The Isoperimetric Number and The Bisection Width of Generalized Cylinders

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Abstract

A d–dimensional generalized cylinder is the Cartesian product of d graphs each of which is either a path graph or a cycle graph. In this paper, we use a simple embedding technique to find exact formulae for the edge–isoperimetric number and the bisection width of a cylinder in certain cases, e.g. when the size of the largest factor is even.

The isoperimetric number and the bisection width of d–dimensional tori (products of cycle graphs) and arrays (products of path graphs) are thus obtained as a byproduct under the same conditions. We also give description of an isoperimetric set as well as a bisection.

Keywords: Isoperimetric number, bisection width, array, torus, edge separator.

1 Introduction

Given a graph G and a subset X of its vertices, let ∂X denote the *edge–boundary* of X, i.e. the set of edges which connect vertices in X with vertices in $V(G) \setminus X$. The *edge-isoperimetric number*, or simply the *isoperimetric number*, of G is defined as

$$i(G) = \min_{1 \leq |X| \leq \frac{|V(G)|}{2}} \frac{|\partial X|}{|X|} . \qquad (1)$$

That is, the set of vertices of G is partitioned into two nonempty sets and the ratio of the number of edges between the two parts and the number of vertices in the smaller one is minimized. As examples of isoperimetric numbers:

- $i(K_k) = \lceil \frac{k}{2} \rceil$ for the complete graph K_k with k vertices,
- $i(P_k) = 1/\lfloor \frac{k}{2} \rfloor$ for the path graph P_k with k vertices,
- $i(C_k) = 2/\lfloor \frac{k}{2} \rfloor$ for the cycle graph C_k with k vertices.

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A subset X which achieves the minimum ratio in (1) is called an *isoperimetric set.* We refer the reader to Mohar [15] or Chung [9] for a discussion of basic results and various interesting properties of $i(G)$ and to Bezrukov [6] for a comprehensive survey of this and related problems.

In this paper, we are interested in the isoperimetric properties of *generalized cylinders* which are the Cartesian product of cycles and paths. The Cartesian product $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which vertices (u, v) and (u', v') are adjacent if and only if u is adjacent to u' in G and $v = v'$, or v is adjacent to v' in H and $u = u'$. The constituent graphs G and H are called *factors.* A *generalized d-dimensional cylinder*, or simply a *cylinder* is a d-fold product $G^d = G_{k_1} \times G_{k_2} \times \cdots \times G_{k_d}$ where each G_{k_i} is either a path P_{k_i} or a cycle C_{k_i} with k_i vertices. Figure 1 illustrates the 2–dimensional cylinder $P_4 \times C_3$. Note that, the *d–dimensional array* $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ and the *d-dimensional torus* $C^d = C_{k_1} \times C_{k_2} \times \cdots \times C_{k_d}$ are special cases of generalized cylinders.

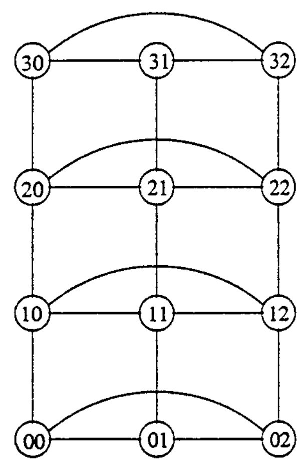

Figure 1: The 2–dimensional cylinder $P_4 \times C_3$.

In this paper, we prove the following theorem by a purely combinatorial method.

Theorem 1 *Given a cylinder* G^d,

- *if* G^d *is a d-dimensional array, i.e.* $G^d = P_{k_1} \times \cdots \times P_{k_d}$ *with* $k_1 \geq k_i$ *and* k_1 *even,*

$$i(G^d) = i(P_{k_1}) = \frac{2}{k_1} \tag{2}$$

- *if* G^d *is a d-dimensional torus, i.e.* $G^d = C_{l_1} \times \cdots \times C_{l_d}$ *with* $l_1 \geq l_i$ *and* l_1 *even,*

$$i(G^d) = i(C_{l_1}) = \frac{4}{l_1} \tag{3}$$

- *if* $G^d = P_{k_1} \times \cdots \times P_{k_r} \times C_{l_1} \times \cdots \times C_{l_s}$ *where* $d = r + s$, $k_1 \geq k_i$, $l_1 \geq l_i$,

$$i(G^d) = \min\{i(P_{k_1}), i(C_{l_1})\} = \begin{cases} \frac{2}{k_1} & \text{if } 2k_1 \geq l_1 \text{ and } k_1 \text{ even,} \\ \frac{4}{l_1} & \text{if } l_1 \geq 2k_1 \text{ and } l_1 \text{ even.} \end{cases} \tag{4}$$

