NORTH-HOLLAND

Parallelogram-Law-Type Identities

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ABSTRACT

Identities generalizing the well-known formula relating the lengths of the sides and the diagonals of a parallelogram in the plane are given. These generalizations all have the flavor of the parallelogram law, and specialize to formulas involving sums of roots of unity, trigonometric functions, binomial coefficients, and permutations over the symmetric and the alternating groups.

The parallelogram law in the complex plane is

$$2\left(|z_1|^2 + |z_2|^2\right) = |z_1 + z_2|^2 + |z_1 - z_2|^2, \tag{1}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, and x_1, y_1, x_2, y_2 are real numbers. The parallelogram law relates the lengths of the diagonals of a parallelogram with vertices $(0,0), (x_1, y_1), (x_2, y_2)$, and $(x_1 + x_2, y_1 + y_2)$ to the lengths of its sides (Figure 1). In general, a Hilbert space is a Banach space whose norm $\|x\|$ satisfies the parallelogram property

$$2\left(\|x\|^2 + \|y\|^2\right) = \|x + y\|^2 + \|x - y\|^2.$$

Consider the binomial identity

$$(n+1)(m+1)^n \binom{2m}{m} = \sum_{0 \le i_0, i_1, \ldots, i_n \le m} \left[(-1)^{i_0}\binom{m}{i_0} + (-1)^{i_1}\binom{m}{i_1} + \cdots + (-1)^{i_n}\binom{m}{i_n}\right]^2, \tag{2}$$

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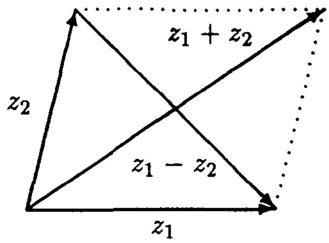

FIG. 1. The parallelogram law in the plane.

and the identity

$$\frac{(n-2)!}{2} n^3 = \sum_{\sigma \in \mathcal{A}_n} |\omega^{1+\sigma_1} + \omega^{2+\sigma_2} + \cdots + \omega^{n+\sigma_n}|^2, \tag{3}$$

where $\mathcal{A}_n$ is the alternating group of degree n and ω is a primitive nth root of unity. It is not immediately clear why (2) or (3) should have any relation to the parallelogram law. However if we first write (1) in an equivalent form as

$$4 \sum_{i=1}^{2} |z_i|^2 = \sum_{\alpha_{i_1}, \alpha_{i_2} \in A} |\alpha_{i_1} z_1 + \alpha_{i_2} z_2|^2 \tag{4}$$

where $A = \{-1, 1\}$, then the right-hand sides of (2), (3), and (4) become sums of squares of norms of certain vectors. Generalizations of complex-number identities of this type based on length-preserving properties of unitary transformations were considered by Klamkin and Murty [6]. In this paper, we take the formulation (4) for the parallelogram law as the starting point, and give elementary proofs as well as a number of specializations of the following theorems of similar flavor.

THEOREM 1. *Assume $A = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is a set of complex numbers with $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 0$. Then for any n complex numbers $z_1, z_2, \ldots, z_n$,*

$$m^{n-1} \left(\sum_{\alpha \in A} |\alpha|^2 \right) \left(\sum_{i=1}^{n} |z_i|^2 \right) = \sum_{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n} \in A} |\alpha_{i_1} z_1 + \alpha_{i_2} z_2 + \cdots + \alpha_{i_n} z_n|^2. \tag{5}$$

THEOREM 2. *Assume $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a set of complex numbers with $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 0$, and let $\mathcal{S}_n$ denote the symmetric group* *of degree n. Then for any n complex numbers $z_1, z_2, \ldots, z_n$,*

$$(n-2)!\left(\sum_{\alpha\in A}|\alpha|^2\right)\left[n\sum_{i=1}^{n}|z_i|^2 - |z_1+z_2+\cdots+z_n|^2\right]$$
$$= \sum_{\sigma\in S_n}|\alpha_{\sigma_1}z_1+\alpha_{\sigma_2}z_2+\cdots+\alpha_{\sigma_n}z_n|^2. \tag{6}$$

