Minimum-energy Broadcast in Simple Graphs with Limited Node Power

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ABSTRACT
The minimum-energy broadcasting problem in wireless networks consists of finding a transmission radius vector for all stations in such a way that the total transmission power of the whole network is least possible. The minimum-energy broadcast problem may by modeled by an edge weighted complete graph in which each vertex in the graph represents a station and the weight of the edge is distance between the two nodes it joins. This is the weighted graph version of the minimum-energy broadcast problem. Wan, Calinescu, Li and Frieder showed that for arbitrary weights, this problem is NP-hard, and also hard to approximate.

In this paper we show that the weighted graph minimum-energy broadcast problem is NP-hard in metric space when transmissions are restricted to a given set of power levels by means of an upper bound d on the allowed transmission radius. This restriction is justified because it is unrealistic to expect transmissions with unbounded power which may be needed in an optimal solution for large diameter networks. We also show that our problem can be solved in polynomial time when there is an optimal solution with a fixed number of transmitter nodes.

KEY WORDS
Broadcast, routing, wireless, transmission, minimum-energy, power, graph.

1. Introduction

We study a graph version of the problem of *broadcasting* from a source station to all the stations in a static ad-hoc wireless network using minimum total energy. We assume the n stations are located in the plane and the *source station* is s. In a wireless network each station is represented by a *node* and there are no fixed links joining pairs of nodes, but when a node transmits with power r^α, all the nodes within distance r will receive the transmission. This power function and the specific value of α, which is normally between 2 and 4, are derived from physical considerations. We state our results for $\alpha = 2$, but can be easily generalized to cover all other values of α. The nodes other than s that transmit with power greater than zero are called *retransmitters* or *relay stations* and their purpose is to receive information and send it to other nodes. The broadcasting problem in static wireless networks consists of finding for each node v a transmission radius $r(v)$ so that s broadcasts to all the nodes either directly or indirectly through the relay nodes. In other words, for every node v there is a sequence of nodes $s = v_{i_1}, v_{i_2}, \ldots, v_{i_l} = v$, for some $l \geq 2$, such that the distance from v_{i_j} to $v_{i_{j+1}}$ is at most $r(v_{i_j})$, for $1 \leq j < l$. The total energy or power used is $\sum_{v \in G} r(v)^2$. Static and slow changing wireless networks have received considerable attention because of their applications to battlefield, emergency disaster relief, etc. as well as in situations when it is not economically practical or physically possible to provide Internet or Intranet connectivity.

Figure 1 depicts a set of 10 points in the plane with circles representing the range of node transmissions where the source node is $s = 10$. Table 1 gives the corresponding transmission vector r for the stations in Figure 1. The source station ($s = 10$) transmits directly to stations 1, 2, 3, 6 and 7. Stations 3, 6 and 7 relay the transmission to 4, 5 and 8, respectively. Node 8 retransmits to station 9.

Table 1. Transmission radius vector r for Figure 1.

	1	2	3	4	5	6	7	8	9	10
Vector r	0	0	2	0	0	2	3	1	0	5

Thus the minimum-energy broadcasting problem in wireless networks consists of finding a transmission radius vector for all vertices in such a way that the total transmission power is least possible. Each transmission radius vector defines a spanning broadcast tree for the network in which the root is the source node s and each path from the root to a vertex v corresponds to the sequence of transmissions by which source s communicates its message to v. Note that there may be many possible transmissions from s to v, but the broadcast tree will just represent one. Internal nodes of the tree are the transmitter nodes (or relays for non-root internal nodes). Figure 2 represents the broadcast spanning tree for the transmission radius vector in Table 1. The cost of transmission from a transmitter node in a

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broadcast tree is the minimum power required by the internal node to transmit to all its children. A minimum-energy broadcast tree is one with the minimum sum of the cost of the transmissions from each internal node in the tree. We call this problem the *geometric minimum-energy broadcast* problem. This problem was initially introduced by Wieselthier, Nguyen and Ephremides [10] and their main result was developing heuristics for its solution.