- *The cardinality of the isoperimetric sets of G^d in equations (2),(3) and (4) is $|V(G^d)|/2$.*

We also give a description of these isoperimetric sets.

As a corollary of Theorem 1, we obtain also obtain formulas for the bisection width of G^d. The notion of bisection width and its relationship with the notion of isoperimetric number is explained in the next section. We start with the motivation behind this work.

1.1 Motivation

The notion of the isoperimetric number of a graph G serves as a measure of connectivity of G as it quantifies the minimal interaction between a set of vertices X and its complement $V(G) \setminus X$ in terms of the number of edges between them. This idea is also important in algorithm design. For instance, the notion of isoperimetric number is implicit in the divide-and-conquer strategy in graph algorithms. To illustrate, consider an algorithm which adopts divide-and-conquer strategy where the set of vertices of the underlying graph is split into two balanced parts such that the algorithm can be run on the two corresponding subgraphs recursively, and the results are combined to get the solution for the original problem. The combining of results at the last step needs to be carried out with minimal effort if such a scheme is expected to be efficient. It is desirable to split the graph in such a way as to keep the interaction between the two partitions (in terms of the number of edges connecting them) as small as possible.

The isoperimetric number is also closely related to the notion of the *bisection width*, $bw(G)$, of a graph G. This is the minimum number of edges that must be removed from G in order to split $V(G)$ into two *equal-sized* (within one if $|V(G)|$ is odd) subsets. If known, one can use the isoperimetric number of a graph G to establish a lower bound for its bisection width using the fact that

$$bw(G) \geq i(G)\left\lfloor \frac{|V(G)|}{2} \right\rfloor . \tag{5}$$

In fact, directly from inequality (5) and Theorem 1 we obtain the lower bounds which yield the exact expressions for the bisection width given as Corollary 1.

Corollary 1 *Given a cylinder G^d with n vertices,*

- *if G^d is a d-dimensional array, i.e. $G^d = P_{k_1} \times \cdots \times P_{k_d}$ with $k_1 \geq k_i$ and k_1 even,*

$$bw(G^d) = \frac{n}{k_1}$$

- *if G^d is a d-dimensional torus, i.e. $G^d = C_{l_1} \times \cdots \times C_{l_d}$ with $l_1 \geq l_i$ and l_1 even,*

$$bw(G^d) = 2\frac{n}{l_1}$$

- *if $G^d = P_{k_1} \times \cdots \times P_{k_r} \times C_{l_1} \times \cdots \times C_{l_s}$ where $d = r + s$, $k_1 \geq k_i$, then*

$$bw(G^d) = \begin{cases} \frac{n}{k_1} & \text{if } 2k_1 \geq l_1 \text{ and } k_1 \text{ even,} \\ 2\frac{n}{l_1} & \text{if } l_1 \geq 2k_1 \text{ and } l_1 \text{ even.} \end{cases}$$

As with the case of isoperimetric sets, we also identify an optimal bisection for G^d.

1.2 Outline

The outline of the rest of this paper is as follows. In Section 2 we give a summary of previous work on isoperimetric properties of various families of product graphs. The proof of our main result appears in Section 3. In Section 4 we give the cardinality and the description of the isoperimetric sets of a cylinder, as well as the subsets achieving the bisection width. In Section 5, we give conclusions and future research considerations.

2 A summary of previous work

There has been a significant amount of research in the area of isoperimetric problems in various popular classes of graphs such as arrays and tori. The isoperimetric number and the bisection width of graphs are intimately related to the theory of *extremal sets* in graphs. An extremal set of a graph for a given m is, in a broad sense, a configuration of m vertices with

- minimum number of boundary edges, or
- maximum number of spanned edges

among all such m-vertex subsets of the given graph. Specifically, one can easily obtain the isoperimetric number of a given graph if the extremal sets minimizing the number of boundary edges are known (and the boundary is actually computable). An extremal set X with $\lfloor \frac{|V(G)|}{2} \rfloor$ vertices in a given graph G is a bisection of G.