More generally,

THEOREM 3. *Assume $A=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ is a set of complex numbers with $\alpha_1+\alpha_2+\cdots+\alpha_n=0$, and let $\mathcal{G}$ be a doubly transitive group of permutations of degree n. Let g_2 denote the size of a stabilizer subgroup of a pair, and let g_1 denote the size of the stabilizer subgroup of a single point. Then for any n complex numbers $z_1, z_2, \ldots, z_n$,*

$$g_2\left(\sum_{\alpha\in A}|\alpha|^2\right)\left[\frac{|\mathcal{G}|}{g_1}\sum_{i=1}^{n}|z_i|^2 - |z_1+z_2+\cdots+z_n|^2\right]$$
$$= \sum_{\sigma\in\mathcal{G}}|\alpha_{\sigma_1}z_1+\alpha_{\sigma_2}z_2+\cdots+\alpha_{\sigma_n}z_n|^2. \tag{7}$$

Before giving the proofs, we look at some special cases. First, some consequences of Theorem 1:

EXAMPLE 1.1. Take $A=\{-1,1\}$ and $n=2$. Then (5) reads

$$\begin{aligned}2\cdot(2)\big(|z_1|^2+|z_2|^2\big) &= |z_1+z_2|^2+|z_1-z_2|^2\\ &\quad+|-z_1-z_2|^2+|-z_1+z_2|^2\\ &= 2|z_1+z_2|^2+2|z_1-z_2|^2,\end{aligned}$$

which simplifies to

$$2\big(|z_1|^2+|z_2|^2\big) = |z_1+z_2|^2+|z_1-z_2|^2.$$

EXAMPLE 1.2. Take $A=\{-1,1\}$ and $n=3$. Then

$$\begin{aligned}8\big(|z_1|^2+|z_2|^2+|z_3|^2\big) &= |z_1+z_2+z_3|^2+|z_1+z_2-z_3|^2\\ &\quad+|z_1-z_2+z_3|^2+|z_1-z_2-z_3|^2\\ &\quad+|-z_1-z_2-z_3|^2+|-z_1-z_2+z_3|^2\\ &\quad+|-z_1+z_2-z_3|^2+|-z_1+z_2+z_3|^2\end{aligned}$$

and therefore

$$4\big(|z_1|^2+|z_2|^2+|z_3|^2\big) = |z_1+z_2+z_3|^2+|z_1+z_2-z_3|^2 \\ +|z_1-z_2+z_3|^2+|z_1-z_2-z_3|^2.$$

EXAMPLE 1.3. Let ω be a primitive cube root of unity and $A = \{1,\omega,\omega^2\}$. For $n=2$ we have

$$\begin{aligned} &3\big(1+|\omega|^2+|\omega^2|^2\big)\big(|z_1|^2+|z_2|^2\big) \\ &= |z_1+z_2|^2+|z_1+\omega z_2|^2+|\omega z_1+z_2|^2 \\ &\quad+|\omega z_1+\omega z_2|^2+\left|\omega z_1+\omega^2 z_2\right|^2+\left|\omega^2 z_1+\omega z_2\right|^2 \\ &\quad+\left|\omega^2 z_1+\omega^2 z_2\right|^2+\left|\omega^2 z_1+z_2\right|^2+\left|z_1+\omega^2 z_2\right|^2. \end{aligned}$$

Thus

$$3\big(|z_1|^2+|z_2|^2\big) = |z_1+z_2|^2+|z_1+\omega z_2|^2+|\omega z_1+z_2|^2.$$

EXAMPLE 1.4. Take A as in Example 1.3 and $n=3$. Then

$$\begin{aligned} &9\big(|z_1|^2+|z_2|^2+|z_3|^3\big) \\ &= |z_1+z_2+z_3|^2+|z_1+z_2+\omega z_3|^2+|z_1+\omega z_2+z_3|^2 \\ &\quad+|\omega z_1-z_2-z_3|^2+\left|z_1+z_2+\omega^2 z_3\right|^2+\left|z_1+\omega^2 z_2+z_3\right|^2 \\ &\quad+\left|\omega^2 z_1+z_2+z_3\right|^2+\left|z_1+\omega z_2+\omega^2 z_3\right|^2+\left|z_1+\omega^2 z_2+\omega z_3\right|^2. \end{aligned}$$

