



q -counting geodesics in the Fibonacci cube

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ARTICLE INFO

Article history:

Received 22 January 2026

Received in revised form 23 September 2026

Accepted 24 September 2026

Keywords:

Fibonacci cube

Hypercube

Geodesic

Alternating permutation

Euler number

q -analogue

ABSTRACT

We study geodesics in the Fibonacci cube Γ_n . For any two vertices $x, y \in V(\Gamma_n)$, we show that the number of geodesics from x to y factors as a multinomial coefficient times a product of Euler numbers, reflecting a decomposition of the differing coordinates into consecutive blocks. This count is a special case of a natural q -analogue based on the flip-time map of a geodesic. In this general case, the generating function factors into a q -multinomial coefficient and a product of q -Euler polynomials. In the simpler case of the hypercube graph, the statistic yields the classical q -factorial.

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1. Introduction

The n -dimensional hypercube Q_n is the simple graph with vertex set

$$V(Q_n) = \{b_1 b_2 \cdots b_n \mid b_i \in \{0, 1\}, 1 \leq i \leq n\}$$

where edges join vertices that differ in exactly one bit. The n -dimensional Fibonacci cube Γ_n is a subgraph of Q_n with vertex set $V(\Gamma_n)$ corresponding to those in Q_n whose binary string representation contains no two consecutive 1s. Therefore the vertices of Γ_n have the property that $b_i b_{i+1} = 0$ for all $i \in \{1, 2, \dots, n-1\}$. For convenience Γ_0 is defined as the graph with a single vertex and no edges.

$V(\Gamma_n)$ is enumerated by the Fibonacci number F_{n+2} , where $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Since its introduction, the Fibonacci cube has served as a rich combinatorial model connecting graph theory, coding theory, and enumerative combinatorics. Many interesting properties of this family of graphs have been studied in the literature. In [2,6], their application as an interconnection network, along with properties important in network design, were presented. An exposition of their recursive structure and additional properties can be found in the survey paper [7]. For a more recent general reference on Fibonacci cubes and their applications, see [4].

A fundamental problem in the study of Fibonacci cubes is the enumeration of shortest paths (geodesics) between two vertices of Γ_n . In the special case of diametral vertices – that is, a pair of vertices that are farthest apart in the graph – the number of such paths is given by an Euler number [5].

In this paper we first extend this phenomenon from diametral pairs to arbitrary vertices. Given $x, y \in V(\Gamma_n)$, the set of coordinates on which they differ decomposes naturally into maximal consecutive blocks. Each block contributes an Euler number as in the diametral case, while the independent interleaving of blocks contributes a multinomial coefficient. This yields a general multinomial-Euler formula for the number of geodesics.

This formula is a special case of a refinement. Using the flip-time map, we keep track of the relative order in which the differing coordinates are flipped along a geodesic. The refinement weights each geodesic by an inversion statistic derived

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Table 1
A geodesic from $x = 01010101$ to $y = 10101010$ in Γ_8 .

Step	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
$y = v_8$	1	0	1	0	1	0	1	0
v_7	1	0	1	0	1	0	0	0
v_6	1	0	0	0	1	0	0	0
v_5	1	0	0	0	1	0	0	1
v_4	0	0	0	0	1	0	0	1
v_3	0	1	0	0	1	0	0	1
v_2	0	1	0	0	0	0	0	1
v_1	0	1	0	0	0	1	0	1
$x = v_0$	0	1	0	1	0	1	0	1

from its flip times, counting only the inversions forced by the block admissibility constraints. The resulting generating function factors into a q -multinomial coefficient times a product of q -Euler polynomials [9,10].

We note that the detailed examples given in Section 9 of this work illustrate various concepts introduced in the body of the paper.

2. Alternating permutations and Euler numbers

Following [9], a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ of $[n]$ is *alternating* if $\sigma_1 > \sigma_2 < \sigma_3 > \cdots$. We denote the set of alternating permutations of $[n] = \{1, 2, \ldots, n\}$ by $\text{Alt}(n)$. A permutation is *reverse-alternating* if $\sigma_1 < \sigma_2 > \sigma_3 < \cdots$, and we denote this set by $\text{RAlt}(n)$. Let $E_n = \#\text{Alt}(n) = \#\text{RAlt}(n)$ denote the Euler number, with $E_0 = 1$.

