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q-cube enumerator polynomial of Fibonacci cubes

Elif Saygı [a,*], Ömer Eğecioğlu [b]

[a] Department of Mathematics and Science Education, Hacettepe University, 06800, Beytepe, Ankara, Turkey

[b] Department of Computer Science, University of California Santa Barbara, Santa Barbara, CA 93106, USA

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ABSTRACT

We consider a q-analogue of the cube polynomial of Fibonacci cubes. These bivariate polynomials satisfy a recurrence relation similar to the standard one. They refine the count of the number of hypercubes of a given dimension in Fibonacci cubes by keeping track of the distances of the hypercubes to the all 0 vertex. For $q = 1$, they specialize to the standard cube polynomials.

We also investigate the divisibility properties of the q-analogues and show that the quotient polynomials for the appropriate indices have nonnegative integral polynomials in q as coefficients. These results have many corollaries which include expressions involving the q-analogues of the Fibonacci numbers themselves and their convolutions as they relate to hypercubes in Fibonacci cubes. Many of our developments can be viewed as refinements of enumerative results given by Klavžar and Mollard in (2012).

1. Introduction

A graph $G = (V, E)$ with vertex set V and edge set E can be used to represent an interconnection network. In this representation, V denotes the processors and E denotes the communication links between processors. The hypercube graph Q_n of dimension n is one of the basic model for interconnection networks. The vertices of Q_n are represented by all binary strings of length n and two vertices are adjacent if and only if they differ in exactly one position. The graph distance between two vertices of a graph is the length of the shortest path connecting these vertices. In Q_n, the graph distance between two vertices is given by the Hamming distance between the corresponding binary strings; this is the number of different bits of their binary representations. In Q_n, the weight of a vertex is defined as the number of ones in the corresponding string, that is, the Hamming weight of the string.

In [12], Fibonacci cubes were introduced as a new model of computation for interconnection networks. The Fibonacci cube Γ_n of dimension n is a subgraph of Q_n, where the vertices correspond to those without two consecutive 1s in their string representation. In other words, if we label the vertices of Γ_n ($n \geq 1$) by using binary strings $b_1 b_2 \ldots b_n$ of length n, then the vertices of Γ_n have the property that $b_i b_{i+1} = 0$ for all $i \in \{1, 2, \ldots, n-1\}$. For convenience, Γ_0 is defined as Q_0, the graph with a single vertex and no edges.

In the literature, many interesting properties and applications of the Fibonacci cubes are presented. Their usage as interconnection networks and properties that are important in network design are given in [12,7]. In [13], the usage in theoretical chemistry and some results on the structure of Fibonacci cubes, including representations, recursive construction, hamiltonicity, the nature of the degree sequence and some enumeration results are presented. Many additional new properties of Fibonacci cubes are given in the literature, see for example [1,15,17,20]. Furthermore, the structure of the

* Corresponding author.
E-mail addresses: esaygi@hacettepe.edu.tr (E. Saygı), omer@cs.ucsb.edu (Ö. Eğecioğlu).

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hypercubes in Γ_n is studied in detail in the literature. The cube polynomial of Γ_n which is the starting point of this paper is studied in [14] and many interesting related results are obtained. The characterization of maximal induced hypercubes in Γ_n appears in [16]. Maximum number of disjoint subgraphs isomorphic to k-dimensional hypercube Q_k in Γ_n and their ratio to the number of vertices in Γ_n are considered in [8,18].

In this paper, we consider the q-analogue of the cube polynomial of the Fibonacci cubes. Many of our results are extensions of the work of Klavžar and Mollard as our q-cube polynomial $c_n(x; q)$ is a refinement of the cube polynomial $c_n(x)$ given in [14]. Furthermore, the q-analogue adds a geometric meaning to the polynomials; the $c_n(x; q)$ satisfy a simple recursion similar to the recursion for the cube polynomial $c_n(x)$ and have a combinatorial interpretation as enumerators of the hypercubes in Γ_n in which distance information of each hypercubes to the all 0 vertex is cataloged. For example,

$$c_2(x) = 3 + 2x$$

since Γ_2 contains three Q_0's and two Q_1's, whereas

$$c_2(x; q) = 1 + 2q + 2x$$

expresses the fact that two of the three Q_0 are at distance 1 from 00 and the other at distance 0; and both Q_1's are at distance 0 from 00 (i.e. they contain 00).

Certain divisibility properties of the cube polynomials were noted in [14]. Our results extend these divisibility properties and also includes information about the nature of the quotient polynomials. Interestingly, the quotients as polynomials in x have coefficients that are polynomials in q which have nonnegative integral coefficients themselves.

The distance information of the hypercubes in Γ_n maintained in $c_n(x; q)$ also has an interpretation in terms of the ranks when Γ_n is viewed as a subposet of the Boolean algebra Q_n, but this is not the emphasis of the present work.

The paper is organized as follows: In Section 2, we give some preliminaries. We present our q-cube enumerator polynomial in Section 3 and investigate divisibility properties in Section 4. In Section 5, we present additional results including the role of Fibonacci numbers and their q-analogues in the construction of $c_n(x; q)$, and a closed form expression for the simple q-analogue of the hypercube's own subcube enumerator.