EXAMPLE 1.5. Let

$$A = \left\{(-1)^i\binom{m}{i} \,\middle|\, i=0,1,\ldots,m\right\}, \qquad m \text{ odd.}$$

Then

$$\sum_{\alpha\in A}|\alpha|^2 = \sum_{i=0}^{m}\binom{m}{i}^2 = \binom{2m}{m}$$

by the Vandermonde identity [2, 7]. Thus

$$\begin{aligned} &(m+1)^n\binom{2m}{m}\sum_{i=0}^{n}|z_i|^2 \\ &= \sum_{0\le i_0,i_1,\ldots,i_n\le m}\left|(-1)^{i_0}\binom{m}{i_0}z_0+(-1)^{i_1}\binom{m}{i_1}z_1+\cdots+(-1)^{i_n}\binom{m}{i_n}z_n\right|^2. \end{aligned}$$

In particular, taking $z_0 = z_1 = \cdots = z_n = 1$,

$$(n+1)(m+1)^n \binom{2m}{m} = \sum_{0 \le i_0, i_1, \ldots, i_n \le m} \left[(-1)^{i_0} \binom{m}{i_0} + (-1)^{i_1} \binom{m}{i_1} + \cdots + (-1)^{i_n} \binom{m}{i_n} \right]^2,$$

which is the identity (2), and taking $n = m$ with

$$z_i = \binom{m}{i}, \qquad i = 0, 1, \ldots, m,$$

one has

$$(n+1)^n \binom{2n}{n}^2 = \sum_{0 \le i_0, i_1, \ldots, i_n \le n} \left[(-1)^{i_0} \binom{n}{i_0}\binom{n}{0} + (-1)^{i_1} \binom{n}{i_1}\binom{n}{1} + \cdots + (-1)^{i_n} \binom{n}{i_n}\binom{n}{n} \right]^2.$$

Next we consider a number of special cases of Theorem 2:

EXAMPLE 2.1. Take $A = \{-1, 1\}$. Then (6) reads

$$4\big(|z_1|^2 + |z_2|^2\big) - 2|z_1 + z_2|^2 = |z_1 - z_2|^2 + |-z_1 + z_2|^2.$$

Consequently

$$2\big(|z_1|^2 + |z_2|^2\big) = |z_1 + z_2|^2 + |z_1 - z_2|^2,$$

which is again the parallelogram law for the plane.

EXAMPLE 2.2. Let ω be a primitive nth root of unity and $A = \{1, \omega, \ldots, \omega^{n-1}\}$. Then Theorem 2 gives

$$(n-2)!n^2 \sum_{i=1}^{n} |z_i|^2 = (n-2)!n|z_1 + z_2 + \cdots + z_n|^2 + \sum_{\sigma \in S_n} \left|\omega^{\sigma_1} z_1 + \omega^{\sigma_2} z_2 + \cdots + \omega^{\sigma_n} z_n\right|^2.$$

EXAMPLE 2.3. Let ω be a primitive nth root of unity, and take $z_i = \omega^i$ for $i = 1, 2, \ldots, n$. Then

$$(n-2)!n^2 \sum_{\alpha \in A} |\alpha|^2 = \sum_{\sigma \in S_n} \left|\alpha_{\sigma_1}\omega + \alpha_{\sigma_2}\omega^2 + \cdots + \alpha_{\sigma_n}\omega^n\right|^2 \qquad (8)$$

for any set of complex numbers $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ whose sum is zero. In particular

$$(n-2)!n^3 = \sum_{\sigma \in \mathcal{S}_n} \left| \omega^{1+\sigma_1} + \omega^{2+\sigma_2} + \cdots + \omega^{n+\sigma_n} \right|^2.$$