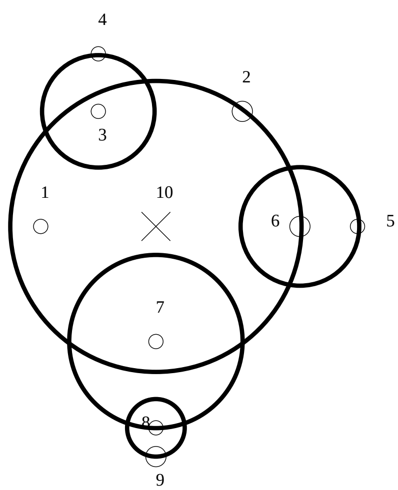

Figure 1. Station 10 broadcasts.

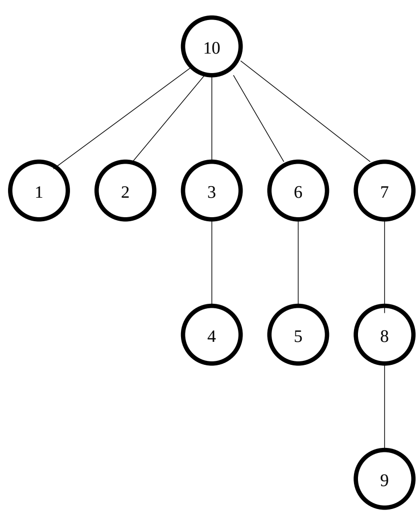

Figure 2. Corresponding spanning tree.

In wired networks that allow only unicasting communications (transmission from one node to just another one) the total cost of the broadcast is the sum of the weights on the links of the spanning tree. Therefore, it can be formulated and solved efficiently by the algorithms for the minimum-cost spanning tree (MST) problem. In wireless networks, there is a significant difference in the calculation of the energy required for the broadcast, since all of the nodes within the communication range of a transmission node may receive a transmission without additional transmitter power. Details of this model and the physical assumptions made can be found in [10]. Wired networks that allow multicasting fall somewhere between the above two communication models.

The minimum-energy broadcast problem may by modeled by an edge weighted complete graph in which each vertex in the graph represents a station and the weight of the edge is the distance between the two nodes it joins. We call this problem with arbitrary weights the *weighted graph minimum-energy broadcast problem*. Wan, Calinescu, Li and Frieder [9] showed that for arbitrary weights, this problem is an NP-hard problem. They also show that approximating this problem is as hard as approximating the dominating set problem which its known to be NP-hard to approximate.

If the set of weights satisfies the triangle inequality, then the problem is said to be in metric space. In this paper we show that the weighted graph version of the problem is NP-hard in metric space when transmissions are restricted to a given set of power levels. This means that the transmission radius of each vertex has to be no more than some bound d. We also show that the weighted graph problem can be solved in polynomial time when there is an optimal solution where only a fixed number of nodes transmit.

A particular case of the weighted graph minimum-energy broadcast problem is obtained by the application of the terms "distance", "diameter", etc., in their usual graph theoretical sense. Here we start with a simple unweighted graph $G = (V, E)$. The distance between $u, v \in V$ is then the length of the shortest path between u and v in G. If vertex u transmits to a distance r in G, this incurs a cost of r^α, and the transmission is received by all $v \in V$ within distance (in the graph theoretical sense) r of u in G. The quantity r is then the transmission radius of u. As before, the total energy (cost) of a broadcast tree is the sum of the energies r^α of the internal nodes of the tree.

One may view our problem as an instance of the metric space problem by applying the following transformation. From a simple graph G we construct a weighted complete graph G' by assigning a weight to each edge in G' equal to the length of the shortest path joining the two vertices in G. The resulting graph satisfies the triangle inequality but does not include all graphs in metric space.

Given a simple graph $G = (V, E)$ and a source vertex $s \in V$, there are two problems of interest:

> **Unrestricted minimum-energy broadcast:** Find a minimum-energy broadcast tree in G from vertex s.
>
> **Restricted minimum-energy broadcast:** Find a minimum-energy broadcast tree in G from vertex s in which the permitted transmission radii are bounded by some $d \geq 1$.