2. Preliminaries

In this section, we present some notation and preliminary results related to Fibonacci cubes. We start with the description of a hypercube. An n-dimensional hypercube (or n-cube) Q_n is the simple graph with vertex set

$$V(Q_n) = \{v_1 v_2 \cdots v_n \mid v_i \in \{0, 1\},\ 1 \le i \le n\}.$$

The number of vertices in Q_n is 2^n and the number of these without two consecutive 1s is enumerated by the Fibonacci numbers. From this point of view, Fibonacci cube Γ_n can be considered as a subgraph of Q_n, obtained from Q_n by removing all vertices containing consecutive 1s. The vertex set of Γ_n can be shown as

$$V(\Gamma_n) = \{v_1 v_2 \cdots v_n \mid v_i \in \{0, 1\} \text{ with } v_i v_{i+1} = 0 \text{ for } 1 \le i < n\}. \tag{1}$$

By convention Γ_0 is defined as K_1. The number of vertices of the Fibonacci cube Γ_n is f_n, where $f_0 = 1, f_1 = 2$ and $f_n = f_{n-1} + f_{n-2}$ for $n \ge 2$. These are the Fibonacci numbers F_n shifted by 2: i.e. $f_n = F_{n+2}$ where $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$. In Fig. 1, we present the first 6 Fibonacci cubes with their vertices labeled with the corresponding binary strings in the hypercube graph. Since there is a close relationship between hypercubes and Fibonacci cubes, it is natural to consider the number of k-dimensional hypercubes in Γ_n in more detail. The enumerator of these subcubes in the Fibonacci cube Γ_n was considered in [14]. Here, we are considering a generalization of these polynomials (see, Section 3).

Fibonacci cubes have a useful decomposition, which is called the "fundamental decomposition" in [13]. For $n \ge 1$, the vertex set (1) of Γ_n can be partitioned into two disjoint subsets A_n and B_n as follows:

$$A_n = \{1v \mid v \in B_{n-1}\} \quad \text{and} \quad B_n = \{0v \mid v \in A_{n-1} \cup B_{n-1}\}$$

with $A_0 = \emptyset$ and $B_0 = \{\epsilon \mid \epsilon \text{ is the empty string}\}$. Note that for $n \ge 2$, we know that the label of any vertex in A_n must start with 10 by the definition of Γ_n. From this decomposition, one can see that A_n and B_n induce subgraphs of Γ_n isomorphic to Γ_{n-2} and Γ_{n-1}, respectively. We will show this fundamental decomposition for $n \ge 2$ as

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}. \tag{2}$$

Note that there are edges between $0\Gamma_{n-1}$ and $10\Gamma_{n-2}$, namely a perfect matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$. We will use this property in the proof of Lemma 1.

In this paper, we consider the q-analogue of the Fibonacci numbers given by $F_0(q) = 0$, $F_1(q) = 1$, and

$$F_n(q) = F_{n-1}(q) + qF_{n-2}(q) \tag{3}$$

for $n \ge 2$. Note that for $q = 1$, $F_n(q)$ gives the Fibonacci numbers. Furthermore, this q-analogue is considered in [10] as Jacobsthal polynomial, and it is simpler than the standard one defined by

$$F'_n = F'_{n-1} + q^{n-2} F'_{n-2} \tag{4}$$

due to Schur, which was studied by Carlitz, Cigler, and others in the literature (see, for example [4–6,9]).

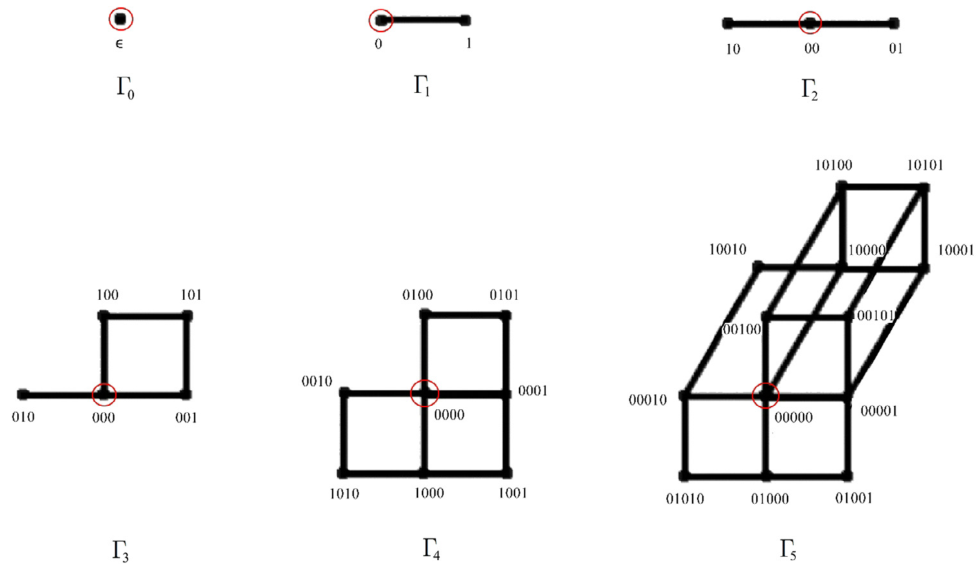

Fig. 1. Fibonacci cubes $\Gamma_0, \Gamma_1, \ldots, \Gamma_5$.

