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k-Fibonacci Cubes: A Family of Subgraphs of Fibonacci Cubes

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Hypercubes and Fibonacci cubes are classical models for interconnection networks with interesting graph theoretic properties. We consider k-Fibonacci cubes, which we obtain as subgraphs of Fibonacci cubes by eliminating certain edges during the fundamental recursion phase of their construction. These graphs have the same number of vertices as Fibonacci cubes, but their edge sets are determined by a parameter k. We obtain properties of k-Fibonacci cubes including the number of edges, the average degree of a vertex, the degree sequence and the number of hypercubes they contain.

Keywords: Hypercube; Fibonacci cube; Fibonacci number.

1. Introduction

An interconnection network can be represented by a graph $G = (V, E)$ with vertex set V denoting the processors and edge set E denoting the communication links between the processors. One of the basic models for these networks is the n-dimensional hypercube graph Q_n, whose vertices are indexed by all binary strings

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of length n and two vertices are adjacent if and only if their Hamming distance is 1. Hsu [4], defined the n-dimensional Fibonacci cubes Γ_n as an alternative model of computation for interconnection networks and showed that they have interesting properties. Γ_n is a subgraph of Q_n, where the vertices are those without two consecutive 1's in their binary string representation. Γ_0 is defined as Q_0, the graph with a single vertex and no edges.

Numerous graph theoretic properties of Fibonacci cubes have been studied. In [8], a survey of the some of the properties including representations, recursive construction, hamiltonicity, degree sequences and other enumeration results are given. The induced d-dimensional hypercubes Q_d in Γ_n are studied in [3, 9, 12, 13, 16, 17] and the boundary enumerator polynomial of hypercubes in Γ_n is considered in [18]. The number of vertex and edge orbits of Fibonacci cubes are determined in [1].

A Fibonacci string of length n which indexes the vertices of Γ_n is a binary string $b_1 b_2 \ldots b_n$ such that $b_i \cdot b_{i+1} = 0$ for $1 \leq i \leq n-1$. The additional requirement $b_1 \cdot b_n = 0$ for $n \geq 2$ defines a subgraph of Γ_n called Lucas cube Λ_n, which was also proposed as a model for interconnection networks [14].

Other classes of graphs have also been defined along the lines of Fibonacci cubes. For instance subgraphs of Q_n where k consecutive 1's are forbidden are proposed in [5] and Fibonacci (p, r)-cubes are presented in [2]. The generalized Fibonacci cube $Q_n(f)$ is defined as the subgraph of Q_n by removing all the vertices that contain some forbidden string f [6]. With this formulation one has $\Gamma_n = Q_n(11)$.

In this paper, we consider a special subgraph of Γ_n which is obtained by eliminating certain edges. These are called k-Fibonacci cubes (or k-Fibonacci graphs) and denoted by Γ_n^k as they depend on a parameter k. The edge elimination which defines k-Fibonacci cubes is carried out at the step analogous to where the fundamental recursion is used to construct Γ_n from the two previous cubes by link edges. The eliminated edges in Γ_n^k then recursively propagate according to the resulting fundamental construction which now depend on the value of the parameter k.

We obtain properties of Γ_n^k including the number of edges, the average degree of a vertex, the degree sequence and the number of hypercubes they contain.

2. Preliminaries

Fibonacci numbers are defined by the recursion $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$, with $f_0 = 0$ and $f_1 = 1$. Similarly, the Lucas numbers L_n are defined by the recursion $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, with $L_0 = 2$ and $L_1 = 1$. Any positive integer can be uniquely represented as a sum of non-consecutive Fibonacci numbers. This representation is usually called the Zeckendorf or canonical representation. Here we note that this representation corresponds to the Fibonacci strings. For some positive integer i assume that $0 \leq i \leq f_{n+2} - 1$. Then i can be written as $i = \sum_{j=1}^{n} b_j \cdot f_{n-j+2}$, where $b_j \in \{0, 1\}$. This gives that the Zeckendorf representation of i is $(b_1, b_2, \ldots, b_n)$ and it corresponds to the Fibonacci string $b_1 b_2 \ldots b_n$. Note that here we assume the integer 0 has Zeckendorf representation $(0, 0, \ldots, 0)$. As an example, for $n = 4$,

$7 = 5 + 2 = 1 \cdot f_5 + 0 \cdot f_4 + 1 \cdot f_3 + 0 \cdot f_2$ gives that the Zeckendorf representation of 7 is $(1, 0, 1, 0)$ and this corresponds to the Fibonacci string 1010.

The distance between two vertices u and v in a connected graph G is defined as the length of a shortest path between u and v in G. For Q_n and Γ_n this distance coincides with the Hamming distance (d_H) which is the number of different bits in the string representation of the vertices. Let $Q_n = (V(Q_n), E(Q_n))$ be the n-dimensional hypercube. Then its vertex set and edge set can be written as

$$V(Q_n) = \{0,1\}^n = \{b_1 b_2 \dots b_n \mid b_i \in \{0,1\}, \quad 1 \le i \le n\}$$

$$E(Q_n) = \{\{u,v\} \mid u, v \in V(Q_n) \text{ and } d_H(u,v) = 1\}.$$

Of course $|V(Q_n)| = 2^n$ and $|E(Q_n)| = n2^{n-1}$. Similarly, we can write the vertex set and the edge set of $\Gamma_n = (V(\Gamma_n), E(\Gamma_n))$ as

$$V(\Gamma_n) = \{b_1 b_2 \dots b_n \mid b_i \in \{0,1\}, \quad 1 \le i \le n \text{ with } b_i \cdot b_{i+1} = 0\}$$

$$E(\Gamma_n) = \{\{u,v\} \mid u, v \in V(\Gamma_n) \text{ and } d_H(u,v) = 1\}.$$

Note that the number of vertices of the Fibonacci cube Γ_n is f_{n+2}.

The fundamental decomposition [8] of Γ_n can be described as follows: Γ_n can be decomposed into the subgraphs induced by the vertices that start with 0 and 10 respectively. The vertices that start with 0 constitute a graph isomorphic to Γ_{n-1} and the vertices that start with 10 constitute a graph isomorphic to Γ_{n-2}. This decomposition can be written symbolically as

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}.$$

In this representation, the vertices are labeled with the Fibonacci strings. In Fig. 1 the first six Fibonacci cubes are presented. The labeling of the vertices follows the fundamental recursion. For $n \ge 2$, Γ_n is built from Γ_{n-1} and Γ_{n-2} by identifying Γ_{n-2} with the Γ_{n-2} in Γ_{n-1} by what we can call *link* edges. The labels of Γ_{n-1} stay unchanged in Γ_n. The labels of the vertices in Γ_{n-2} (which are circled in Fig. 1) are shifted up by f_{n+1} in Γ_n.

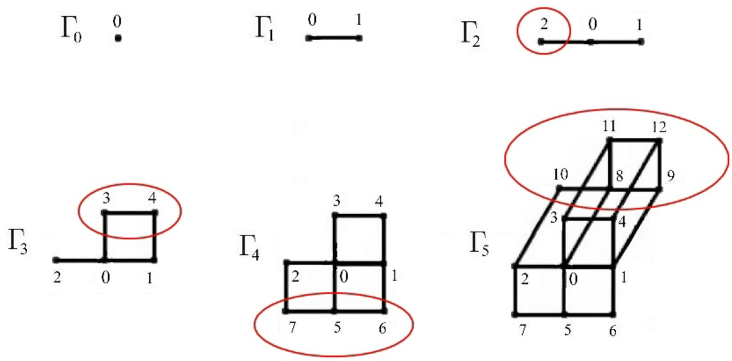

Fig. 1. Fibonacci cubes $\Gamma_0, \Gamma_1, \dots, \Gamma_5$.