Désiré André [1] proved that the exponential generating function of Euler numbers is

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x = 1 + x + \frac{x^2}{2!} + 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + 16\frac{x^5}{5!} + \cdots \quad (1)$$

The sequence (A000111 in the OEIS [8]) starts with

$$1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, \dots$$

3. Diametral paths in Γ_n

The relationship between diametral paths in the Fibonacci cube and alternating permutations was discussed in [5]. A *diametral path* in Γ_n is a shortest path between two vertices whose distance equals the diameter of Γ_n . Hsu noted in [6] that $\text{diam}(\Gamma_n) = n$, and therefore these special geodesic paths are of length n . As paths in Γ_n , they realize the maximum possible Hamming distance permitted by the Fibonacci cube's no-consecutive-1s constraint. It is known that for every n , there is a unique diametral pair of vertices in Γ_n [5, Proposition 1].

For completeness, the construction for the case of Γ_8 in which the unique diametral pair is $x = 01010101$, $y = 10101010$ is included here. We view the edges on each geodesic to be directed from x to y . For a path

$$x = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_8 = y,$$

each vertex v_{i+1} is obtained from v_i by flipping a 0 to a 1, or a 1 to a 0, with the proviso that no consecutive 1s appear in any v_i .

Writing x at the bottom row and y at the top, the i th step in Table 1 shows the vertex v_i on the path after i edges on the path have been traversed. In this representation, both the path and the row indices are read from bottom to top.

Marking the first 1 in each odd-indexed column and the first 0 in each even-indexed column as we go up the table gives the visual representation of the corresponding alternating permutation shown in Table 2.

4. Flip-sequences of geodesics

For $x, y \in V(\Gamma_n)$, let

$$D = D(x, y) = \{i : x_i \neq y_i\} \quad (2)$$

with $m = |D(x, y)|$. Every geodesic P from x to y in Γ_n has length m . P is uniquely encoded by its *flip sequence*

$$\pi(P) = (p_1, p_2, \dots, p_m),$$

where p_t is the coordinate flipped at step t .

As an example, the geodesic in Table 1 has flip sequence $(4, 6, 5, 2, 1, 8, 3, 7)$.

Table 2

The alternating permutation 5 4 7 1 3 2 8 6 corresponding to the diametral path in Table 1.

Step	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
$v = v_8$	1	0	1	0	1	0	1	0
v_7	1	0	1	0	1	0	0	0
v_6	1	0	0	0	1	0	0	0
v_5	1	0	0	0	1	0	0	1
v_4	0	0	0	0	1	0	0	1
v_3	0	1	0	0	1	0	0	1
v_2	0	1	0	0	0	0	0	1
v_1	0	1	0	0	0	1	0	1
$u = v_0$	0	1	0	1	0	1	0	1

5. Block decomposition and the multinomial–Euler formula

For arbitrary $x, y \in V(\Gamma_n)$, define $D(x, y)$ as in (2). Consider the path graph on the vertex set $[n]$ and edges $\{i, i + 1\}$ for $1 \leq i \leq n - 1$. The induced subgraph whose vertex set is given by $D(x, y)$ is a disjoint union of path components. This is equivalent to the unique decomposition of the set $D(x, y)$ into maximal intervals of consecutive indices

$$D = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_b, \quad |I_j| = m_j, \quad \sum_{j=1}^b m_j = m, \quad (3)$$

ordered so that every coordinate in block I_r is smaller than every coordinate in block I_s whenever $r < s$.

Remark 5.1 (Block Orientation). Here we record the orientation rule which is a consequence in the single-block treatment of [5] where the block is the whole coordinate set $[n]$. Let $I = \{t, t + 1, \dots, t + k - 1\} \subseteq D(x, y)$ be one of the blocks in the decomposition (3). In a geodesic from x to y in Γ_n , consider the value of the bit x_t in $x = x_1 x_2 \cdots x_n$ at the step the path first begins to flip coordinates in I .

- If $x_t = 0$, then the admissible internal flip orders in I are encoded by $\text{Alt}(k)$.
- If $x_t = 1$, then the admissible internal flip orders in I are encoded by $\text{RAlt}(k)$.

In either case the number of admissible internal orders on I is E_k .