In this paper we consider the decision versions of these problems and show that the restricted problem is NP-hard. This restriction is justified because it is unrealistic to expect transmissions with unlimited power which may be needed in an optimal solution for large diameter networks.

We formally define the restricted minimum-energy broadcast (RMEB) decision problem as follows:

RESTRICTED MINIMUM-ENERGY BROADCAST PROBLEM (RMEB):
INSTANCE: A 4–tuple (G, s, d, K) where $G = (V, E)$ is a simple graph, $s \in V$ is the source node, $d < |V|$, $K < |V|^2$ are positive integers.
QUESTION: Is there a spanning broadcast tree rooted at s and with total energy K or less in which each transmission radius used is at most d?

Thus in RMEB, we permit nontrivial transmission radii only from the set $\{1, 2, \ldots, d\}$. If the radius is unrestricted, then the decision problem is UMEB defined as

UNRESTRICTED MINIMUM-ENERGY BROADCAST PROBLEM (UMEB):
INSTANCE: A 3–tuple (G, s, K) where $G = (V, E)$ is a simple graph, $s \in V$ is the source node, $K < |V|^2$ is a positive integer.
QUESTION: Is there a spanning broadcast tree rooted at s and with total energy K or less?

In Section 2., we present our reduction to establish the NP-completeness of RMEB. Issues relating to UMEB, as well as the the corresponding approximation problems are discussed in Section 3..

2. NP-completeness of RMEB

We begin by defining the vertex cover problem (VC). Let $G = (V, E)$ be an undirected graph with vertex set V, and edge set E. A subset $V' \subseteq V$ of vertices is said to be a vertex cover for G iff every edge in E is incident to at least one vertex in V'.

VERTEX COVER (VC):
INSTANCE: A pair (G, K), where $G = (V, E)$ is an undirected graph, and $K < |V|$ is an integer.
QUESTION: Does G have a vertex cover with at most K vertices?

VC was among the first set of problems shown to be NP-complete [4, 6]. The class of NP-complete problems is the set of all decision problems $Q \in$ NP such that SAT $\propto Q$, where SAT is the satisfiability problem defined in [4], and $\propto$ represents polynomial time reducibility [4]. The class of NP-complete problems is very rich. We refer the reader to [3, 4, 6] for additional details about the theory of NP-completeness. We establish our intractability results by constructing a polynomial time reduction from VC to the RMEB problem.

A *Turing reduction* from a search problem R to a problem R' is an algorithm A that solves R by using a hypothetical subprogram S for solving R', such that, if S were a polynomial time algorithm for R', then A would be a polynomial time algorithm for R. A search problem R is called NP-hard if there exists some NP-complete problem R' that Turing-reduces to R. Thus if R is NP-hard, then it cannot be solved in polynomial time unless $P =$ NP. In particular, all NP-complete problems are NP-hard.

2.1 A special case: RMEB with $d = 1$

There is an easy Turing reduction that shows that RMEB is NP-hard when the transmission radius is always 1. We describe this next. The maximum leaf spanning tree (MLST) ([4], Problem ND2, p. 206) is an NP-complete problem. This problem is defined as follows:

MAXIMUM LEAF SPANNING TREE (MLST):
INSTANCE: Graph $G = (V, E)$, positive integer $K \leq |V|$.
QUESTION: Is there a spanning tree for G in which K or more vertices have degree 1?

Proposition 1 *The MLST problem Turing reduces to RMEB with $d = 1$. Thus RMEB is NP-hard if a vertex broadcasts to its neighbors only.*

Proof The restricted minimum-energy broadcast problem RMEB with $d = 1$ is one in which all the transmissions are to nodes at a distance one. This is equivalent to finding a tree with root s (the source) such the number of internal nodes (transmitters) is as few as possible. This is the same as finding a spanning tree with root s in which the number of leaf nodes is as large as possible. Thus if there is a polynomial time algorithm S for RMEB with $d = 1$, then we can use this to solve the MLST by running S a total of $|V|$ times, each time with a different source vertex s. •

2.2 The general reduction for RMEB

We construct a direct polynomial time reduction from VC to RMEB. This type of reduction is stronger than Turing reduction we considered for the $d = 1$ case, and works for $d = 1$ as well.