Using (3), first $F_n(q)$ for $n \geq 2$ are computed as

$$1,\ 1+q,\ 1+2q,\ 1+3q+q^2,\ 1+4q+3q^2,\ 1+5q+6q^2+q^3,\ \ldots.$$

It is well known that

$$\sum_{n\geq 0} F_n t^n = \frac{t}{1-t-t^2}.$$

Similarly, one can easily obtain the generating function of $F_n(q)$ (see, for example [10]) as

$$\sum_{n\geq 0} F_n(q) t^n = \frac{t}{1-t-qt^2}. \tag{5}$$

Let $h_{n,k}$ denote the number of k-dimensional hypercubes in the Fibonacci cube Γ_n. Then the cube polynomial, or the cube enumerator polynomial $c_n(x)$ of Γ_n is defined as

$$c_n(x) = \sum_{k\geq 0} h_{n,k} x^k\ . \tag{6}$$

A few of these cube polynomials are given below:

$$\begin{aligned}
c_0(x) &= 1,\\
c_1(x) &= 2+x,\\
c_2(x) &= 3+2x,\\
c_3(x) &= 5+5x+x^2,\\
c_4(x) &= 8+10x+3x^2,\\
c_5(x) &= 13+20x+9x^2+x^3,\\
c_6(x) &= 21+38x+22x^2+4x^3,\\
c_7(x) &= 34+71x+51x^2+14x^3+x^4.
\end{aligned}$$

Evidently, the constant terms are the number of Q_0's, i.e. the number of vertices of Γ_n. Therefore,

$$c_n(0) = f_n = F_{n+2}.$$

Many interesting results on $c_n(x)$ and $h_{n,k}$ in (6) are given in [14]. It can be observed that the numbers in Table 1 satisfy the recursion

$$h_{n,k} = h_{n-1,k} + h_{n-2,k} + h_{n-2,k-1}\ . \tag{7}$$

The first column entries ($k=0$) of the table are $f_0, f_1, f_2, \ldots$ and the diagonal entries are $1, 1, 0, 0, \ldots$. After these, the other entries can be filled row by using the recursion (7).

Next, we use the q-analogue (3) to study generalization of $c_n(x)$.

Table 1
The table of coefficients of the cube polynomials $c_n(x)$ by rows. The entry in row n, column k is the coefficient $h_{n,k}$, the number of k-dimensional hypercubes in Γ_n.

n	k						
	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	2	1	0	0	0	0	0
2	3	2	0	0	0	0	0
3	5	5	1	0	0	0	0
4	8	10	3	0	0	0	0
5	13	20	9	1	0	0	0
6	21	38	22	4	0	0	0
7	34	71	51	14	1	0	0

3. *q*-cube polynomials of the Fibonacci cubes

In this section, we define the polynomials $c_n(x; q)$ of the Fibonacci cube Γ_n. They will be defined in terms of the distance of the corresponding k-dimensional hypercubes to the all 0 vertex for $n \geq 1$. Recall that the distance between two subgraphs of a graph is the smallest graph distance between pairs of vertices taken one from each.

The polynomial $c_n(x; q)$ is defined as the sum of terms of the form $q^d x^k$, one for each hypercube subgraph of Γ_n. The exponent k is the dimension of the hypercube under consideration, and the exponent d is its distance to the all 0 vertex in Γ_n. By convention, we define $c_0(x; q) = 1$.

It is useful to think of $c_n(x; q)$ as a polynomial in x whose coefficients are polynomials in q. First, few $c_n(x; q)$ are as follows:

$$\begin{aligned}
c_0(x; q) &= 1, \\
c_1(x; q) &= 1 + q + x, \\
c_2(x; q) &= 1 + 2q + 2x, \\
c_3(x; q) &= 1 + 3q + q^2 + (3 + 2q)x + x^2, \\
c_4(x; q) &= 1 + 4q + 3q^2 + (4 + 6q)x + 3x^2.
\end{aligned}$$

Now, we illustrate the structure of $c_2(x; q)$ and $c_3(x; q)$ in more detail. Recall that there are 3 vertices and 2 edges in the graph of Γ_2 as in Fig. 1. The 0-dimensional hypercubes are the vertices of the graph. There is a single vertex having distance 0 to the vertex 00 (i.e. 00 itself), and there are 2 vertices having distance 1. Therefore, the coefficient of x^0 in $c_2(x; q)$ is $1+2q$. Similarly, 1-dimensional hypercubes are the edges of the graph, and there are a total of 2 of those, each having distance zero to the vertex 00. Therefore, the coefficient of x^1 is 2. This gives $c_2(x; q) = 1 + 2q + 2x$.

Similarly, to construct $c_3(x; q)$, we consider all hypercubes in Γ_3 having dimension $k < 3$ and their distances to 000. For $k = 0$, we know that there are 5 vertices in the graph giving 0-dimensional hypercubes. The vertex 000 has distance 0, the vertices 010, 100 and 001 each have distance 1 and the vertex 101 has distance 2 to 000. So the coefficient of x^0 is $1+3q+q^2$.

Now, consider $k = 1$, that is, 1-dimensional hypercubes in the graph. We know that they are the edges of the graph and from Fig. 2 we see that there are 3 with distance 0 and 2 with distance 1 to the vertex 000. So the coefficient of x^1 in $c_3(x; q)$ is $3 + 2q$.