EXAMPLE 2.4. Let $\alpha_k = z_k = \cos(2\pi k/n)$ for $k = 1, 2, \ldots, n$. Then

$$\sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} z_k = 0.$$

Since

$$\sum_{k=1}^{n} \cos^2 kx = \frac{n}{2} + \frac{\cos(n+1)x \sin nx}{2 \sin x}$$

(see [5], for example), we have

$$\sum_{k=1}^{n} |\alpha_k|^2 = \sum_{k=1}^{n} |z_k|^2 = \frac{n}{2}.$$

This gives the identity

$$(n-2)!\frac{n^3}{4} = \sum_{\sigma \in \mathcal{S}_n} \left[\cos \frac{2\pi\sigma_1}{n} \cos \frac{2\pi}{n} + \cos \frac{2\pi\sigma_2}{n} \cos \frac{4\pi}{n} + \cdots + \cos \frac{2\pi\sigma_n}{n} \cos \frac{2n\pi}{n} \right]^2.$$

Similarly, since

$$\sum_{k=1}^{n} \sin^2 kx = \frac{n}{2} - \frac{\cos{(n+1)x} \sin nx}{2 \sin x},$$

by taking $\alpha_k = \cos{(2\pi k/n)}, z_k = \sin{(2\pi k/n)}$ we obtain the trigonometric identity

$$(n-2)!\frac{n^3}{4} = \sum_{\sigma \in \mathcal{S}_n} \left[\sin \frac{2\pi\sigma_1}{n} \cos \frac{2\pi}{n} + \sin \frac{2\pi\sigma_2}{n} \cos \frac{4\pi}{n} + \cdots + \sin \frac{2\pi\sigma_n}{n} \cos \frac{2n\pi}{n} \right]^2,$$

and by taking $\alpha_k = \sin{(2\pi k/n)}, z_k = \sin{(2\pi k/n)}$ for $k = 1, 2, \ldots, n$, we obtain the identity

$$(n-2)!\frac{n^3}{4} = \sum_{\sigma \in \mathcal{S}_n} \left[\sin \frac{2\pi\sigma_1}{n} \sin \frac{2\pi}{n} + \sin \frac{2\pi\sigma_2}{n} \sin \frac{4\pi}{n} + \cdots + \sin \frac{2\pi\sigma_n}{n} \sin \frac{2n\pi}{n} \right]^2.$$

EXAMPLE 2.5. Let

$$A = \left\{ (-1)^i \binom{n}{i} \,\middle|\, i = 0, 1, \ldots, n \right\}, \qquad n \text{ odd}.$$

If we take

$$z_i = (-1)^i \binom{n}{i}, \qquad i = 0, 1, \ldots, n,$$

and use the Vandermonde convolution identity, we find that

$$\begin{aligned}&\frac{(n+1)!}{n}\binom{2n}{n}^2\\ &= \sum_{\sigma \in \mathcal{S}_{\{0,1,\ldots,n\}}} \left[(-1)^{\sigma_0}\binom{n}{\sigma_0}\binom{n}{0} + (-1)^{1+\sigma_1}\binom{n}{\sigma_1}\binom{n}{1} + \cdots + (-1)^{n+\sigma_n}\binom{n}{\sigma_n}\binom{n}{n}\right]^2,\end{aligned}$$

where $\mathcal{S}_{\{0,1,\ldots,n\}}$ denotes the permutation group on $\{0, 1, \ldots, n\}$. Taking

$$z_i = \binom{n}{i}, \qquad i = 0, 1, \ldots, n,$$

we obtain

$$\begin{aligned}&\frac{(n+1)!}{n}\binom{2n}{n}\left[\binom{2n}{n} - 4^n\right]\\ &= \sum_{\sigma \in \mathcal{S}_{\{0,1,\ldots,n\}}} \left[(-1)^{\sigma_0}\binom{n}{\sigma_0}\binom{n}{0} + (-1)^{\sigma_1}\binom{n}{\sigma_1}\binom{n}{1} + \cdots + (-1)^{\sigma_n}\binom{n}{\sigma_n}\binom{n}{n}\right]^2.\end{aligned}$$

Finally we consider some special cases of Theorem 3:

EXAMPLE 3.1. Let $\mathcal{G} = \mathcal{S}_n$. Then $g_2 = (n-2)!, g_1 = (n-1)!$, and Theorem 3 specializes to Theorem 2.