Throughout the paper we use the Fibonacci string representation and integer representation of the vertices interchangeably. For example, the first 8 vertices of Γ_5 are the ones with labels $0, 1, \ldots, 7$ whose string representations are 00000, 00001, 00010, 00100, 00101, 01000, 01001, 01010.

In terms of the adjacency matrix A_n of Γ_n, the link edges are the edges corresponding to the identity matrices that appear in the matrix decomposition of A_n as shown on the left in Fig. 3.

3. *k*-Fibonacci Cubes

In this section we introduce k-Fibonacci cubes, which are special subgraphs of Fibonacci cubes. We will indicate the dependence on k by a superscript and denote these graphs by Γ_n^k.

For $k \geq 1$, Γ_n^k is defined for $n \geq 0$ by removing certain edges of Γ_n. We let $\Gamma_0^k = \Gamma_0$ and $\Gamma_1^k = \Gamma_1$. We define Γ_n^k for $n \geq 2$ in terms of Γ_{n-1}^k and Γ_{n-2}^k, in a manner that is similar to the fundamental decomposition of Γ_n. The difference is as follows:

> *In the construction of* Γ_n *from* $00\Gamma_{n-2}$ *and its copy* $10\Gamma_{n-2}$ *in* Γ_{n-1}, *there are* f_n *link edges* uv *with* $u = 00\alpha \in 00\Gamma_{n-2}$ *and* $v = 10\alpha \in 10\Gamma_{n-2}$, *which are added to the edge set for every Fibonacci string* α *of length* $n-2$. *The various* α *here are the Fibonacci representation of the integers* $0, 1, \ldots, f_n - 1$, *which are the vertices of* Γ_{n-2}.
>
> *In our construction of* Γ_n^k *from* $00\Gamma_{n-2}^k$ *and its copy* $10\Gamma_{n-2}^k$ *in* Γ_n^k, *we only include the link edges* uv *with* $u = 00\alpha \in 00\Gamma_{n-2}^k$ *and* $v = 10\alpha \in 10\Gamma_{n-2}^k$ *corresponing to the Fibonacci strings* α *that are the representations of* $0, 1, \ldots, k-1$ *in* Γ_{n-2}^k. *In other words, only the first* k *vertices in* $00\Gamma_{n-2}^k$ *and its copy* $10\Gamma_{n-2}^k$ *in* Γ_n^k *contribute link edges, and the remaining link edges that would have been included in the construction of* Γ_n *are discarded.*

As long as $f_n \leq k$, the construction for the k-Fibonacci cubes is identical to the construction of the Fibonacci cubes with the same initial graphs, and therefore for $f_n \leq k$ we have $\Gamma_n^k = \Gamma_n$. Let $n_0 = n_0(k)$ be the smallest integer for which $f_{n_0} > k$. For a given k, $n_0(k)$ is the smallest integer n for which $\Gamma_n^k \neq \Gamma_n$. First few values are $n_0(1) = 3$, $n_0(2) = 4$, $n_0(3) = 5$, $n_0(4) = 5$. Using the Binet formula for the Fibonacci numbers, we calculate that $n_0 = n_0(k)$ is given explicitly by

$$n_0 = 1 + \left\lfloor \log_\phi \left(\sqrt{5}k + \sqrt{5} - \frac{1}{2} \right) \right\rfloor \tag{1}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. This sequence starts as

$$3, 4, 5, 5, 6, 6, 6, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 8, 8, 8, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 10, \ldots$$

in which the lengths of the runs are the Fibonacci numbers.

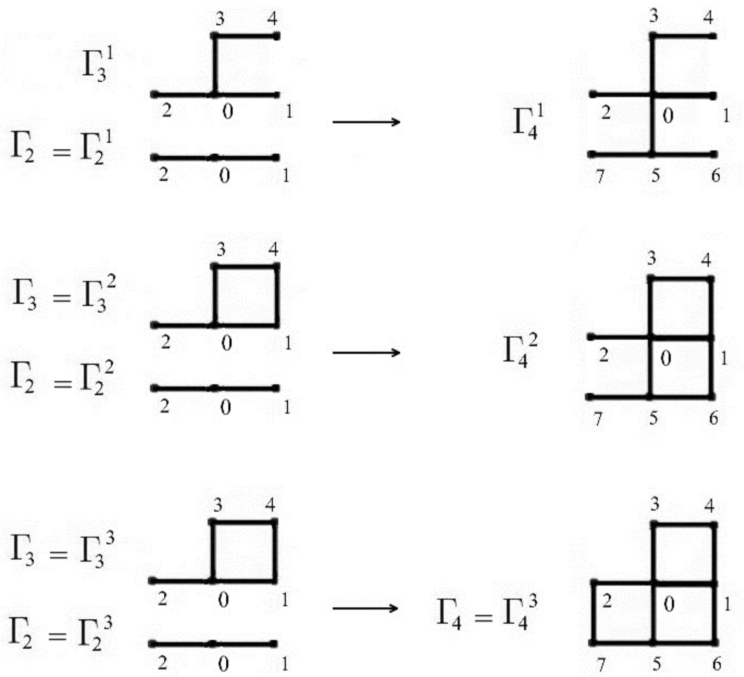

Fig. 2. The construction of Γ_4^k from Γ_3^k and Γ_2^k for $k = 1, 2, 3$.

We remark that an equivalent definition of the graphs Γ_n^k could be made by starting the construction from the two Fibonacci cubes Γ_{n_0-1} and Γ_{n_0} where n_0 is the smallest integer for which $f_{n_0} > k$, and then constructing only k link edges for the construction of the Γ_n^k from the two previous ones after that.

In Fig. 2 we show the construction of the k-Fibonacci cubes Γ_4^k from Γ_3^k and Γ_2^k for $k = 1, 2, 3$.

Similar to the adjacency matrix A_n of Γ_n we can write the adjacency matrix A_n^k of Γ_n^k. In Γ_n^k, we have k link edges instead of the f_n link edges of Γ_n. Hence in A_n^k the link edges are the edges corresponding to the $k \times k$ identity matrices that appear in the matrix decomposition of A_n^k shown on the right in Fig. 3.

Examples for small k

For $k = 1$, the graphs Γ_n^1 are all trees. If we think of them as rooted at the all zero vertex, then the next tree is obtained by making the root of the previous tree a child of the root of the present tree, i.e. by making the previous tree a principal subtree of the current one. Note that this is different from the usual definition of Fibonacci trees, where traditionally the two previous trees are made left and right principal subtrees of a new root vertex.

The number of vertices in Γ_n^1 is $|\Gamma_n| = f_{n+2}$. The height h_n of Γ_n^1 satisfies $h_0 = 0$, $h_1 = 1$ and $h_n = \min\{h_{n-1}, 1 + h_{n-2}\}$. Therefore the height is given by $h_n = \lceil n/2 \rceil$.