First we define a certain gadget that we build to transform an edge between two vertices u, v. Given positive integers c and d, the graph $g(c, d)$ is constructed as follows.

Case 1: d is even

1. Make c copies of a path with $d/2$ nodes.
2. Construct the multigraph obtained by making c edges between u and v.
3. Subdivide each of these c edges by inserting $d - 1$ additional vertices (this turns each edge between u and v into a path with $d + 1$ vertices).
4. Connect one extreme vertex of each of the c paths to the central vertex of the subdivisions in a one-to-one manner to obtain $g(c, d)$.

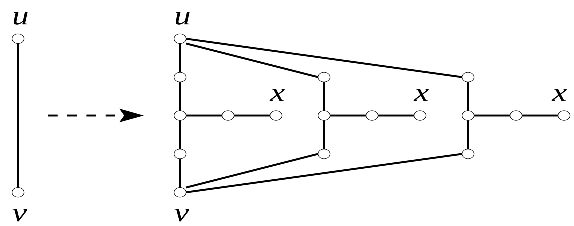

Figure 3. The gadget $g(3, 4)$.

As an example, the gadget $g(3, 4)$ is shown in Figure 3.

Case 2: d is odd

1. Make c copies of a path with $(d+1)/2$ nodes.
2. Construct the multigraph obtained by making c edges between u and v.
3. Subdivide each of these c edges by inserting $d-1$ additional vertices (this turns each edge between u and v into a path with $d+1$ vertices).
4. Connect one extreme vertex of each of the c paths to the two central vertices of the subdivisions in a one-to-one manner to obtain $g(c, d)$.

As an example, the gadget $g(3, 5)$ is shown in Figure 4.

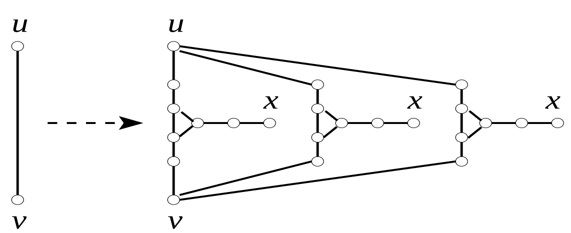

Figure 4. The gadget $g(3, 5)$.

If we construct $g(c, d)$ starting with the edge $\{u, v\}$, we say that the gadget is *built* on the edge $\{u, v\}$. In our reduction, we will be using the gadgets $g(c, d)$ with $c \gg d$. We observe that for any gadget $g(c, d)$ built on the edge $\{u, v\}$, the minimum energy required to transmit from either u or v (or from a vertex outside $g(c, d)$) to all of the extreme vertices of $g(c, d)$ labeled as x in Figures 3 and 4, is at least $\min\{d^2, c\}$. The quantity c comes from at least c transmissions of radius 1 or more required if neither u nor v has transmission radius d.

Proposition 2 *VC polynomial time reduces to RMEB. Thus RMEB is NP-hard.*

Proof Given an instance of the vertex cover problem (G, K) with $G = (V, E)$ and $K < |V|$, we construct an instance of an RMEB problem $(G_2, s, d, (1+K)d^2)$ as follows. Suppose $n = |V|, e = |E|$. Construct a graph G_2 from G by using the gadgets $g(c, d)$ and following the steps below:

1. First add an extra vertex s and add the edges $\{s, v\}$ for all $v \in V$ to E. Call the resulting graph $G_1 = (V_1, E_1)$. Then $|V_1| = 1 + n$ and $|E_1| = n + e$.

2. Construct $G_2 = (V_2, E_2)$ from G_1 by replacing each edge $\{u, v\} \in E_1$ by the gadget $g(c, d)$.

We show later on that we can set c equal to nd^2 in this construction.