EXAMPLE 3.2. Take $\alpha_i = z_i = \omega^i$ for $i = 1, 2, \ldots, n$, where ω is a primitive nth root of unity. Then

$$n^2(g_1 + g_2) = \sum_{\sigma \in \mathcal{G}} |\omega^{1+\sigma_1} + \omega^{2+\sigma_2} + \cdots + \omega^{n+\sigma_n}|^2$$

for any doubly transitive group of permutations $\mathcal{G}$ of degree n.

EXAMPLE 3.3. Let $\mathcal{G} = \mathcal{A}_n$ be the alternating group of degree n. Then $g_2 = (n-2)!/2, g_1 = (n-1)!/2$, and (7) becomes

$$\begin{aligned}&\frac{(n-2)!}{2}\left(\sum_{\alpha \in A} |\alpha|^2\right)\left[n\sum_{i=1}^{n} |z_i|^2 - |z_1 + z_2 + \cdots + z_n|^2\right]\\ &\quad = \sum_{\sigma \in \mathcal{A}_n} |\alpha_{\sigma_1} z_1 + \alpha_{\sigma_2} z_2 + \cdots + \alpha_{\sigma_n} z_n|^2.\end{aligned}$$

Taking $z_i = \omega^i$ for $i = 1, 2, \ldots, n$, where ω is a primitive nth root of unity, analogous to (8) we obtain

$$\frac{(n-2)!}{2} n^2 \sum_{\alpha \in A} |\alpha|^2 = \sum_{\sigma \in \mathcal{A}_n} \left| \alpha_{\sigma_1} \omega + \alpha_{\sigma_2} \omega^2 + \cdots + \alpha_{\sigma_n} \omega^n \right|^2$$

for any set of complex numbers $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ whose sum is zero. In particular

$$\frac{(n-2)!}{2} n^3 = \sum_{\sigma \in \mathcal{A}_n} \left| \omega^{1+\sigma_1} + \omega^{2+\sigma_2} + \cdots + \omega^{n+\sigma_n} \right|^2,$$

which is the identity given in (3).

Now we give proofs of Theorems 1–3. These proofs are essentially based on the fact that unitary transformations are length-preserving.

Proof of Theorem 1. Consider the $m^n \times n$ matrix $\mathbf{M}$ whose rows consists of all distinct vectors $(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n})$ with $\alpha_{i_k} \in A$, $k = 1, 2, \ldots, n$. Let $\mathbf{z} = (z_1, z_2, \ldots, z_n)^T$. Then the right-hand side of (5) is simply

$$\|\mathbf{M}\,\mathbf{z}\|^2. \tag{9}$$

Let $\mathbf{u} = (u_1, u_2, \ldots, u_{m^n})^T$ and $\mathbf{v} = (v_1, v_2, \ldots, v_{m^n})^T$ denote two distinct columns of $\mathbf{M}$. Let $\chi(S)$ be the indicator of the statement S: $\chi(S) = 1$ if S is true and $\chi(S) = 0$ if S is false. Then for any $\alpha \in A$

$$\sum_{i=1}^{m^n} \chi(\alpha = u_i) = m^{n-1}.$$

Similarly, given $\alpha, \beta \in A$,

$$\sum_{i=1}^{m^n} \chi(\alpha = u_i)\chi(\beta = v_i) = m^{n-2}.$$

Now

$$\langle \mathbf{u},\, \mathbf{v} \rangle = \sum_{i=1}^{m^n} u_i \overline{v_i} = \sum_{\alpha \in A} \sum_{\beta \in A} \sum_{i=1}^{m^n} \alpha \overline{\beta} \chi(\alpha = u_i)\chi(\beta = v_i).$$

Thus

$$\begin{aligned} \langle \mathbf{u},\, \mathbf{v} \rangle &= \sum_{\alpha \in A} \sum_{\beta \in A} \alpha \overline{\beta} \sum_{i=1}^{m^n} \chi(\alpha = u_i)\chi(\beta = v_i) \\ &= \sum_{\alpha \in A} \sum_{\beta \in A} \alpha \overline{\beta} m^{n-2} \\ &= m^{n-2} |\alpha_1 + \alpha_2 + \cdots + \alpha_m|^2 = 0. \end{aligned} \tag{10}$$