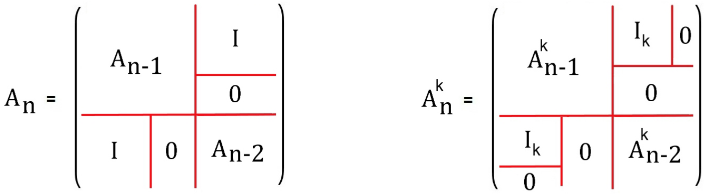

Fig. 3. Left: the structure of the adjacency matrix A_n of the Fibonacci cube Γ_n in terms of A_{n-1} and A_{n-2} in accordance with the fundamental decomposition. Here $\mathbf{I}$ is the $f_n \times f_n$ identity matrix, and the remaining elements are zeros. Right: the structure of the adjacency matrix A_n^k of Γ_n^k in terms of A_{n-1}^k and A_{n-2}^k where I_k is now the $k \times k$ identity matrix, and the remaining elements are zeros.

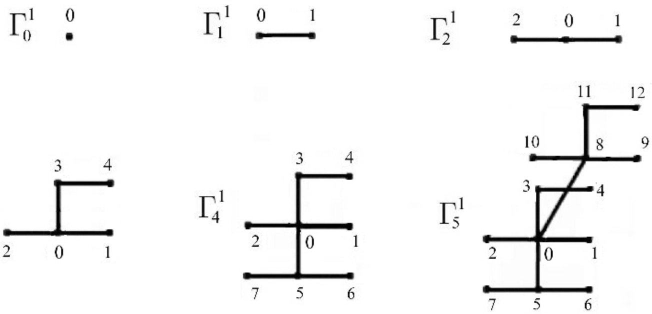

Fig. 4. The first six k-Fibonacci cubes for $k = 1$.

Fig. 4 shows the first six k-Fibonacci cubes (trees) for $k = 1$. In the Appendix we redraw these trees in another form (Fig. 7). Note that our k-Fibonacci cubes for $k = 1$ are the "Fibonacci Trees" as given in Hsu's original paper on Fibonacci cubes [4].

For $n \geq 3$, Γ_n^2 consists of $f_n - 1$ squares (Q_2's, or 4-cycles) glued by their edges and f_{n-1} pendant vertices in the manner shown in Fig. 8 in the Appendix.

For $n \geq 4$, Γ_n^3 is constructed from $f_{n+1} - 2$ squares and for $n \geq 4$, Γ_n^4 is constructed from $2f_n - 3$ squares in the manner shown in Figs. 9 and 10 in the Appendix, respectively.

We consider the number of induced hypercubes in Γ_n^k in Sec. 4.4 in more detail.

To demonstrate the structure of Γ_n^k for other values of n and k, the graphs of Γ_{12}^4 and Γ_{12}^{13} are depicted in Fig. 11 in the Appendix.

4. Basic Properties of k-Fibonacci Cubes Γ_n^k

In this section, we present the basic properties of k-Fibonacci cubes. The results include the number of edges, the average degree of a vertex, degree sequence and the number of hypercubes contained.

By definition of Γ_n^k we know that $|V(\Gamma_n^k)| = |V(\Gamma_n)| = f_{n+2}$.

4.1. *The number of edges of k-Fibonacci cubes*

Let $E(n)$ denote the number of edges of Γ_n and $E_k(n)$ denote the number of edges of Γ_n^k. $E(n)$ is given by (see, [4, Lemma 5])

$$E(n) = \frac{1}{5}(2(n+1)f_n + nf_{n+1}). \tag{2}$$

Clearly, $E_k(n) = E(n)$ for $n < n_0$ and $E_k(n)$ satisfies the recursion

$$E_k(n) = E_k(n-1) + E_k(n-2) + \min\{k, f_n\}. \tag{3}$$

For $n \geq n_0$ the recursion in (3) reduces to

$$E_k(n) = E_k(n-1) + E_k(n-2) + k. \tag{4}$$

Then by induction on n and using the recursion (4), we directly obtain the following result.

Proposition 1. *The number of edges of* Γ_n^k *is given by*

$$E_k(n) = (k + E(n_0-2))f_{t+3} + (E(n_0-1) - E(n_0-2))f_{t+2} - k,$$

where $t = n - n_0$.

By using the classical identity $L_t = f_{t-1} + f_{t+1}$ of the Lucas numbers, along with Proposition 1 and (2) we obtain the following result.

Corollary 2. *For* $n \geq n_0$, $E_k(n)$ *is given in closed form by*

$$\begin{aligned} E_k(n) &= \frac{1}{2}(L_t + 3f_t)E(n_0-1) + \frac{1}{2}(L_t + f_t)E(n_0-2) + (L_t + 2f_t - 1)k \\ &= \frac{1}{5}(n_0 f_{n_0-1} L_{t+1} + (n_0-1) f_{n_0} L_{t+2}) + (f_{t+3} - 1)k \end{aligned}$$

where $t = n - n_0$.

4.2. *Average degree of a vertex in* Γ_n^k

In [10] it is calculated that

$$\lim_{n\to\infty} \frac{2E(n)}{nf_{n+2}} = \lim_{n\to\infty} \frac{\frac{2}{5}(2(n+1)f_n + nf_{n+1})}{nf_{n+2}} = 1 - \frac{1}{\sqrt{5}}.$$

It follows that the average degree of a vertex in Γ_n is asymptotically given by

$$\left(1 - \frac{1}{\sqrt{5}}\right) n \approx 0.553n. \tag{5}$$

Now we consider the analogous problem for the k-Fibonacci cubes Γ_n^k for a fixed k and prove that the average degree of a vertex in Γ_n^k is independent of n.

Proposition 3. *For a fixed k the average degree of a vertex in Γ_n^k is asymptotically given by*

$$\begin{aligned}\frac{1}{5}(3+\sqrt{5}) + \left(1-\frac{1}{\sqrt{5}}\right)\log_\phi\left(\sqrt{5}k+\sqrt{5}-\frac{1}{2}\right) \\ \approx 1.047 + 0.553\log_\phi\left(\sqrt{5}k+\sqrt{5}-\frac{1}{2}\right),\end{aligned}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Proof. By the properties of the Fibonacci numbers we have

$$\lim_{n\to\infty}\frac{f_{n-n_0+3}}{f_{n+2}} = \phi^{1-n_0}, \quad \lim_{n\to\infty}\frac{f_{n-n_0+2}}{f_{n+2}} = \phi^{-n_0}. \tag{6}$$

For k fixed, using Proposition 1 and (6), the average degree of a vertex in Γ_n^k is given by

$$\begin{aligned}\lim_{n\to\infty}\frac{2E_k(n)}{f_{n+2}} &= 2\phi^{-n_0}((k+E(n_0-2))\phi + E(n_0-1) - E(n_0-2)) \\ &= 2\phi^{-n_0}\left(\phi k + \frac{n_0}{10}(5-\sqrt{5})f_{n_0-1} + \frac{n_0-1}{\sqrt{5}}f_{n_0}\right) \qquad (7)\end{aligned}$$

where we used the expression for $E(n)$ in (2) for $n = n_0 - 1$ and $n = n_0 - 2$. To obtain the rate of growth of the exact formula for the asymptotic average degree given in (7), we further use the approximations

$$f_{n_0} \approx \frac{\phi^{n_0}}{\sqrt{5}}, \quad n_0 \approx 1 + \log_\phi\left(\sqrt{5}k+\sqrt{5}-\frac{1}{2}\right).$$

Using these in the formula (7) gives the approximation for the average degree of a vertex in Γ_n^k as desired. □

In Proposition 3, we show that the average degree of a vertex in Γ_n^k is independent of n, to illustrate this we compute asymptotic values of the average degree of a vertex in Γ_n^k for some values of k in Fig. 5, where $1 \le k \le 300$.