The problem of finding extremal sets of the first (or, second) type is called *the minimum-boundary-edge problem* (or, *the maximum-induced-edge problem*). It has been shown that the minimum–boundary–edge and the maximum–induced–edge problems are equivalent for regular graphs [8].

The maximum–induced–edge problem (hence the minimum–boundary–edge problem, because of its regularity) for the hypercube (d-dimensional binary Hamming graph) was solved by Harper [11] and extended by Lindsey [14] to the d–dimensional k–ary Hamming graph. In both instances, there is a nested structure of solutions, and the set of the first m vertices in *lexicographic order* constitutes an extremal set. The maximum–induced–edge problem for the d–dimensional k–ary array A_k^d was first solved by Bollobás and Leader [8]. Since A_k^d is not regular, this result does not automatically give a solution to the minimum–boundary–edge problem. It was later extended to general arrays by Ahlswede and Bezrukov [1] who also gave a solution for $P_{k_1} \times P_{k_2}$ for the minimum–boundary–edge problem.

The first nontrivial bounds on the minimum–boundary–edge problem for the d–dimensional k–ary arrays were given by Bollobás and Leader [8]. In fact, the results given in this paper are implied ([12]) by the results obtained by Bollobás and Leader in [8]. The technique used in [8] involves the solution to the continuous analogue of the minimum–boundary–edge problem and, for a subset X, $|X| \leq |V(G^d)|/2$, results in lower bounds of the form

$$|\partial X| \geq \min\{|X|^{1-1/r} r k^{(n/r)-1} : r = 1, \ldots, d\} \tag{6}$$

which are tight for the cases we are interested in. Note that Bollobás and Leader's result is more general than finding the isoperimetric number. However, as we shall show in this paper, if one is merely interested in obtaining an exact formula for the isoperimetric number itself then a direct combinatorial embedding technique suffices.

Finally, we remark that similar problems have been defined in the literature for the vertex-boundary of a given configuration of vertices.

3 The isoperimetric number of a cylinder

We shall focus on proving equation (4) of Theorem 1, since equations (2) and (3) are proved using essentially the same technique. Specifically, we will prove the following.

Proposition 1 *Given a cylinder* $G^d = P_{k_1} \times \cdots \times P_{k_r} \times C_{l_1} \times \cdots \times C_{l_s}$ *with* $d = r + s$; $k_1 \geq k_i$, $l_1 \geq l_i$,

$$i(G^d) = \min\{i(P_{k_1}), i(C_{l_1})\} = \begin{cases} \frac{2}{k_1} & \text{if } 2k_1 \geq l_1 \text{ and } k_1 \text{ even,} \\ \frac{4}{l_1} & \text{if } l_1 \geq 2k_1 \text{ and } l_1 \text{ even.} \end{cases} \tag{7}$$

Proof We prove this by showing that the expressions on the right–hand side are both lower and upper bounds for $i(G^d)$. We use the following well-known result to prove the upper bound.

$$i(G_1 \times G_2 \times \cdots \times G_r) \leq \min\{i(G_1), i(G_2), \ldots, i(G_r)\} \tag{8}$$

See Chung [9] for a proof of (8). Clearly, $i(P_{k_1}) = \min_i\{i(P_{k_i})\}$ and $i(C_{l_1}) = \min_i\{i(C_{l_i})\}$. If $2k_1 \geq l_1$ then $i(P_{k_1}) \leq i(C_{l_1})$ and for even k_1 we have,

$$i(G^d) \leq \frac{2}{k_1} = i(P_{k_1}) = \min\{i(P_{k_1}), \ldots, i(P_{k_r}), i(C_{l_1}), \ldots, i(C_{l_s})\}. \tag{9}$$

Similarly, if $l_1 \geq 2k_1$ and l_1 is even then,

$$i(G^d) \leq \frac{4}{l_1} = i(C_{l_1}) = \min\{i(P_{k_1}), \ldots, i(P_{k_r}), i(C_{l_1}), \ldots, i(C_{l_s})\}, \tag{10}$$

since $i(C_{l_1}) \leq i(P_{k_1})$.