On the other hand,

$$\langle \mathbf{u}, \mathbf{u} \rangle = \sum_{\alpha \in A} \sum_{\beta \in A} \alpha \overline{\beta} \sum_{i=1}^{m^n} \chi(\alpha = \beta = u_i)$$

$$= \sum_{\alpha \in A} |\alpha|^2 \sum_{i=1}^{m^n} \chi(\alpha = u_i) = m^{n-1} \sum_{\alpha \in A} |\alpha|^2. \tag{11}$$

It follows that

$$\mathbf{M}^*\mathbf{M} = \left(m^{n-1} \sum_{\alpha \in A} |\alpha|^2 \right) \mathbf{I},$$

where $\mathbf{M}^*$ is the conjugate transpose of $\mathbf{M}$ and $\mathbf{I}$ is the $n \times n$ identity matrix. Since [3]

$$\|\mathbf{M}\,\mathbf{z}\|^2 = \langle \mathbf{M}\,\mathbf{z}, \mathbf{M}\,\mathbf{z} \rangle = \langle \mathbf{M}^*\mathbf{M}\,\mathbf{z}, \mathbf{z} \rangle, \tag{12}$$

combining (10), (11), and (12), we have

$$\|\mathbf{M}\,\mathbf{z}\|^2 = m^{n-1} \left(\sum_{\alpha \in A} |\alpha|^2 \right) \langle \mathbf{z}, \mathbf{z} \rangle,$$

which is the content of Theorem 1. ■

Proof of Theorem 2. For this proof we take $\mathbf{M}$ to be the $n! \times n$ matrix with rows $(\alpha_{\sigma_1}, \alpha_{\sigma_2}, \ldots, \alpha_{\sigma_n})$, for $\sigma \in \mathcal{S}_n$. Let $\mathbf{z} = (z_1, z_2, \ldots, z_n)^T$. The right-hand side of (6) is again given by (9). Let $\mathbf{u} = (u_1, u_2, \ldots, u_{n!})^T$ and $\mathbf{v} = (v_1, v_2, \ldots, v_{n!})^T$ denote two distinct columns of $\mathbf{M}$. Then for any $\alpha \in A$

$$\sum_{i=1}^{n!} \chi(\alpha = u_i) = (n-1)!,$$

and for any distinct pair $\alpha, \beta \in A$,

$$\sum_{i=1}^{n!} \chi(\alpha = u_i)\chi(\beta = v_i) = (n-2)!.$$

Similar to the computation of (10) and (11), we find that

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle &= (n-2)! \sum_{i \neq j} \alpha_i \overline{\alpha_j} \\ &= (n-2)! \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} - (n-2)! \sum_{i=1}^{n} \alpha_i \overline{\alpha_i} \\ &= (n-2)! |\alpha_1 + \alpha_2 + \cdots + \alpha_n|^2 - (n-2)! \sum_{\alpha \in A} |\alpha|^2 \\ &= -(n-2)! \sum_{\alpha \in A} |\alpha|^2\end{aligned}$$

and

$$\langle \mathbf{u}, \mathbf{u}\rangle = (n-1)! \sum_{\alpha \in A} |\alpha|^2.$$

Thus

$$\mathbf{M}^*\mathbf{M} = \left((n-2)! \sum_{\alpha \in A} |\alpha|^2 \right) (n\mathbf{I} - \mathbf{J}),$$

where $\mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{J}$ is the $n \times n$ matrix of 1's. Since

$$\langle \mathbf{J}\,\mathbf{z}, \mathbf{z}\rangle = |z_1 + z_2 + \cdots + z_n|^2,$$

we have

$$\begin{aligned}\|\mathbf{M}\,\mathbf{z}\|^2 &= \langle \mathbf{M}\,\mathbf{z}, \mathbf{M}\,\mathbf{z}\rangle = \langle \mathbf{M}^*\mathbf{M}\,\mathbf{z}, \mathbf{z}\rangle \\ &= \left((n-2)! \sum_{\alpha \in A} |\alpha|^2 \right) \langle (n\mathbf{I} - \mathbf{J})\mathbf{z}, \mathbf{z}\rangle\end{aligned}$$