We view V_2 as consisting of $1+n$ *old* vertices which are s and the original vertices in V, and a number of *new* vertices which are the vertices introduced by the gadgets $g(c, d)$ (these are the vertices distinct from u, v in Figures 3, and 4). We claim that in the resulting instance G_2, it is possible to broadcast from s with total energy $(1+K)d^2$ to all vertices in V_2 with transmission radius bounded by d iff G has a vertex cover of size at most K.

First assume that G has a vertex cover V' of size K. With radius d, s can transmit to all of the new vertices on the gadgets formed from edges $\{s, v\}$ in G_1, and to all of the vertices in the vertex cover V' (in fact to all vertices in V_1). Now each $v \in V'$ can broadcast with radius d to all other vertices. Thus in G_2, it is possible to broadcast from s by using total energy $(1+K)d^2$.

Suppose now that in G_2, s can broadcast with total energy $(1+K)d^2$ or less for some $K < n$. Note that just as VC always has a solution for $K = n$, s can always broadcast in G_2 using total energy energy $(1+n)d^2$.

Let W be the set of vertices that are transmitters in such a broadcast tree of total energy $(1+K)d^2$ with root s. We'll show that in addition to s, W contains K old vertices (i.e. vertices in V_1) that have transmission radius d, and that these K vertices form a VC for G. We'll also show that all transmitters in the broadcast tree have radius d.

First we show that s has transmission radius $r = d$. Since the allowed radii are at most d in RMEB, it is enough to show that $r \geq d$. By way of contradiction, assume that s transmits with radius $r < d$. Then by our previous observation on the power required to reach extreme vertices labeled x in a gadget, we need at least an additional nd^2 energy to send to all gadgets built on edges $\{s, v\}, v \in V$, or one of them requires power c. Choose $c = nd^2$. Then the total power required to broadcast from s in G_2 is at least $r^2 + nd^2$. Since $K < n$, $r^2 + nd^2 \geq 1 + nd^2 > (1+K)d^2$, which is a contradiction. Thus s has transmission radius d.

Now given that $c = nd^2$ and s has transmission radius d, it follows that if W contains some new vertex y in some $g(c, d)$ built on an edge $\{u, v\} \in E_1$, then it contains either u or v (or both), since the only way to reach y from the source s is by relaying the transmission through u or v, for otherwise the power required will at least be c, which is too large. Therefore for every edge $\{u, v\} \in E_1$, either u or v is a transmitter. It follows that $V' = W \setminus \{s\}$ is a vertex cover for G of cardinality at most K.

The number of vertices of G_2 is found to be $|V_2| = O(d^3 ne)$. Note also that the reduction itself can be carried out in polynomial time with respect to n, i.e. the reduction is polynomial time. •

3. Polynomial Time Algorithm for the Weighted Graph Version

In this section we present a simple $O(n^{k+2})$ algorithm that finds an optimal solution to the UMEB problem when there is an optimal solution in which no more than k nodes are transmitters.

The algorithm is simple. First it will try all subsets of at most k nodes. For each node there are at most n different power levels at which it may transmit, since transmission at intermediate levels will not reach other stations. We then try all these $O(n^k)$ possible power level choices for all the nodes selected and check to see if it is a solution and if so keep track of the one with smallest objective function value. Clearly this can be done in $O(n^2)$, so the total time complexity bound becomes $O(n^{k+2})$. Since there are $\binom{n}{k}$ subsets with k stations, the total time complexity bound is $O(n^{2k+2})$.

4. Discussion

We have shown that the weighted graph minimum-energy broadcast problem is NP-hard in metric space when transmissions are restricted to a given set of power levels by means of an upper bound d on the allowed transmission radius. This restriction is justified because it is unrealistic to expect transmissions with unlimited power which may be needed in an optimal solution for large diameter networks. We have also shown that our problem can be solved in polynomial time when there is an optimal solution with a fixed number of transmitter nodes.

The RMEB problem with $d = 1$ is as hard to approximate as the set cover problem, which is known to be NP-hard to approximate. This reduction is not in this paper but it is similar to the ones in [5].

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