To prove the lower bound, we extend an embedding technique which Leighton used in [13] to obtain a lower bound for the bisection width of arrays. Specifically, we embed a directed complete graph K_n with n vertices into G^d where n is the number of vertices of G^d. Analogous to the case of an undirected graph, for a directed graph G, we define ∂X to be the the number of directed edges which connect a vertex in X with a vertex in $V(G) \setminus X$, or vice versa. Any partition of vertices in K_n induces a partition in G^d as a result of the embedding. Using this, we argue that the number of boundary edges in a partition of vertices in G^d cannot be less than that of K_n divided by the *congestion* of the embedding. The congestion is the maximum number of edges of K_n routed through any edge of G^d.

Before we describe our embedding, consider a nonempty subset X of vertices of G^d with $|X| \leq n/2$. Let X' be the subset of vertices of the complete graph associated with the vertices in X as a result of the embedding. Note that $|\partial X'| = 2|X'|(n - |X'|)$ and $|\partial X| \geq |\partial X'|/c$ where c is the congestion of the embedding. Therefore

$$\frac{|\partial X|}{|X|} \geq \frac{|\partial X'|}{c|X|} \geq \frac{2|X'|(n - |X'|)}{c|X|}$$

Since $|X| = |X'|$ and $1 \leq |X| \leq n/2$,

$$\frac{2|X'|(n - |X'|)}{c|X|} = \frac{2(n - |X'|)}{c} \geq \frac{2\frac{n}{2}}{c} = \frac{n}{c}. \tag{11}$$

Thus we have,

$$i(G^d) \geq \frac{n}{c} \tag{12}$$

The embedding we describe next has congestion c with

$$c = \begin{cases} n\frac{k_1}{2} & \textit{if } 2k_1 \geq l_1 \textit{ and } k_1 \textit{ even}, \\ n\frac{l_1}{4} & \textit{if } l_1 \geq 2k_1 \textit{ and } l_1 \textit{ even}. \end{cases}$$

which, when combined with (12), yields the desired lower bound. □

The details of the embedding are given next.

3.1 Embedding a directed complete graph into a cylinder

Given a generalized cylinder $G^d = P_{k_1} \times \cdots \times P_{k_r} \times C_{l_1} \times \cdots \times C_{l_s}$ where $d = r + s$, for notational convenience let $k_{r+i} := l_i$ for $1 \leq i \leq s$. We identify each vertex v by a label $(v_1, \ldots, v_d)$ where $0 \leq v_i \leq k_i - 1$ for $1 \leq i \leq d$ as shown in Figure 1. Note that there are a total of $n = k_1 k_2 \cdots k_d$ vertices.

We embed into G^d a directed complete graph K_n with n vertices where the vertices are identified with the vertices of G^d. The edge from node $u = (u_1, \ldots, u_d)$ to node $v = (v_1, \ldots, v_d)$ of K_n is embedded by using a left–to–right dimensional routing scheme. That is, first the value u_1 is "corrected" into v_1, then u_2 into v_2, and so on, until all u_i have been corrected.

When correcting u_i into v_i for $1 \leq i \leq r$, (i.e. when the factor in the i^{th} dimension of the cylinder is a path) the correction is done by taking the (unique) shortest path between u_i and v_i. In other words, u_i is incremented (or decremented) until it becomes equal to v_i. If $r+1 \leq i \leq r+s$, since the i^{th} factor is a cycle, there are exactly two shortest paths between u_i and v_i when they are diametrically opposite *and* k_i is even. For instance, take $k_i = 6$, $u_i = 2$ and $v_i = 5$ as shown in Figure 2. The shortest paths between vertices 2 and 5 are 2–3–4–5 and 2–1–0–5. In all other

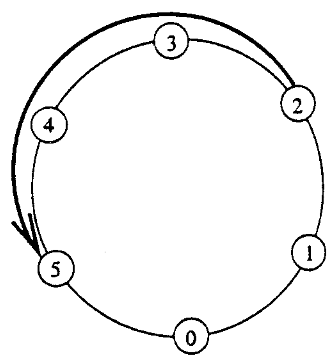

Figure 2: The cycle with 6 vertices and embedding of edge $\langle 2, 5 \rangle$.

cases the shortest path is unique. To avoid ambiguity when there are multiple shortest paths, we always take the one with increasing label values (transition from $k_i - 1$ to 0 is assumed to be increasing, whereas a transition from vertex 0 to $k_i - 1$ is decreasing.) Thus, in our example, the path 2–3–4–5 is taken as shown in Figure 2. We next prove the congestion lemma.