It is shown in [4] that Γ_n contains about $\frac{4}{5}$ the number of edges of the hypercube for the same number of vertices. The analogous ratio goes to zero with increasing n for Γ_n^k since the average degree is independent of n. We can calculate the rate of convergence by using the asymptotic expressions that appear in the proof of Proposition 3. For fixed k, the number of edges of Γ_n^k is about

$$\begin{aligned}&\frac{\frac{1}{5}(3+\sqrt{5}) + \frac{1}{5}(5-\sqrt{5})\log_\phi(\sqrt{5}k+\sqrt{5}-\frac{1}{2})}{\log_2 f_{n+2}} \\ &\approx \frac{1}{n}\left(1.508 + 0.796\log_\phi\left(\sqrt{5}k+\sqrt{5}-\frac{1}{2}\right)\right)\end{aligned}$$

times the number of edges of Γ_n.

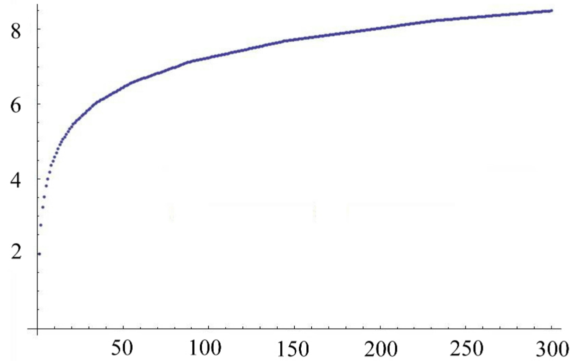

Fig. 5. Computational asymptotic values of the average degree of a vertex in Γ_n^k as a function of k, $1 \leq k \leq 300$.

4.3. *Degree polynomials of* $\boldsymbol{\Gamma_n^k}$

We start with the following known result for Γ_n that will be useful in our calculations. For the degree sequence of Γ_n^k we need a refinement of this theorem.

Theorem 4 ([11]). *For all $n \geq m \geq 0$ the number of vertices of Γ_n having degree m is given by*

$$f_{n,m} = \sum_{i=\lceil (n-m)/2 \rceil}^{s} \binom{n-2i}{m-i} \binom{i+1}{n-m-i+1},$$

where $s = \min(m, n-m)$.

We first define the bivariate degree enumerator polynomial $D_n^k(x,y)$ for Γ_n^k in which the degrees of the k vertices with labels $0, 1, \ldots, k-1$ are kept track of by the variable y, while the others are kept track of by the variable x.

More precisely

$$D_n^k(x,y) = \sum_{i=0}^{k-1} y^{d_i} + \sum_{i=k}^{f_{n+2}-1} x^{d_i}, \tag{8}$$

where d_i is the degree of the vertex labeled i. This polynomial can be seen as a refinement of the boundary enumerator polynomial $\mathbb{D}_{n,0}$ given in [18, Section 3].

As an example consider $\Gamma_4^2 = \Gamma_3^2 + \Gamma_2^2$ as shown in Fig. 6 in which the first $k = 2$ vertices are circled. From (8), we have $D_2^2(x,y) = y^2 + y + x$ and $D_3^2(x,y) = y^3 + y^2 + 2x^2 + x$. Using $D_2^2(x,y)$ and $D_3^2(x,y)$ we can now develop $D_4^2(x,y)$ as follows:

By the definition of Γ_4^2 there are $k = 2$ link edges between Γ_3^2 and Γ_2^2. The first one is the edge between the vertices labeled $0 \in \Gamma_3^2$ and $0 \in \Gamma_2^2$, whereas the second one is the edge between the vertices labeled $1 \in \Gamma_3^2$ and $1 \in \Gamma_2^2$. Therefore, only the degrees of these first 2 vertices in Γ_3^2 and first 2 vertices in Γ_2^2 increase by 1 in Γ_4^2.

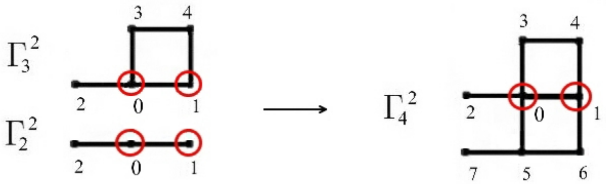

Fig. 6. Construction of Γ_4^2 from Γ_2^2 and Γ_3^2.

The degrees of the other vertices remain the same. The vertices labeled $0, 1 \in \Gamma_4^2$ are the vertices $0, 1 \in \Gamma_3^2$ whose degree information in $D_4^2(x, y)$ should be kept track of by the variable y. Therefore we can write,

$$\begin{aligned} D_4^2(x,y) &= yD_3^2(0,y) + D_3^2(x,0) + xD_2^2(0,x) + D_2^2(x,0) \\ &= y^4 + y^3 + x^3 + 3x^2 + 2x, \end{aligned}$$

and this can also be seen from Fig. 6 using (8).

By generalizing the above idea for fixed k, we have the following result on degree enumerator polynomial $D_n^k(x, y)$ for Γ_n^k.

Theorem 5. *Assume that* $D_{n_0-1}^k(x, y) = p_{n_0-1}(y) + q_{n_0-1}(x)$ *and* $D_{n_0-2}^k(x, y) = p_{n_0-2}(y) + q_{n_0-2}(x)$. *Then the degree enumerator polynomial* $D_n^k(x, y)$ *for* Γ_n^k *satisfies the recursion*

$$\begin{aligned} D_n^k(x,y) &= y^{t+1}p_{n_0-1}(y) + f_{t+2}q_{n_0-1}(x) + f_{t+1}q_{n_0-2}(x) \\ &\quad + f_{t+1}xp_{n_0-2}(x) + xp_{n_0-1}(x)\sum_{i=0}^{t} f_{t-i}x^i \end{aligned}$$

for $n \geq n_0$ *with* $t = n - n_0$.

Proof. We know that $\Gamma_n^k = 0\Gamma_{n-1}^k + 10\Gamma_{n-2}^k$ and there are k link edges between Γ_{n-1}^k and Γ_{n-2}^k for $n \geq n_0$. This means that the degrees of the first k vertices of Γ_{n-1}^k and Γ_{n-2}^k increase by 1 in Γ_n^k. But the degrees of the other vertices remain the same. So the first k vertices of Γ_n^k are the first k vertices of Γ_{n-1}^k whose degree information in $D_n^k(x, y)$ should be kept track of by the variable y. Therefore, for a fixed k, $D_n^k(x, y)$ satisfies the recursion

$$D_n^k(x,y) = yD_{n-1}^k(0,y) + D_{n-1}^k(x,0) + xD_{n-2}^k(0,x) + D_{n-2}^k(x,0) \tag{9}$$

for $n \geq n_0$. Now, writing

$$D_n^k(x,y) = p_n(y) + q_n(x),$$

and using (9) we see that the polynomials $p_n(y)$ and $q_n(x)$ must individually satisfy the recursions

$$p_n(y) = yp_{n-1}(y) \tag{10}$$

$$q_n(x) = q_{n-1}(x) + q_{n-2}(x) + xp_{n-2}(x) \tag{11}$$

for $n \geq n_0$ with initial values

$$p_{n_0-1}(y),\ p_{n_0-2}(y) \quad \text{and} \quad q_{n_0-1}(x),\ \ q_{n_0-2}(x).$$