Theorem 5.2 (Geodesics in Γ_n). The number of geodesics between $x, y \in V(\Gamma_n)$ is given by

$$\#\text{Geo}(x, y) = \binom{m}{m_1, m_2, \dots, m_b} \prod_{j=1}^b E_{m_j}. \quad (4)$$

Proof. A geodesic flips exactly the coordinates in D , hence it consists of m_j flips in block I_j for each j . By the single-block bijections in [5], each block I_j contributes E_{m_j} admissible internal flip orders of alternating type $\text{Alt}(m_j)$ or reverse-alternating type $\text{RAlt}(m_j)$, depending on the block orientation, but counted by E_{m_j} in either case. Because blocks are separated, flips in distinct blocks do not affect the local admissibility conditions in other blocks. Therefore the choices of internal orders are independent and may be interleaved arbitrarily. The possible number of interleavings of sequences of lengths $m_1, m_2, \dots, m_b$ is given by the multinomial coefficient in (4). $\square$

6. Flip–time statistics

Let P be a geodesic with flip sequence $\pi(P) = (p_1, p_2, \dots, p_m)$. Define the *block word*

$$w(P) = w_1 w_2 \cdots w_m$$

by setting $w_t = j$ iff $p_t \in I_j$. The inversion number $\text{inv}(w(P))$ records the inversions created by shuffling steps from different blocks.

For a geodesic P from x to y , define the *flip time map*

$$\tau = \tau_P : D(x, y) \longrightarrow [m]$$

by setting $\tau(i) = t$ if and only if the coordinate i is flipped at step t of P , that is, if $p_t = i$ in the flip sequence $\pi(P)$.

For each block $I_i = \{t_i, t_i + 1, \dots, t_i + m_i - 1\}$, define the *block time word*

$$\tau_i(P) = (\tau(t_i), \tau(t_i + 1), \dots, \tau(t_i + m_i - 1)),$$

i.e., the flip times of the coordinates in I_i read in increasing index order. Standardizing $\tau_i(P)$ (by replacing its smallest entry by 1, the next smallest by 2, and so on, preserving relative order), yields a permutation $\text{std}(\tau_i(P)) \in S_{m_i}$. We denote the inversion number of this permutation by $\text{inv}(\text{std}(\tau_i(P)))$.

Remark 6.1. A geodesic P from x to y may be described by choosing, for each block I_j , an admissible internal order (that is, a choice of $\text{std}(\tau_j(P)) \in \text{Alt}(m_j)$ or $\text{RAlt}(m_j)$ according to Remark 5.1), and then shuffling these blockwise orders. Equivalently, in terms of flip times, the standardized block time permutation $\text{std}(\tau_j(P))$ is alternating or reverse-alternating according to the orientation of I_j , while the interleaving of blocks is encoded by the block word $w(P)$.

Next, we define the statistic we will use on geodesics in Γ_n .

Definition 6.2.

$$\text{inv}_{\text{flip}}(P) = \text{inv}(w(P)) + \sum_{i=1}^b \text{inv}(\text{std}(\tau_i(P))). \quad (5)$$

This statistic depends on the flip sequence $\pi(P)$ through the induced flip time map τ_P and the block structure. In particular, $\text{inv}_{\text{flip}}(P)$ is not the ordinary inversion number of $\pi(P)$. It records only those inversions imposed by block admissibility constraints.

Concerning the first summand in (5), this is precisely MacMahon's standard combinatorial interpretation of the q -multinomial coefficient as the generating function of inversion numbers of words with prescribed multiplicities (see, for example, [3, Theorem 2.6.1]).

Lemma 6.3 (*q -Multinomial as Inversion Enumerator*). Let $m_1, m_2, \dots, m_b$ be nonnegative integers with sum m . Then

$$\sum_w q^{\text{inv}(w)} = \binom{m}{m_1, m_2, \dots, m_b}_q = \frac{[m]_q!}{[m_1]_q! [m_2]_q! \cdots [m_b]_q!},$$

where the sum ranges over all words $w \in [b]^m$ in which letter j occurs exactly m_j times.

7. The hypercube q -analogue

Definition 7.1. For a positive integer n , the q -factorial polynomial $[n]_q!$ is defined by

$$[n]_q! := \prod_{i=1}^n \frac{1 - q^i}{1 - q} = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1})$$

with $[0]_q! := 1$.

We briefly record what the flip-time inversion statistic yields in Q_n , where no block decomposition is present, for comparison with the Fibonacci cube.

Let $x, y \in Q_n$ and let

$$D(x, y) = \{i_1 < i_2 < \cdots < i_m\}, \quad m = H(x, y).$$

Every geodesic P from x to y flips exactly the coordinates in $D(x, y)$, each exactly once.