Finally, consider $k = 2$. There is only one 2-dimensional hypercube in Γ_3, and this hypercube contains the vertex 000. So the contribution from 2-dimensional subcubes is x^2. Therefore, we get $c_3(x; q) = (1+3q+q^2)+(3+2q)x+x^2$. A graphical presentation of these hypercubes in Γ_3 and their contribution to $c_3(x, q)$ is presented in Fig. 2.

We next determine the generating function for the q-cube polynomial $c_n(x; q)$ and relate it to the q-analogues of the Fibonacci numbers in (3). Before this result however, we present the following recursion which allows for the calculation of the polynomials and which is central to what follows. Note that this result is the q-analogue of [14, Equation (1)].

Lemma 1. *For $n \geq 2$, the q-cube polynomial $c_n(x; q)$ satisfies*

$$c_n(x; q) = c_{n-1}(x; q) + (q + x)c_{n-2}(x; q) \tag{8}$$

with $c_0(x; q) = 1$ and $c_1(x; q) = 1 + q + x$.

Proof. We see that

$$c_2(x; q) = 1 + 2q + 2x = c_1(x; q) + (q + x)c_0(x; q).$$

Now, we can use induction on n. Assume that the recursion holds for $n - 1$. From the fundamental decomposition (2) of Γ_n, we know that $\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}$. Similarly, we can write $0\Gamma_{n-1} = 00\Gamma_{n-2} + 010\Gamma_{n-3}$, and there is a matching between the corresponding vertices of $10\Gamma_{n-2}$ and $00\Gamma_{n-2}$. Furthermore, note that all 0 vertex belongs to $0\Gamma_{n-1}$ (and hence $00\Gamma_{n-2}$) in Γ_n. It follows that there are only three kinds of hypercubes in Γ_n (see also, proof of [14, Proposition 3.1]):

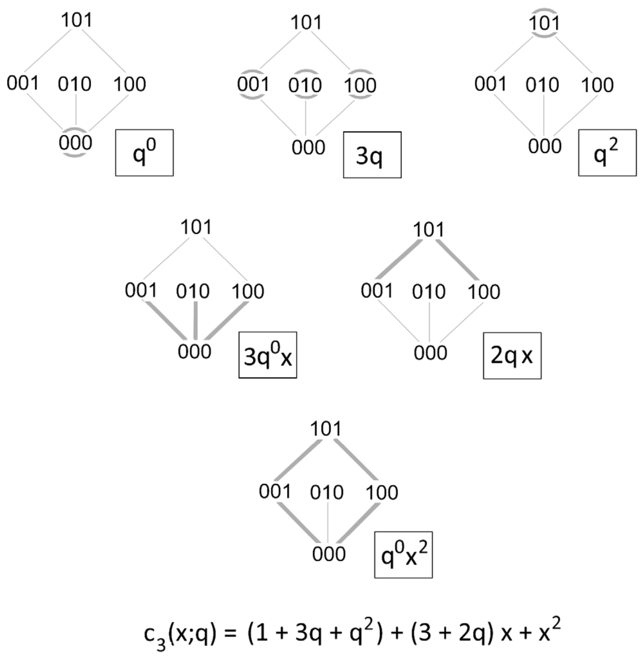

Fig. 2. The elements of the q-cube polynomial $c_3(x; q)$.

Case 1: A k-dimensional hypercube in $0\Gamma_{n-1}$ remains a k-dimensional hypercube in Γ_n and the distances of these cubes to the all 0 vertex remain unchanged. By induction, these are enumerated by $c_{n-1}(x; q)$.

Case 2: Any k-dimensional hypercube in $10\Gamma_{n-2}$ is again a k-dimensional hypercube in Γ_n, and the distances of these cubes in Γ_n to the all 0 vertex go up by 1 due to the edges identifying the corresponding vertices in $10\Gamma_{n-2}$ and $00\Gamma_{n-2}$ (note that, the all 0 vertex is in $00\Gamma_{n-2}$; it is not in $10\Gamma_{n-2}$). This increase in the distance to the all 0 vertex by 1 means multiplication by q, and the contribution of these hypercubes is $qc_{n-2}(x; q)$.

Case 3: A k-dimensional hypercube in $10\Gamma_{n-2}$ has an isomorphic copy in $00\Gamma_{n-2}$ and all the corresponding vertices of these k-dimensional hypercubes are connected by a matching. Therefore these two k-dimensional hypercubes together with the edges connecting them gives a $(k + 1)$-dimensional hypercube in Γ_n. Also, note that the distances of these cubes to the all 0 vertex remain unchanged. The contribution of these hypercubes is $xc_{n-2}(x; q)$, since multiplication by x has the effect of increasing the dimension by 1. Adding up these three contributions, we obtain recursion (8). •

Now, we consider the generating function of the $c_n(x; q)$.

Proposition 1. *The generating function of the q-cube polynomial $c_n(x; q)$ is*

$$\sum_{n\geq 0} c_n(x; q)t^n = \frac{1 + t(q + x)}{1 - t - t^2(q + x)}.$$

Proof. Let $S = \sum_{n\geq 0} c_n(x; q)t^n$. We know that $c_0(x; q) = 1$, $c_1(x; q) = 1 + q + x$ thus, by summation of the recursion (8) for $n \geq 2$, S satisfies

$$S - 1 - t(1 + q + x) = t(S - 1) + t^2(q + x)S,$$

which gives the desired result. •

Next, we solve the recursion in (8) directly to find $c_n(x; q)$ in explicit form.