From (10) we immediately obtain the formula for $p_n(y)$ for $n \geq n_0$ as

$$p_n(y) = y^{n-n_0+1} p_{n_0-1}(y).$$

To solve the recursion for $q_n(x)$ from (11), we form the generating function

$$Q(z) = \sum_{n \geq n_0} q_n(x) z^n.$$

Note first that

$$\begin{aligned}
\sum_{n \geq n_0} x p_{n-2}(x) z^n &= x p_{n_0-2}(x) z^{n_0} + x \sum_{n > n_0} x^{n-n_0-1} p_{n_0-1}(x) z^n \\
&= x p_{n_0-2}(x) z^{n_0} + \frac{x p_{n_0-1}(x) z^{n_0+1}}{1-xz}.
\end{aligned}$$

Multiplying (11) with z^n and summing both sides for $n \geq n_0$, $Q(z)$ is found to be

$$Q(z) = \frac{z^{n_0}}{1-z-z^2}\left((1+z)q_{n_0-1}(x) + q_{n_0-2}(x) + x p_{n_0-2}(x) + \frac{x p_{n_0-1}(x) z}{1-xz}\right)$$

Using the expansions

$$\frac{1}{1-z-z^2} = \sum_{n \geq 0} f_{n+1} z^n, \qquad \frac{z}{1-z-z^2} = \sum_{n \geq 0} f_n z^n$$

and

$$\frac{z}{(1-z-z^2)(1-xz)} = \sum_{n \geq 0} \left(\sum_{i=0}^{n} f_{n-i} x^i \right) z^n$$

and setting $t = n - n_0$, we finally obtain

$$q_n(x) = f_{t+2} q_{n_0-1}(x) + f_{t+1} q_{n_0-2}(x) + f_{t+1} x p_{n_0-2}(x) + x p_{n_0-1}(x) \sum_{i=0}^{t} f_{t-i} x^i. \qquad \square$$

Using Theorem 5 with the initial conditions from Theorem 4 we obtain the following special cases.

Corollary 6. *For $k \in \{1,2,3\}$ the degree enumerator polynomials $D_n^k(x,y)$ for Γ_n^k are given by*

$$D_n^1(x,y) = y^n + f_{n+1} x + \sum_{i=2}^{n-1} f_{n-i} x^i$$

$$D_n^2(x,y) = y^n + y^{n-1} + f_{n-1} x + \sum_{i=2}^{n-1} f_{n-i+1} x^i$$

$$D_n^3(x,y) = y^n + y^{n-1} + y^{n-2} + f_1 x^{n-1} + L_{n-1}x^2 + 2\sum_{i=3}^{n-2} f_{n-i}x^i$$

where L_n is the nth Lucas number.

4.4. *Number of induced hypercubes in* Γ_n^k

In this section, we count the number of d-dimensional hypercubes induced in Γ_n^k. We know that for $d=0$ and $d=1$ these numbers are equal to the number of vertices and number of edges in Γ_n^k respectively.

We first consider the number of squares (4-cycles) in Γ_n^k. This is the case where $d=2$.

Definition 7. Let $Z(i)$ denote the number of 1's in the Zeckendorf representation of i for $i \geq 0$ and define for $m \geq 0$, the partial sums

$$P(m) = \sum_{i=0}^{m} Z(i). \tag{12}$$

These sequences start as

$$0,1,1,1,2,1,2,2,1,2,2,2,3,1,2,2,2,3,2,3,3,1,\ldots$$

for $Z(i)$ and

$$0,1,2,3,5,6,8,10,11,13,15,17,20,21,23,25,27,30,32,35,38,39,\ldots$$

for $P(m)$.

Let $S(n)$ denote the number of squares of Γ_n and $S_k(n)$ denote the number of squares of Γ_n^k for $k \geq 2$. From [7] we know that

$$S(n) = \frac{1}{50}((5n+1)(n-2)f_n + 6nf_{n-2}).$$

For $n < n_0$ we have $S_k(n) = S(n)$. By the fundamental decomposition of Γ_n^k, the number of squares in Γ_n^k is the sum of three quantities: the number of squares in Γ_{n-1}^k, the number of squares in Γ_{n-2}^k and the number of squares that are created by the addition of k link edges between Γ_{n-1}^k and Γ_{n-2}^k involving the vertices with labels $0,1,\ldots,k-1$.

The number of squares of the last type above is equal to the number of edges in the subgraph of Γ_n^k induced by the first k vertices $0,1,\ldots,k-1$.

Lemma 8. *The number of edges in the subgraph of Γ_n^k induced by the first k vertices $0,1,\ldots,k-1$ is $P(k-1)$.*

Proof. For a vertex i in Γ_n^k with $i \in \{0,1,\ldots,k-1\}$, switching a 1 in the Zeckendorf representation to a 0 gives an adjacent vertex to i in $\{0,1,\ldots,k-1\}$. So i has $Z(i)$

neighbors in the subgraph induced by the vertices 0 through $k-1$. Summing the contributions over i gives the lemma. □

It follows that $S_k(n)$ satisfies the recursion

$$S_k(n) = S_k(n-1) + S_k(n-2) + P(k-1). \tag{13}$$

Using (13) and induction on n we directly obtain that for $n \geq n_0$

$$\begin{aligned} S_k(n) = {} & (P(k-1) + S(n_0-2))f_{t+3} \\ & + (S(n_0-1) - S(n_0-2))f_{t+2} - P(k-1), \end{aligned} \tag{14}$$

where $t = n - n_0$.

Numerical values of the number of squares for small values of k and n can be found in Table 1, where the entries given in boldface are the number of squares in the Fibonacci cube Γ_n itself. Note that this sequence for $n \geq 0$ is $0, 0, 0, 1, 3, 9, 22, 51, 111, 233, 474, 942, \ldots$ and these numbers are the triple convolution of the Fibonacci numbers. Their generating function is $\frac{x^3}{(1-x-x^2)^3}$ and the closed formula is given explicitly in [7] as $\frac{1}{50}((5n+1)(n-2)f_n + 6nf_{n-2})$.

Let $Q_d^k(n)$ denote the number of d-dimensional hypercubes in Γ_n^k. Thus using this notation, $Q_1^k(n) = E_k(n)$ and $Q_2^k(n) = S_k(n)$ as they appear in (3) and (13), respectively. Furthermore, let $P_d(k-1, n)$ denote the number of d-dimensional hypercubes contained in the subgraph of Γ_n^k induced by the vertices with labels $0, 1, \ldots, k-1$. Note that with this notation, $P_1(k-1, n) = P(k-1)$ as it appears in recursion (14) and $P_0(k-1, n) = k$, as it appears as the nonhomogeneous part of recursion (4).

Table 1. Counting squares in Γ_n^k.