Consequently

$$\|\mathbf{Mz}\|^2 = n\left((n-2)! \sum_{\alpha \in A} |\alpha|^2 \right) \sum_{i=1}^{n} |z_i|^2 - \left((n-2)! \sum_{\alpha \in A} |\alpha|^2 \right) |z_1 + z_2 + \cdots + z_n|^2,$$

which proves Theorem 2. ∎

Proof of Theorem 3. If $\mathcal{G}$ is doubly transitive, then the stabilizers of pairs of points are all conjugate subgroups of $\mathcal{G}$. Similarly, the subgroups fixing a point are all conjugates. Thus it makes sense to talk about g_2 and

g_1. Let $\mathbf{M}$ to be the $|\mathcal{G}| \times n$ matrix with rows $(\alpha_{\sigma_1}, \alpha_{\sigma_2}, \ldots, \alpha_{\sigma_n})$, for $\sigma \in \mathcal{G}$. Then for any two column vectors $\mathbf{u}, \mathbf{v}$ of $\mathbf{M}$ and $\alpha \in A$

$$\sum_{i=1}^{|\mathcal{G}|} \chi(\alpha = u_i) = g_1,$$

and for any distinct pair $\alpha, \beta \in A$,

$$\sum_{i=1}^{|\mathcal{G}|} \chi(\alpha = u_i)\chi(\beta = u_i) = g_2.$$

In this case we compute that

$$\langle \mathbf{u},\, \mathbf{v}\rangle = -g_2 \sum_{\alpha \in A} |\alpha|^2 \quad \textit{and} \quad \langle \mathbf{u},\, \mathbf{u}\rangle = g_1 \sum_{\alpha \in A} |\alpha|^2.$$

Thus

$$\mathbf{M}^*\mathbf{M} = \left(\sum_{\alpha \in A} |\alpha|^2 \right) [(g_1 + g_2)\mathbf{I} - g_2\mathbf{J}].$$

Therefore

$$\begin{aligned} \|\mathbf{M}\,\mathbf{z}\|^2 &= \langle \mathbf{M}^*\mathbf{M}\,\mathbf{z}, \mathbf{z}\rangle \\ &= \left(\sum_{\alpha \in A} |\alpha|^2 \right) \left((g_1 + g_2) \sum_{i=1}^{n} |z_i|^2 - g_2 |z_1 + z_2 + \cdots + z_n|^2 \right). \end{aligned}$$

Now Theorem 3 follows from the relation

$$g_1 + g_2 = \frac{g_2}{g_1} |\mathcal{G}|$$

satisfied by every doubly transitive permutation group $\mathcal{G}$ [4]. ∎

REMARKS. The identities derived here are of the same type as consequences of a general theorem of Brauer and Coxeter [1]:

THEOREM 4. *Suppose $\mathcal{G}$ is an absolutely irreducible finite group of homogeneous linear transformations of an n-dimensional complex vector space U. Pick an h-dimensional subspace V_1 together with its complementary subspace W_1, and suppose the pairs $\{(V_1, W_1), (V_2, W_2), \ldots, (V_k, W_k)\}$ form an orbit under $\mathcal{G}$. If $\mathbf{p}_i$ denotes the vector of V_i obtained from a given vector $\mathbf{z} \in U$ by projection parallel to W_i, then*

$$\frac{h}{n}\mathbf{z} = \frac{1}{k}(\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_k). \tag{13}$$

As an example of a special case of this result, Brauer and Coxeter obtain Schönhardt's theorem [8] that if a vector $\mathbf{z}$ in the plane is projected orthogonally on the sides of a regular k-gon, then the arithmetic mean of these k projections is $\mathbf{z}/2$. If in Theorem 4 the group $\mathcal{G}$ and the subspace V_1 can be picked in such a way as to guarantee that the subspaces $V_1, V_2, \ldots, V_k$ are pairwise orthogonal, then $\langle \mathbf{p}_i, \mathbf{p}_j \rangle = 0$ for $i \neq j$ and from (13) we obtain

$$\frac{h^2k^2}{n^2}\|\mathbf{z}\|^2 = \sum_{i=1}^{k} \|\mathbf{p}_i\|^2,$$

which would furnish further identities of the type given here.

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