Define the flip-time map

$$\tau_P : D(x, y) \longrightarrow [m],$$

where $\tau_P(i) = t$ if and only if coordinate i is flipped at step t . Since the hypercube has no adjacency constraints, the set $D(x, y)$ forms a single unconstrained block. Reading the flip times in index order,

$$\tau_1(P) = (\tau_P(i_1), \tau_P(i_2), \dots, \tau_P(i_m)),$$

the hypercube flip-time inversion statistic becomes

$$\text{inv}_Q(P) = \text{inv}(\text{std}(\tau_1(P))).$$

Theorem 7.2 (*Hypercube q -Factorial Formula*). For any $x, y \in Q_n$ with $H(x, y) = m$,

$$\sum_{P \in \text{Geo}_{Q_n}(x, y)} q^{\text{inv}_Q(P)} = [m]_q!. \quad (6)$$

Proof. Because the hypercube has no adjacency constraints, every ordering of the flip times is admissible. Thus, as P ranges over all geodesics from x to y , the standardized flip-time permutation $\text{std}(\tau_1(P))$ ranges over all permutations in S_m . Summing $q^{\text{inv}(\pi)}$ over all $\pi \in S_m$ yields the classical Mahonian generating function in (6). $\square$

Remark 7.3. The hypercube case represents the degenerate situation in which the block decomposition consists of a single unconstrained block. Accordingly, the flip-time statistic reduces to the ordinary inversion number on permutations, and no Euler numbers appear.

7.1. q -Euler numbers

The q -Euler numbers (polynomials) $E_n(q)$ may be defined as the inversion generating function of alternating permutations by setting

$$E_n(q) = E_n^{\text{Alt}}(q) = \sum_{\sigma \in \text{Alt}(n)} q^{\text{inv}(\sigma)}, \quad E_n^{\text{RAlt}}(q) = \sum_{\sigma \in \text{RAlt}(n)} q^{\text{inv}(\sigma)}.$$

By equidistribution, $E_n(q)$ (with some notational differences) coincides with the classical major-index generating function of alternating permutations [10, Chapter 1]. There is also an exponential generating function for the sequence $E_n(q)$ due to Stanley, in which the usual trigonometric series in (1) are replaced by q -trigonometric series [9,10].

The first few of the polynomials $E_n(q)$ are as follows:

$$E_1(q) = 1,$$

$$E_2(q) = q,$$

$$E_3(q) = q + q^2,$$

$$E_4(q) = q^2 + q^3 + 2q^4 + q^5,$$

$$E_5(q) = q^2 + 2q^3 + 3q^4 + 4q^5 + 3q^6 + 2q^7 + q^8,$$

$$E_6(q) = q^3 + 2q^4 + 5q^5 + 7q^6 + 9q^7 + 10q^8 + 10q^9 + 8q^{10} + 5q^{11} + 3q^{12} + q^{13}.$$

Via the complement map on permutations we have the reciprocity relation

$$E_n^{\text{RAlt}}(q) = q^{\binom{n}{2}} E_n(q^{-1}). \quad (7)$$

8. A multinomial-Euler q -formula

Theorem 8.1 (Flip-Time q -Multinomial-Euler Formula). Let $x, y \in V(\Gamma_n)$ with block decomposition

$$D(x, y) = I_1 \sqcup \cdots \sqcup I_b, \quad |I_i| = m_i,$$

and block orientations as in Remark 5.1. Then

$$\sum_{P \in \text{Geo}(x, y)} q^{\text{inv}_{\text{flip}}(P)} = \binom{m}{m_1, \dots, m_b}_q \prod_{i=1}^b \begin{cases} E_{m_i}^{\text{Alt}}(q), & I_i \text{ alternating,} \\ E_{m_i}^{\text{RAlt}}(q), & I_i \text{ reverse-alternating.} \end{cases}$$

Proof. At $q = 1$, geodesics from x to y are obtained by choosing:

- a shuffle of the blocks $I_1, I_2, \dots, I_b$, and
- for each block I_i , a permutation in $\text{Alt}(m_i)$ or $\text{RAlt}(m_i)$ according to its orientation.