Theorem 1. *For any nonnegative integer n, the q-cube polynomial $c_n(x; q)$ has degree $\left\lfloor \frac{n+1}{2} \right\rfloor$ in x and*

$$c_n(x; q) = \frac{1}{2^{n+1}} \sum_{i=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n+2}{2i+1} (1 + 4(q + x))^i . \tag{9}$$

Table 2
The table of coefficients of the q-cube polynomials $c_n(x; q)$ by rows. The entry in row n, column k is the coefficient polynomial $h_{n,k}(q)$.

n	k				
	0	1	2	3	4
0	1	0	0	0	0
1	$1+q$	1	0	0	0
2	$1+2q$	2	0	0	0
3	$1+3q+q^2$	$3+2q$	1	0	0
4	$1+4q+3q^2$	$4+6q$	3	0	0
5	$1+5q+6q^2+q^3$	$5+12q+3q^2$	$6+3q$	1	0
6	$1+6q+10q^2+4q^3$	$6+20q+12q^2$	$10+12q$	4	0
7	$1+7q+15q^2+10q^3+q^4$	$7+30q+30q^2+4q^3$	$15+30q+6q^2$	$10+4q$	1

Proof. We know that the characteristic equation of the recursion in (8) is

$$r^2 - r - (q+x) = 0.$$

This equation gives an explicit expression in the form

$$c_n(x; q) = \frac{(1+\theta)^{n+2} - (1-\theta)^{n+2}}{2^{n+2}\theta} \tag{10}$$

where $\theta = \sqrt{1+4(q+x)}$. Using binomial expansions for $(1 \pm \theta)^{n+2}$ and after some algebraic manipulation, we obtain (9). ●

In particular, writing

$$c_n(x; q) = \sum_{k \geq 0} h_{n,k}(q) x^k ,$$

we obtain the following expression for the coefficient polynomials $h_{n,k}(q)$.

Corollary 1. *For any nonnegative integer n, the coefficient polynomials of the q-cube polynomial $c_n(x; q)$ are*

$$h_{n,k}(q) = \frac{1}{2^{n+1}} \left(\frac{4}{1+4q} \right)^k \sum_{i=k}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n+2}{2i+1} \binom{i}{k} (1+4q)^i .$$

A few of the polynomials $h_{n,k}(q)$ are given in Table 2. Here, we note that as a consequence of Lemma 1 the coefficients $h_{n,k}(q)$ satisfy

$$h_{n,k}(q) = h_{n-1,k}(q) + q h_{n-2,k}(q) + h_{n-2,k-1}(q)$$

which is the q-analogue of (7).

Using the properties of convolutions, we obtain the following result relating the coefficient polynomials $h_{n,k}(q)$ of the q-cube $c_n(x; q)$ and the q-analogue of the Fibonacci numbers given in (3).

Proposition 2. *The coefficient polynomials $h_{n,k}(q)$ of the q-cube enumerator $c_n(x; q)$ are given by*

$$h_{n,k}(q) = \sum F_{i_0}(q) F_{i_1}(q) \cdots F_{i_k}(q) \tag{11}$$

where the summation is over all $i_0, i_1, \ldots, i_k \geq 0$ *with* $i_0 + i_1 + \cdots + i_k = n - k + 2$.

Proof. Recall from Proposition 1 that the generating function of the $c_n(x; q)$ is

$$\sum_{n \geq 0} c_n(x; q) t^n = \frac{1 + t(q+x)}{1 - t - t^2(q+x)}. \tag{12}$$

On the other hand, using (5), the $(k+1)$-fold convolutions of the $F_n(q)$ have the generating function

$$\frac{t^{k+1}}{(1-t-qt^2)^{k+1}}.$$

Setting

$$g_k(t; q) = \frac{t^{2k-1}}{(1-t-qt^2)^{k+1}}$$

for $k \geq 1$ with

$$g_0(t;q) = \frac{t^{-1}}{(1-t-qt^2)} - \frac{1}{t}$$

and calculating directly, we find

$$\begin{aligned}\sum_{k\geq 0} g_k(t;q)x^k &= -\frac{1}{t} + \frac{1}{t(1-t-qt^2)} \sum_{k\geq 0}\left(\frac{xt^2}{1-t-qt^2}\right)^k \\ &= \frac{1+t(q+x)}{1-t-t^2(q+x)}.\end{aligned}$$

This is the generating function of the $c_n(x;q)$ of (12). Therefore, the $g_k(t;q)$ are the generating functions of the columns of Table 2. This proves the proposition by equating the coefficients of $t^n x^k$ in the two expressions. •

Note that the coefficient of q^i in polynomials in the first column ($k=0$) of Table 2 corresponds to the number of vertices in Γ_n at distance i from the all 0 vertex, this is clearly equal to the number of Fibonacci words of length n and weight i. So, one can explicitly write (see, for example [10])

$$F_{n+2}(q) = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i+1}{i} q^i. \tag{13}$$

The second column ($k=1$) polynomials are

$$\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i}{i+1} (i+1)\, q^i,$$

and in general, we have the following expression for the entry in row n, column k:

Proposition 3. *The coefficient polynomials $h_{n,k}(q)$ of the q-cube enumerator $c_n(x;q)$ of the Fibonacci cube Γ_n are given explicitly by*

$$h_{n,k}(q) = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i+1-k}{i+k} \binom{i+k}{k} q^i.$$

Note that Proposition 3 is the q-analogue of [14, Corollary 3.3] and an immediate proof is already contained in the alternative proof of [14, Corollary 3.3]. Furthermore, it can be proved directly from the recurrence in (8), by using induction on k and verifying a binomial identity.