Lemma 1 *Let c be the congestion of the embedding of an n-vertex directed complete graph K_n into a cylinder $G^d = P_{k_1} \times \cdots \times P_{k_r} \times C_{l_1} \times \cdots \times C_{l_s}$ with n vertices, as described above, where*

$d = r + s$, $k_1 \geq k_i$ *and* $l_1 \geq l_i$. *Then we have,*

$$c = \begin{cases} n^{\frac{k_1}{2}} & \text{if } 2k_1 \geq l_1 \text{ and } k_1 \text{ even,} \\ n^{\frac{l_1}{4}} & \text{if } l_1 \geq 2k_1 \text{ and } l_1 \text{ even.} \end{cases}$$

Proof For convenience, we again take $k_{r+i} = l_i$ for $1 \leq i \leq s$. Define c_i for $1 \leq i \leq d$ to be the maximum number of edges of K_n embedded in any edge *in the* i^{th} *dimension of* G^d. It suffices to show, for $1 \leq i \leq r$,

$$c_i = \begin{cases} n^{\frac{k_i}{2}} & \text{if } k_i \text{ is even,} \\ n^{\frac{k_i^2-1}{2k_i}} & \text{if } k_i \text{ is odd.} \end{cases}$$

and for $r + 1 \leq i \leq r + s$,

$$c_i = \begin{cases} n^{\frac{k_i}{4}} & \text{if } k_i \text{ is even,} \\ n^{\frac{k_i^2-1}{4k_i}} & \text{if } k_i \text{ is odd,} \end{cases}$$

since, if these hold, then

$$c = \max_{1 \leq i \leq d}\{c_i\} = \begin{cases} n^{\frac{k_1}{2}} & \text{if } 2k_1 \geq l_1 \text{ and } k_1 \text{ even,} \\ n^{\frac{l_1}{4}} & \text{if } l_1 \geq 2k_1 \text{ and } l_1 \text{ even} \end{cases}$$

as desired.

Let $e = \langle(e_1, \ldots, e_i, \ldots, e_d), (e_1, \ldots, e_i + 1, \ldots, e_d)\rangle$ be an edge of G^d in the i^{th} dimension. Also, let $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ be two distinct nodes of the complete graph for which the embedding results in the directed edge from u to v to be embedded through e. Note that c_i is the maximum number of such (u, v) pairs. Since routing is done in a left-to-right fashion, we must have

$$\begin{array}{ll} u_{i+1} = e_{i+1} \quad , & v_1 = e_1 \\ u_{i+2} = e_{i+2} \quad , & v_2 = e_2 \\ \vdots & \vdots \\ u_d = e_d \quad , & v_{i-1} = e_{i-1} \end{array}$$

along with either $u_i \leq e_i$ and $v_i \geq e_i + 1$ or $v_i \leq e_i$ and $u_i \geq e_i + 1$ depending on the direction of the embedded edge, i.e. $\langle u, v\rangle$ or $\langle v, u\rangle$. This leaves $k_1 \cdots k_{i-1}k_{i+1} \cdots k_d$ choices for $u_1, \ldots, u_{i-1}, v_{i+1}, \ldots, v_d$.