$k\backslash n$	3	4	5	6	7	8	9	10	Closed form	$n_0(k)$
2	**1**	2	4	7	12	20	33	54	$f_{n-1} + f_{n-2} - 1$	4
3	**1**	**3**	6	11	19	32	53	87	$3f_{n-2} + 4f_{n-3} - 2$	5
4	**1**	**3**	7	13	23	39	65	107	$4f_{n-2} + 4f_{n-3} - 3$	5
5	**1**	**3**	**9**	17	31	53	89	147	$8f_{n-3} + 12f_{n-4} - 5$	6
6	**1**	**3**	**9**	18	33	57	96	159	$9f_{n-3} + 12f_{n-4} - 6$	6
7	**1**	**3**	**9**	20	37	65	110	183	$11f_{n-3} + 12f_{n-4} - 8$	6
8	**1**	**3**	**9**	**22**	41	73	124	207	$19f_{n-4} + 31f_{n-5} - 10$	7
9	**1**	**3**	**9**	**22**	42	75	128	214	$20f_{n-4} + 31f_{n-5} - 11$	7
10	**1**	**3**	**9**	**22**	44	79	136	228	$22f_{n-4} + 31f_{n-5} - 13$	7
11	**1**	**3**	**9**	**22**	46	83	144	242	$24f_{n-4} + 31f_{n-5} - 15$	7
12	**1**	**3**	**9**	**22**	48	87	152	256	$26f_{n-4} + 31f_{n-5} - 17$	7
13	**1**	**3**	**9**	**22**	**51**	93	164	277	$42f_{n-5} + 73f_{n-6} - 20$	8
14	**1**	**3**	**9**	**22**	**51**	94	166	281	$43f_{n-5} + 73f_{n-6} - 21$	8

Proposition 9. *Let $Q_d^k(n)$ be the number of d-dimensional hypercubes in Γ_n^k, then $Q_d^k(n)$ satisfies the recurrence relation*

$$Q_d^k(n) = Q_d^k(n-1) + Q_d^k(n-2) + P_{d-1}(k-1)$$

where

$$P_{d-1}(k-1) = \sum_{i=0}^{k-1} \binom{Z(i)}{d-1}.$$

Proof. There are three types of d-dimensional hypercubes that contribute to $Q_d^k(n)$: those coming from Γ_{n-1}^k, those coming from Γ_{n-2}^k, and the ones formed by the k link vertices used in the construction of Γ_n^k. The d-dimensional hypercubes of the last type are counted by the number of $(d-1)$-dimensional hypercubes contained in the subgraph of Γ_{n-1}^k induced by the vertices with labels $0, 1, \ldots, k-1$. For any of these vertices i we need to select $d-1$ ones among the $Z(i)$ ones in i. Then by varying these $d-1$ ones we obtain 2^{d-1} vertices with labels in $\{0, 1, \ldots, k-1\}$ each giving a $(d-1)$-dimensional hypercube in Γ_{n-1}^k. Therefore, a simple generalization of Lemma 8 gives the number of such hypercubes as

$$P_{d-1}(k-1, n-1) = \sum_{i=0}^{k-1} \binom{Z(i)}{d-1}.$$

We can denote the above quantity as $P_{d-1}(k-1)$ as the sum on the right does not depend on n. This completes the proof. □

Let $Q_d(n)$ denote the number of d-dimensional hypercubes in Γ_n. This number and its q-analogue are determined in [9, Corollary 3.3] and [16, Proposition 3], respectively. Using these result we have the following.

Corollary 10. *Let $Q_d(n)$ denote the number of d-dimensional hypercubes in Γ_n. Then $Q_d(n)$ is given explicitly by*

$$Q_d(n) = \sum_{i=d}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i+1}{i} \binom{i}{d}.$$

In particular, we have the formulas for the first few dimensions d as follows:

$$Q_1(n) = E(n) = \frac{1}{5}(2(n+1)f_n + nf_{n+1}),$$

$$Q_2(n) = S(n) = \frac{1}{50}((5n+1)(n-2)f_n + 6nf_{n-2}),$$

$$Q_3(n) = \frac{n(n-2)}{150}(4(n-4)f_{n-3} + 3(n-3)f_{n-4}).$$

Using Corollary 10 and by solving the recurrence relation satisfied by $Q_d^k(n)$ given in Proposition 9 by induction, similar to the case of Proposition 1 and (14), we obtain

Theorem 11. *Let $Q_d^k(n)$ denote the number of d-dimensional hypercubes in Γ_n^k. Then*

$$Q_d^k(n) = (P_{d-1}(k-1) + Q_d(n_0 - 2))f_{t+3} \\ + (Q_d(n_0 - 1) - Q_d(n_0 - 2))f_{t+2} - P_{d-1}(k-1),$$

for $n \geq n_0$ with $t = n - n_0$, where $Q_d(n)$ denotes the number of d-dimensional hypercubes in Γ_n.

5. Remarks

We end this work with three remarks. The first one is on a number of additional graph theoretic properties of Γ_n^k. After that we present an observation on the computation of the partial sum $P(k)$ defined in (12) for a given k. Finally we consider a class of special subgraphs of Fibonacci cubes which are inspired by the Γ_n^k.

Remark 12. The diameter and the radius of Γ_n^k are directly related to the same properties of the Fibonacci cubes themselves. We start by noting the following nested structure of k-Fibonacci cubes.

For non-negative integers n and k we know that Γ_n^k can be obtained directly from Γ_n. It is either equal to Γ_n, or for $n \geq n_0$, it is obtained from Γ_n by removing certain edges. Furthermore, for $n \geq n_0$, Γ_n^k can also be obtained from Γ_n^{k+1} by removing edges. Therefore

$$\Gamma_n^1 \subseteq \cdots \subseteq \Gamma_n^k \subseteq \cdots \subseteq \Gamma_n. \tag{15}$$

We know that Γ_n^1 is a tree with root 0^n (the vertex with integer label 0). It follows that for $u, v \in V(\Gamma_n^1)$

$$d(u,v) \leq d(u, 0^n) + d(v, 0^n) = w_H(u) + w_H(v), \tag{16}$$

where w_H denotes the Hamming weight. We have $diam(\Gamma_n) = n$ as given in [4]. For $n < n_0$ $diam(\Gamma_n^k) = diam(\Gamma_n) = n$. For $n \geq n_0$, we know that Γ_n^k is a subgraph of Γ_n, the vertices of Γ_n^k and Γ_n are the same and Γ_n^k has fewer edges. Therefore, $diam(\Gamma_n^k) \geq diam(\Gamma_n) = n$. On the other hand, using (15) and (16) for any $u, v \in V(\Gamma_n^k)$ we have

$$d(u,v) \leq d(u, 0^n) + d(v, 0^n) = w_H(u) + w_H(v) \leq n.$$

Therefore, we have $diam(\Gamma_n^k) = n$.

By a similar argument one can show that the radius of Γ_n^k is equal to the radius of Γ_n, which is obtained in [15] as $\lceil \frac{n}{2} \rceil$.

Remark 13. To our knowledge there is no direct formula for the partial sum $P(k)$ defined in (12) for a given k. The trivial recurrence

$$P(k) = P(k-1) + Z(k)$$

seems to require the computation of the Zeckendorf representation of every number up to k. Interestingly, the structure of the edges in k-Fibonacci cubes allows for a way to compute $P(k)$.

Assume that $k = f_m$. Then we know that the vertices of $\Gamma_{m-2} = \Gamma^k_{m-2}$ are labeled with the Zeckendorf representation of the integers $0, 1, \ldots, k-1$. From Lemma 8 we observe that $P(k-1)$ equals to the number of edges of the Γ_{m-2}, that is,

$$P(k-1) = P(f_m - 1) = |E(\Gamma_{m-2})| = \frac{1}{5}(2(m-1)f_{m-2} + (m-2)f_{m-1}).$$

Now we can easily generalize this idea. Assume that $f_m < k < f_{m+1}$. Then write $k = f_m + r$ for some positive integer r. This means that $P(k-1)$ is equal to the number of edges of the subgraph of Γ_{m-1} induced by the first k vertices labeled by $0, 1, \ldots, k-1$ from Lemma 8. Using the decomposition of $\Gamma_{m-1} = 0\Gamma_{m-2} + 10\Gamma_{m-3}$, this number is equal to the sum of the number of edges in the subgraph of Γ_{m-1} induced by the first f_m vertices labeled by $0, 1, \ldots, f_m - 1$ (which is Γ_{m-2}), the number of edges in the subgraph of Γ_{m-1} induced by the vertices labeled by $f_m, f_m + 1, \ldots, k-1$ (which is equal to the number of edges in the subgraph of Γ_{m-3} induced by the vertices labeled by $0, 1, \ldots, r-1$) and r which is the number of link edges between vertices labeled $0, 1, \ldots, r-1$ and the corresponding vertices in Γ_{m-2}.