Remark 1. From the two different expressions for $h_{n,k}(q)$ in Corollary 1 and Proposition 3, we obtain the following identity for $n \geq 0$ and $k \leq \lfloor \frac{n+1}{2} \rfloor$:

$$\frac{1}{2^{n+1}}\left(\frac{4}{1+4q}\right)^k \sum_{i=k}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+2}{2i+1}\binom{i}{k}(1+4q)^i = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i+1-k}{i+k}\binom{i+k}{k} q^i.$$

Note that, for $k=0$, this equality reduces to [10, Equation (4.2)].

4. Divisibility properties of the q-cube polynomials

In this section, we present divisibility properties of the q-cube polynomials of the Fibonacci cubes. We start by some examples.

The polynomial $c_1(x;q) = 1+q+x$ divides $c_4(x;q),\ c_7(x;q),\ c_{10}(x;q),\ \ldots$ since

$$\begin{aligned} c_4(x;q) &= c_1(x;q)\,(1+3q+3x)\,, \\ c_7(x;q) &= c_1(x;q)\left(1+6q+9q^2+q^3+(6+18q+3q^2)x+(9+3q)x^2+x^3\right), \\ c_{10}(x;q) &= c_1(x;q)\left(1+9q+27q^2+29q^3+6q^4+(9+54q+87q^2+24q^3)x \right.\\ &\quad \left. +(27+87q+36q^2)x^2+(29+24q)x^3+6x^4\right). \end{aligned}$$

Similarly, $c_2(x) = 1 + 2q + 2x$ divides $c_6(x; q)$, $c_{10}(x; q)$, $c_{14}(x; q)$, ... since

$$c_6(x; q) = c_2(x; q)\left(1 + 4q + 2q^2 + (4 + 4q)x + 2x^2\right),$$
$$c_{10}(x; q) = c_2(x; q)\left(1 + 8q + 20q^2 + 16q^3 + 3q^4 + (8 + 40q + 48q^2 + 12q^3)x + (20 + 48q + 18q^2)x^2 + (16 + 12q)x^3 + 3x^4\right).$$

Next, $c_3(x) = 1 + 3q + q^2 + (3 + 2q)x + x^2$ divides $c_8(x; q)$, $c_{13}(x; q)$, $c_{18}(x; q)$, ... since

$$c_8(x; q) = c_3(x; q)\left(1 + 5q + 5q^2 + (5 + 10q)x + 5x^2\right),$$
$$c_{13}(x; q) = c_3(x; q)\left(1 + 10q + 35q^2 + 50q^3 + 25q^4 + q^5 + (10 + 70q + 150q^2 + 100q^3 + 5q^4)x + (35 + 150q + 150q^2 + 10q^3)x^2 + (50 + 100q + 10q^2)x^3 + (25 + 5q)x^4 + x^5\right).$$

The above examples hint at certain divisibility properties for the q-cube polynomials of the Fibonacci cubes. Also, the coefficients of the polynomials x^k on the right hand side seem to be polynomials in q with nonnegative integral coefficients. These observations are proved in the following theorem.

Theorem 2. *For any $m \geq 0$, $c_m(x; q)$ divides $c_{(m+2)n+m}(x; q)$ as a polynomial in x for $n \geq 0$. Furthermore, the coefficients of powers of x in the quotient are polynomials in q with nonnegative integral coefficients.*

Proof. To prove the theorem, we go back to the expression in (10) and write

$$\frac{c_{(m+2)n+m}(x; q)}{c_m(x; q)} = \frac{(1+\theta)^{(m+2)n+m+2} - (1-\theta)^{(m+2)n+m+2}}{2^{(m+2)n}\left((1+\theta)^{m+2} - (1-\theta)^{m+2}\right)} \tag{14}$$

where $\theta = \sqrt{1 + 4(q + x)}$. For a fixed m, denote the quotient on the left of (14) by $\alpha_n(x; q)$. Putting

$$P = (1+\theta)^{m+2} \quad \text{and} \quad Q = (1-\theta)^{m+2}, \tag{15}$$

we can write

$$\alpha_n(x; q) = \frac{P^{n+1} - Q^{n+1}}{2^{(m+2)n}(P - Q)} = \frac{1}{2^{(m+2)n}}\sum_{k=0}^{n} P^k Q^{n-k}. \tag{16}$$

