda)!$, respectively, to get rid of any fractions. Note that this is not possible if we use our first interpretation of $U^{-1}_{\lambda,\nu}$ in (2.32), i.e., in

$$\chi(\lambda=\mu)=\sum_{\nu\vdash n} \frac{A_{\lambda,\nu}}{\nu!} |OD_{\nu,\mu}|. \tag{2.34}$$

Thus from the point of view of combinatorial proofs, the matrix identities $UU^{-1}=I$ and $U^{-1}U=I$ cannot be proved by symmetry, and having two distinct combinatorial interpretations for $U^{-1}_{\lambda,\mu}$ is crucial.

3. The Forgotten Basis and Concluding Remarks

In this section, we shall give a combinatorial interpretation of $F^{\lambda}=\mathbf{F}^{-1}(f_{\lambda})$, where $\{f_{\lambda}\}_{\lambda\vdash n}$ is the so called forgotten basis of H^n studied by Doubilet in [1]. To this end, we need to define a fundamental involution θ on $A(S_n)$. Recall that we think of each $\sigma\in S_n$ as an element of the group algebra as $\sigma=\sum_{\tau\in S_n}\chi(\tau=\sigma)\tau$. Thus $\{\sigma\}_{\sigma\in S_n}$ is a basis for $A(S_n)$. We define θ on this basis by

$$\theta(\sigma)=\operatorname{sign}(\sigma)\sigma \tag{3.1}$$

and extend it to all of $A(S_n)$ by linearity. Now by the Frobenius map, we define a corresponding involution $\bar{\theta}$ on H^n by declaring for $p\in H^n$ that

$$\bar{\theta}(p)=\mathbf{F}\circ\theta\circ\mathbf{F}^{-1}(p). \tag{3.2}$$

Note that $\theta(p^\lambda) = n^\lambda$ so that $\bar{\theta}(h_\lambda) = e_\lambda$. $\bar{\theta}$ is the so-called *Hall involution*; see [1]. It is proved in [5] that $\bar{\theta}$ satisfies

$$\bar{\theta}(\psi_\lambda) = (-1)^{n-k(\lambda)} \psi_\lambda, \tag{3.3}$$

$$\bar{\theta}(e_\lambda) = h_\lambda, \quad \bar{\theta}(h_\lambda) = e_\lambda, \tag{3.4}$$

$$\bar{\theta}(S_\lambda) = S_{\lambda'}. \tag{3.5}$$

We can define a sixth basis of H^n, $\{f_\lambda\}_{\lambda \vdash n}$, by

$$f_\lambda = \bar{\theta}(m_\lambda). \tag{3.6}$$

$\{f_\lambda\}_{\lambda \vdash n}$ is called the forgotten basis by MacDonald [7], and was studied by Doubilet [1], where it is given a combinatorial interpretation. An alternate combinatorial definition for f_λ is proved in [3] where it is shown that

$$f_\lambda = (-1)^{n-k(\lambda)} \sum_{\mu \vdash n} |D_{\lambda,\mu}| m_\mu. \tag{3.7}$$

Since for any $f, g \in A(S_n)$

$$\langle f, g \rangle = \langle \theta(f), \theta(g) \rangle, \tag{3.8}$$

it follows that

$$\langle e_\lambda, f_\lambda \rangle = \langle \bar{\theta}(e_\lambda), \bar{\theta}(f_\mu) \rangle = \langle h_\lambda, m_\mu \rangle = \chi(\lambda = \mu). \tag{3.9}$$

In other words, $\{f_\lambda\}_{\lambda \vdash n}$ is the dual basis for $\{e_\lambda\}_{\lambda \vdash n}$.

Of course, in the light of (3.6), we have

$$F^\lambda = \theta(M^\lambda), \tag{3.10}$$

so that we immediately have a combinatorial interpretation for $F^\lambda(\sigma)$. That is, (3.10) and Theorem 1 easily imply

THEOREM 8. *Let $F^\lambda = \mathbf{F}^{-1}(f_\lambda)$ and suppose $\lambda(\sigma) = \mu$. Then*

$$F^\lambda(\sigma) = (-1)^{n-k(\mu)} \sum_{T \in D_{\lambda,\mu}} w(T). \tag{3.11}$$

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