Let $k - 1 = \sum_{s=1}^{\ell} f_{i_s}$ denote the Zeckendorf representation of $k-1$ with $i_1 < i_2 < \cdots < i_\ell = m$. We can write $r = 1 + \sum_{s=1}^{\ell-1} f_{i_s}$ since $k = f_m + r$. Then we have

$$\begin{aligned}
P(k-1) &= P(f_m - 1 + r) \\
&= P(f_m - 1) + P(r-1) + r \\
&= \sum_{s=1}^{\ell} P(f_{i_s} - 1) + \sum_{j=1}^{\ell-1} \left(1 + \sum_{s=1}^{\ell-j} f_{i_s}\right) \\
&= \sum_{s=1}^{\ell} |E(\Gamma_{i_s-2})| + \sum_{j=1}^{\ell-1} j \cdot f_{i_{\ell-j}} + r \\
&= \sum_{s=1}^{\ell} \frac{1}{5}(2(i_s - 1)f_{i_s-2} + (i_s - 2)f_{i_s-1}) + \sum_{j=1}^{\ell-1} j \cdot f_{i_{\ell-j}} + r.
\end{aligned}$$

Using this expansion, one can easily compute $P(k-1)$ using the Zeckendorf representation of $k-1$.

Example:

Let $k - 1 = 15$. The Zeckendorf representation of 15 is (100010). Next we need to find the number of edges of the subgraph of Γ_6 induced by the first k vertices labeled $0, 1, \ldots, 15$. By the fundamental decomposition $\Gamma_6 = 0\Gamma_5 + 10\Gamma_4$. Now we can partition the vertex set of this graph as $\{0, 1, \ldots, 12\}$ and $\{13, 14, 15\}$. The subgraph with vertices $\{0, 1, \ldots, 12\}$ is isomorphic to the $\Gamma_5 \subseteq \Gamma_6$ and the subgraph with vertices $\{13, 14, 15\}$ is isomorphic to the $\Gamma_2 \subseteq \Gamma_4 \subseteq \Gamma_6$. Note that there are 3 link edges between these subgraphs. Therefore, the total number of edges that we are looking for is equal to

$$|E(\Gamma_5)| + |E(\Gamma_2)| + 3 = 20 + 2 + 3 = 25,$$

which directly gives that $P(15) = 25$.

Remark 14. In the construction of Γ_n there are f_n link edges between Γ_{n-1} and Γ_{n-2}. In this paper, we considered k-Fibonacci cubes Γ_n^k, in which the number of link edges between Γ_{n-1}^k and Γ_{n-2}^k is a fixed number k for all $n \geq n_0(k)$, where $n_0(k)$ is as given in (1).

A simple variant of these graphs can also be defined by taking the number of link edges between Γ_{n-1} and Γ_{n-2} as a variable instead of a fixed number k. Given $0 \leq \lambda \leq 1$, we loosely define here a class of graphs Γ_n^λ for $n \geq 0$, whose construction closely resembles that of the Fibonacci cubes themselves. We start with $\Gamma_0^\lambda = \Gamma_0$, $\Gamma_1^\lambda = \Gamma_1$. For $n \geq 2$, Γ_n^λ is defined in terms of Γ_{n-1}^λ and Γ_{n-2}^λ, similar to the fundamental decomposition of Γ_n but by taking only the first λf_n link edges that exist between $00\Gamma_{n-2}^\lambda$ and its copy $10\Gamma_{n-2}^\lambda$ in Γ_{n-1}^λ, instead of all f_n of them like we do for Γ_n or first k of them like we do for Γ_n^k.

We have an explicit expression for the number of edges $E(n)$ of Γ_n as given in (2). Let $E_\lambda(n)$ denote the number of edges of Γ_n^λ. Then $E_\lambda(n)$ satisfies the recursion

$$E_\lambda(n) = E_\lambda(n-1) + E_\lambda(n-2) + \lambda f_n$$

for $n \geq 2$ and therefore

$$E_\lambda(n) = (1-\lambda) f_n + \lambda E(n).$$

Asymptotically, the average degree of a vertex in Γ_n is the expression in (5). By taking limits and using (5), the average degree of a vertex in Γ_n^λ is found to be

$$\left(1 - \frac{1}{\sqrt{5}}\right) \lambda n,$$

which reduces to the average degree in Γ_n for $\lambda = 1$, as expected.

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Appendix A. Figures of Some k-Fibonacci Cubes

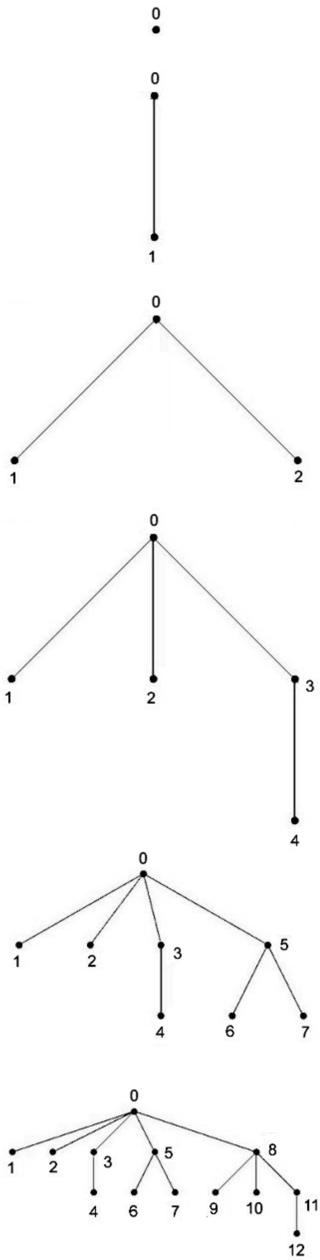

Fig. 7. The first six trees in Fig. 4 redrawn.

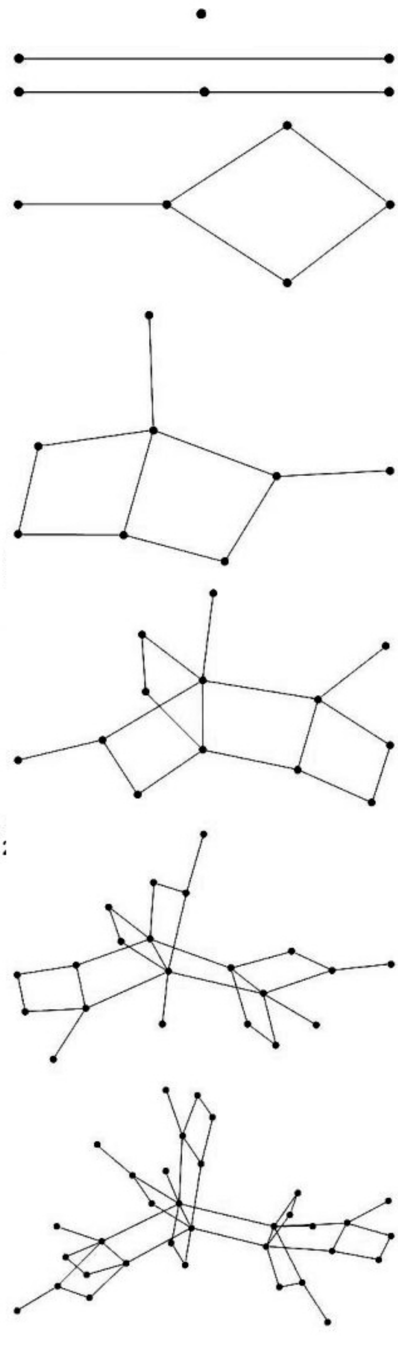

Fig. 8. The first eight k-Fibonacci cubes for $k = 2$.