For $n = 0, 1$, we obtain

$$\alpha_0(x; q) = 1, \qquad \alpha_1(x; q) = \frac{P + Q}{2^{m+2}}. \tag{17}$$

Writing

$$\alpha_n(x; q) = \left(\frac{P}{P - Q}\right)\left(\frac{P}{2^{m+2}}\right)^n + \left(\frac{-Q}{P - Q}\right)\left(\frac{Q}{2^{m+2}}\right)^n$$

we see that $\alpha_n(x; q)$ is the solution to the second order linear recursion with characteristic equation

$$\left(r - \frac{P}{2^{m+2}}\right)\left(r - \frac{Q}{2^{m+2}}\right) = 0.$$

This recursion is

$$\alpha_n(x; q) = \left(\frac{P}{2^{m+2}} + \frac{Q}{2^{m+2}}\right)\alpha_{n-1}(x; q) - \left(\frac{PQ}{2^{2m+4}}\right)\alpha_{n-2}(x; q) \tag{18}$$

for $n \geq 2$ with initial values as given in (17). Using (15) and $\theta = \sqrt{1 + 4(q + x)}$, the recurrence (18) can be written directly in terms of x as

$$\alpha_n(x; q) = \alpha_1(x; q)\alpha_{n-1}(x; q) + (-1)^{m+1}(q + x)^{m+2}\alpha_{n-2}(x; q) \tag{19}$$

for $n \geq 2$. Here, $\alpha_0(x;q) = 1$ and

$$\begin{aligned}\alpha_1(x;q) &= \frac{P+Q}{2^{m+2}} \\ &= \frac{1}{2^{m+2}}\left(\left(1+\sqrt{1+4(q+x)}\right)^{m+2} + \left(1-\sqrt{1+4(q+x)}\right)^{m+2}\right) \\ &= \frac{1}{2^{m+1}}\sum_{i\geq 0}\binom{m+2}{2i}\,(1+4(q+x))^i.\end{aligned}$$

Since $\alpha_1(x;q)$ is a polynomial in x, it follows from the recursion (19) that $\alpha_n(x;q)$ is a polynomial in x for all $n \geq 0$.

Next, we show that the coefficients of $\alpha_n(x;q)$ are polynomials in q whose coefficients are nonnegative integers. For integrality, it suffices to show that the coefficients of $\alpha_1(x;q)$ are integral polynomials and then make use of the recursion (19). To do this, we now take into account the dependence on m and write $\alpha_1(x;q)$ as $\beta_m(x;q)$. Thus,

$$\begin{aligned}\beta_m(x;q) &= \frac{P+Q}{2^{m+2}} \\ &= \frac{1}{2^{m+2}}\left(\left(1+\sqrt{1+4(q+x)}\right)^{m+2} + \left(1-\sqrt{1+4(q+x)}\right)^{m+2}\right).\end{aligned}$$

For $m = 0, 1$, we have

$$\beta_0(x;q) = 1+2q+2x, \qquad \beta_1(x;q) = 1+3q+3x\,. \tag{20}$$

Writing

$$\beta_m(x;q) = \frac{1}{4}(1+\sqrt{1+4(q+x)})^2\left(\frac{1+\sqrt{1+4(q+x)}}{2}\right)^m + \frac{1}{4}(1-\sqrt{1+4(q+x)})^2\left(\frac{1-\sqrt{1+4(q+x)}}{2}\right)^m$$

and going through the calculations with the appropriate characteristic equation again, we find that $\beta_m(x;q)$ is the solution to the second degree linear recurrence equation

$$\beta_m(x;q) = \beta_{m-1}(x;q) + (q+x)\beta_{m-2}(x;q)$$

for $m \geq 2$ with initial values as $\beta_0(x;q)$ and $\beta_1(x;q)$ as given in (20). From this recursion, it is evident that the coefficients of $\alpha_1(x;q)$ are polynomials in q and their coefficients are nonnegative integers. It is curious that the recurrence satisfied by the $\beta_m(x;q)$ is the same as the recursion (8) satisfied by the q-cube polynomials $c_n(x;q)$, albeit with different initial conditions.

The coefficient of x^j in $\alpha_1(x;q)$ is explicitly given by

$$\frac{1}{2^{m+1}}\sum_{i\geq 0}\binom{m+2}{2i}\binom{i}{j}\,4^j(1+4q)^{i-j}$$

and by the above reasoning this is always a polynomial in q with integer coefficients.

We proved that the coefficients of all of the $\alpha_n(x;q)$ are polynomials in q with integer coefficients. The nonnegativity of the coefficients is clear from (19) for m odd. For m even, we use the expression (16) for $\alpha_n(x;q)$. It suffices to show that the coefficients polynomials of the powers of x have nonnegative coefficients in

$$\sum_{k=0}^{n} P^k Q^{n-k}. \tag{21}$$

For $2k < n$, consider the pair

$$\begin{aligned}P^kQ^{n-k} + P^{n-k}Q^k &= \left(1-\theta^2\right)^{(m+2)k}\left((1+\theta)^{(m+2)\,(n-2k)} + (1-\theta)^{(m+2)\,(n-2k)}\right) \\ &= (-4(q+x))^{(m+2)k}\sum_{i\geq 0}2\binom{(m+2)\,(n-2k)}{2i}\,(1+4(q+x))^i.\end{aligned}$$

For m even, the coefficients are nonnegative. In case n is even, there is a single central term in the sum (21), which is P^kQ^k for $k = n/2$. For m even, using (15), we obtain that the coefficients of this term are also nonnegative. Therefore, the coefficient polynomials in $\alpha_n(x;q)$ are always sum of polynomials in q with nonnegative coefficients. •

Remark 2. Using (13), the constant term of the quotient can be written as

$$\frac{c_{(m+2)n+m}(0;q)}{c_m(0;q)} = \frac{F_{(m+2)n+m+2}(q)}{F_{m+2}(q)}\,.$$

The integrality of the quotient implies that in particular $F_m(q) \mid F_{mn}(q)$ for $m, n \geq 1$. If we take $q = 1$ this reduces to the well-known divisibility result [19] of Fibonacci numbers $F_m \mid F_{mn}$ for $m, n \geq 1$.