The number of choices for u_i and v_i depends on whether the i^{th} factor of G^d is a path or cycle. If it is a path then u_i and v_i can be chosen one of

$$(e_i + 1)(k_i - (e_i + 1)) + (k_i - (e_i + 1))(e_i + 1) = 2(e_i + 1)(k_i - e_i - 1)$$

different ways. From an analysis of the extreme values of $2(x + 1)(k_i - x - 1)$ on $0 \leq x < k_i$, we have

$$2(e_i + 1)(k_i - e_i - 1) \leq \begin{cases} \frac{k_i^2}{2} & \text{if } k_i \text{ is even,} \\ \frac{k_i^2-1}{2} & \text{if } k_i \text{ is odd} \end{cases}$$

Therefore for $1 \leq i \leq r$,

$$c_i = \max_{e_i}\left\{k_1 \cdots k_{i-1}k_{i+1} \cdots k_d\Big(2(e_i + 1)(k_i - e_i - 1)\Big)\right\} = \begin{cases} n^{\frac{k_i}{2}} & \text{if } k_i \text{ is even,} \\ n^{\frac{k_i^2-1}{2k_i}} & \text{if } k_i \text{ is odd.} \end{cases}$$

as desired. If the i^{th} factor of G^d is a cycle then the number of ways u_i and v_i can be chosen is exactly

$$\frac{\frac{k_i}{2}\left(\frac{k_i}{2}+1\right)}{2}+\frac{\frac{k_i}{2}\left(\frac{k_i}{2}-1\right)}{2}=\frac{k_i^2}{4}$$

when k_i is even, and

$$\frac{\frac{k_i-1}{2}\frac{k_i+1}{2}}{2}+\frac{\frac{k_i-1}{2}\frac{k_i+1}{2}}{2}=\frac{k_i^2-1}{4}$$

when k_i is odd. Thus, for $r+1 \leq i \leq d$, we have

$$c_i = \begin{cases} n\frac{k_i}{4} & if\ k_i\ is\ even, \\ n\frac{k_i^2-1}{4k_i} & if\ k_i\ is\ odd \end{cases}$$

as needed. □

The proof of our main Proposition 1 now follows by combining the congestion lemma with the inequality (12), which gives the lower bound $i(G^d) \geq \min\{i(P_k), i(C_l)\}$, and the two inequalities (9), and (10) which give the reverse inequality $i(G^d) \leq \min\{i(P_k), i(C_l)\}$.

4 Isoperimetric sets, their cardinalities and the bisection width

Proposition 2 *The cardinality of the isoperimetric sets of G^d in equation (7) is $|V(G^d)|/2$.*

Proof An isoperimetric set must make the two sides of inequality (11) equal, which can happen only if it has $|V(G^d)|/2$ vertices. □

Even though any isoperimetric set must have cardinality $|V(G^d)|/2$ vertices, because of the structural symmetry of cylinders, there may be multiple isoperimetric sets. For $G^d = P_{k_1} \times \cdots \times P_{k_r} \times C_{l_1} \times \cdots \times C_{l_s}$ where $d = r + s$, the set X given below is an isoperimetric set.

$$X = \begin{cases} \{u = (u_1, \ldots, u_d) \mid u_1 < \frac{k_1}{2}\} & if\ 2k_1 \geq l_1\ and\ k_1\ even, \\ \{u = (u_1, \ldots, u_d) \mid u_{r+1} < \frac{l_1}{2}\} & if\ l_1 \geq 2k_1\ and\ l_1\ even. \end{cases}$$

It can be seen that the partition $(X, \overline{X})$ is indeed a bisection achieving the bisection width of G^d given in Corollary 1.

5 Summary and future considerations

We have used an embedding technique to calculate the isoperimetric number of a large subclass of generalized cylinders. This class includes d–dimensional tori and d–dimensional arrays in which the size of the largest factor is even.

We remark that this embedding technique does not work when the largest factor in the product has an odd number of vertices. Interestingly, the inequalities (6) are not sharp enough to obtain the isoperimetric number in this case either. Ultimately it would be desirable to show that $G^d = G_{k_1} \times G_{k_2} \times \cdots \times G_{k_d}$,

$$i(G^d) = \min_j\{i(G_{k_j})\}$$

for any generalized cylinder regardless of the parities of the factors involved.

We have recently proved in [5] that for d-dimensional arrays the above formula holds, i.e. $i(P_{k_1} \times \cdots \times P_{k_d}) = \min_j\{i(P_{k_j})\}$. In order to obtain this result, a different notion of extremal sets, namely extremal sets minimizing dimension–normalized boundary in Hamming graphs [4] is utilized. The technique used involves embedding a Hamming graph into the array and associating the extremal sets of the Hamming graph with the isoperimetric sets of the array. We suspect that this technique generalizes to the case of the generalized cylinders as well. This work is in progress.

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