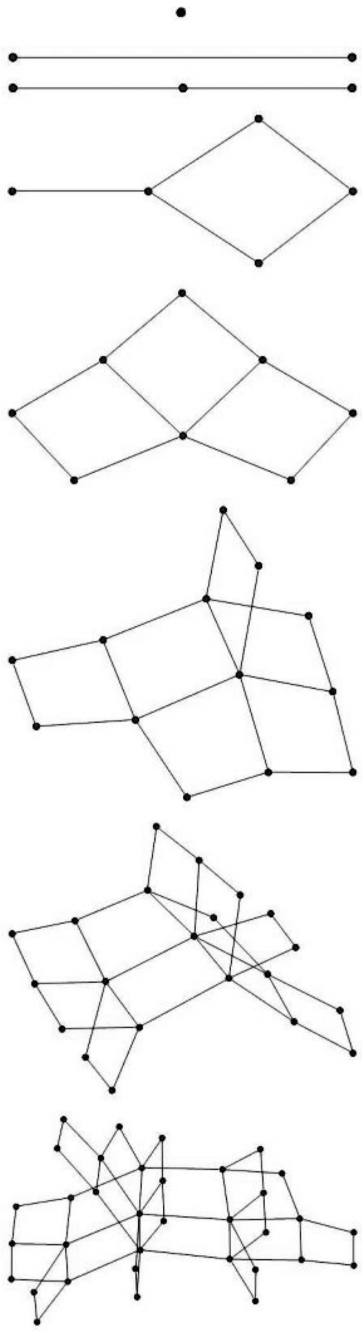

Fig. 9. The first eight k-Fibonacci cubes for $k = 3$.

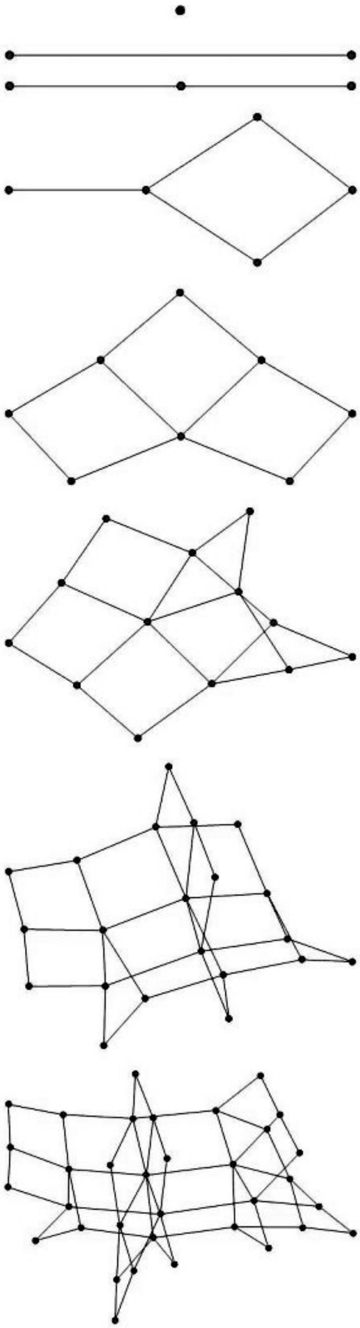

Fig. 10. The first eight k-Fibonacci cubes for $k = 4$.

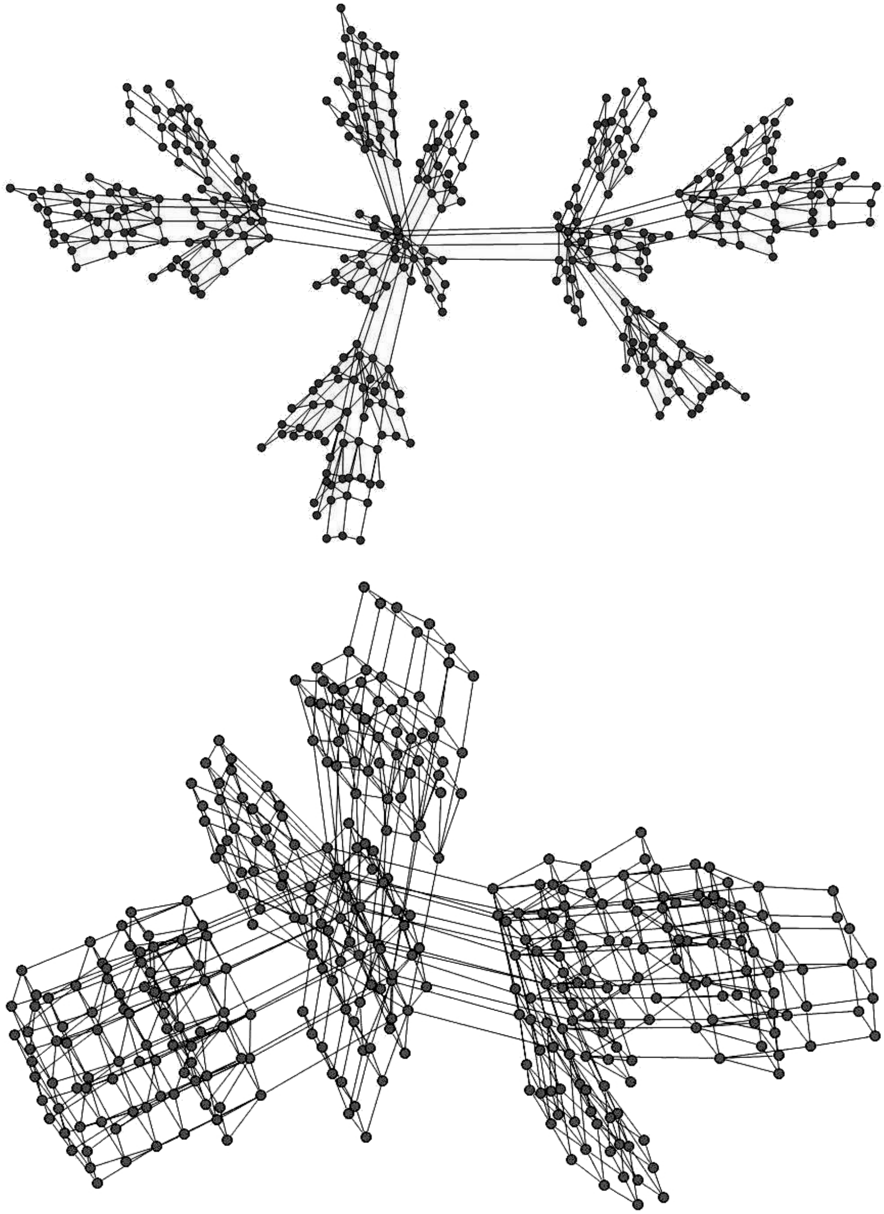

Fig. 11. The structure of Γ_{12}^{4} (top) and Γ_{12}^{13} (bottom).