Remark 3. Letting $q = 1$, we obtain from Theorem 2 that for any $m \geq 0$, $c_m(x) \mid c_{(m+2)n+m}(x)$ for $n \geq 0$ which was given in [14, Corollary 6.3, (ii)]. For the numerical case, Theorem 2 also shows that the quotient polynomials have only nonnegative integer coefficients.

5. Observations and specializations

For completeness, we present the enumerator for the original hypercubes and some special results on $c_n(x; 1)$.

5.1. Hypercube's own q-cube enumerator polynomial

We consider a statistic similar to the one we used for Γ_n for the hypercube graph Q_n itself. Our result is the q-analogue of the cube polynomial for hypercubes (see [3,14]). Identify Q_n with the Hasse diagram of the poset of all binary strings of length n with 0^n at the bottom and 1^n at the top and where the covering relation is flipping a 0 to a 1 (This is the lattice of subsets of an n-element set.) The rank of a vertex $v = v_1 v_2 \cdots v_n$ is its Hamming weight. Denote this by $|v|_1$. A k-dimensional subcube of Q_n can be identified with a k-subset of the set of indices $1, 2, \ldots, n$ which is allowed to vary all possible ways. For any k-dimensional subcube H of Q_n, assign the weight

$$w(H) = q^i x^k$$

where i is the smallest rank of the vertices of H. This is the same as the distance between H and the all 0 vertex when we consider the Hasse diagram of Q_n as a graph. For the Fibonacci cube, we have obtained in Theorem 1 that

$$\sum_H w(H) = c_n(x; q) = \frac{1}{2^{n+1}} \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+2}{2i+1} (1 + 4(q+x))^i.$$

The expression for the hypercube Q_n itself is simpler.

Proposition 4. *For the n-dimensional hypercube graph Q_n,*

$$\sum_H w(H) = (1 + q + x)^n,$$

where the summation is over all subcubes H of Q_n.

Proof. If H is k-dimensional, then its vertices are of the form

$$v_0 x_1 v_1 x_2 \cdots x_k v_k$$

for $x_1, \ldots, x_k \in \{0, 1\}$ and $v_0 \cdots v_k$ a fixed string of length $n - k$ over $\{0, 1\}$. Clearly, the lowest ranked vertex in H has $x_1 = \cdots = x_k = 0$, and the highest one has $x_1 = \cdots = x_k = 1$. Therefore,

$$\begin{aligned} \sum_H w(H) &= \sum_{k=0}^{n} x^k \sum_{\substack{v_0, \ldots, v_k \in \{0,1\}^* \\ |v_0| + \cdots + |v_k| = n-k}} q^{|v_0 \cdots v_k|_1} \\ &= \sum_{k=0}^{n} x^k \sum_{i \geq 0} \binom{n}{k} \binom{n-k}{i} q^i = (1 + q + x)^n. \quad \bullet \end{aligned}$$

5.2. Results for $q = 1$

In this section, we present some special results for the case $q = 1$, which is the case considered in [14]. Besides the known results, we also present some new ones.

The constant term in $c_n(x; 1)$ is the number of vertices f_n of Γ_n, which is obtained by taking $x = 0$ in (9). Curiously, this gives

$$f_n = \frac{1}{2^{n+1}} \sum_{i \geq 0} \binom{n+2}{2i+1} 5^i.$$

In terms of the Fibonacci numbers $F_n = f_{n-2}$, we obtain the following well known formula (see, for example Identity 91 in [19] or Identity 235 in [2]).

Proposition 5. *The Fibonacci numbers* $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ *are given by*

$$F_n = \frac{1}{2^{n-1}} \sum_{i \geq 0} \binom{n}{2i+1} 5^i.$$

Also, from the expression (9) for the $c_n(x; q)$, we immediately get the specialization

$$c_n\left(-\frac{1+4q}{4}; q\right) = \frac{n+2}{2^{n+1}}.$$

This can of course be obtained from the original recursion (8) by setting $x = -\frac{1+4q}{4}$ and solving the resulting recursion. Also note that the equality $c_n\left(-\frac{5}{4}; 1\right) = \frac{n+2}{2^{n+1}}$ is given in [14, p. 103]. The values $a_n = c_n(1; 1)$ satisfy the recurrence

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 1, a_1 = 3$, giving the Jacobstahl sequence [11]:

$$1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, \ldots.$$

Remark 4. Evidently, the quotients of the q-cube polynomials carry interesting combinatorial information as the coefficients polynomials are all integral and have nonnegative coefficients. It should also be possible to pursue this venue of investigation for Lucas cubes for their q-analogues. In another direction, the q-cube polynomials themselves can be further refined by making use of the q-Fibonacci numbers defined by (4) instead of the ones in (3) that we have used. They would then carry extra information concerning the subcubes of Γ_n, in terms of their creation history when we look at the repeated fundamental decomposition of each Γ_n into Γ_{n-1} and Γ_{n-2}. There are also interesting questions we are considering which are related to the interpretation of the distances of subcubes to the all 0 vertex in Γ_n as rank information when the graphs are viewed as partially ordered sets.

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