The statistic inv_{flip} splits additively as in (5) into a shuffle part $\text{inv}(w(P))$ and independent internal parts $\text{inv}(\text{std}(\tau_i(P)))$. Summing over all choices yields the stated product. $\square$

The indices $1, 2, \dots, b$ of the blocks can be partitioned into two subsets A and R , where the indices in A are those for which the block is alternating, and the ones in R are the ones in which the block is reverse-alternating. We can write A and R in terms of the bit structure of $x = x_1 x_2 \cdots x_n \in V(\Gamma_n)$. If t_i denotes the minimum element of I_i for $i \in [b]$, then

$$A = \{i \in [b] : x_{t_i} = 0\} \quad R = \{i \in [b] : x_{t_i} = 1\}.$$

With this notation, the theorem can be written as

$$\sum_{P \in \text{Geo}(x, y)} q^{\text{inv}_{\text{flip}}(P)} = \binom{m}{m_1, \dots, m_b}_q \prod_{i \in A} E_{m_i}^{\text{Alt}}(q) \prod_{i \in R} E_{m_i}^{\text{RAlt}}(q).$$

Using the reciprocity relation (7), we can restate Theorem 8.1 as

Corollary 8.2 (Alternate Form of [Theorem 8.1](#)).

$$\sum_{P \in \text{Geo}(x,y)} q^{\text{inv}_{\text{flip}}(P)} = \binom{m}{m_1, \dots, m_b}_q q^{\sum_{i \in R} \binom{m_i}{2}} \prod_{i \in A} E_{m_i}(q) \prod_{j \in R} E_{m_j}(q^{-1}).$$

9. Examples

9.1. A two-block example

Let $x, y \in V(\Gamma_8)$ be given by

$$y = 01010001,$$

$$x = 00100010.$$

Then

$$D(x, y) = \{2, 3, 4, 7, 8\} = I_1 \sqcup I_2, \quad I_1 = \{2, 3, 4\}, \quad I_2 = \{7, 8\},$$

so $m_1 = 3, m_2 = 2$ and $m = 5$. Since $x_2 = 0$ and $x_7 = 1$, the block orientations are

$$A = \{1\}, \quad R = \{2\}.$$

Therefore [Corollary 8.2](#) gives

$$\sum_{P \in \text{Geo}(x,y)} q^{\text{inv}_{\text{flip}}(P)} = \binom{5}{3, 2}_q q^{\binom{2}{2}} E_3(q) E_2(q^{-1}) = \binom{5}{3, 2}_q q E_3(q) E_2(q^{-1}).$$

Since $E_3(q) = q + q^2$ and $E_2(q) = q$, we have $q E_2(q^{-1}) = q \cdot q^{-1} = 1$, and hence

$$\sum_{P \in \text{Geo}(x,y)} q^{\text{inv}_{\text{flip}}(P)} = \binom{5}{3, 2}_q (q + q^2).$$

Using [Lemma 6.3](#) and expanding,

$$\sum_{P \in \text{Geo}(x,y)} q^{\text{inv}_{\text{flip}}(P)} = q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 3q^6 + 2q^7 + q^8. \quad (8)$$

We now interpret this expansion using the flip-time definition of inv_{flip} . For a geodesic P with flip sequence $\pi(P) = (p_1, \dots, p_5)$, let $\tau_P : D(x, y) \rightarrow [5]$ be the flip-time map, i.e., $\tau_P(i) = t$ if and only if $p_t = i$. For each block I_i , recall that

$$\tau_i(P) = (\tau(t_i), \tau(t_i + 1), \dots, \tau(t_i + m_i - 1))$$

is the list of flip times in that block, read in increasing index order, and the internal contribution of the block is $\text{inv}(\text{std}(\tau_i(P)))$.

In this example the block decomposition is

$$I_1 = \{2, 3, 4\}, \quad I_2 = \{7, 8\},$$

with I_1 alternating and I_2 reverse-alternating. For $I_2 = \{7, 8\}$ with reverse-alternating orientation, admissibility forces the flip at coordinate 7 to occur before the flip at coordinate 8 on every geodesic. Hence

$$\tau(7) < \tau(8), \quad \text{std}(\tau_2(P)) = 12, \quad \text{inv}(\text{std}(\tau_2(P))) = 0$$

for every geodesic P .

For the block $I_1 = \{2, 3, 4\}$, admissibility of a geodesic P implies that the flip-time map τ_P satisfies

$$\tau(2) > \tau(3) \quad \text{and} \quad \tau(3) < \tau(4),$$

since I_1 is oriented alternating. Thus there are exactly two possible relative orders of $(\tau(2), \tau(3), \tau(4))$ when read in index order $(2, 3, 4)$. After standardization these give

$$\text{std}(\tau_1(P)) = 213 \quad (\text{inv} = 1) \quad \text{or} \quad \text{std}(\tau_1(P)) = 312 \quad (\text{inv} = 2).$$

Thus, for every geodesic P , $\text{inv}_{\text{flip}}(P) = \text{inv}(w(P)) + 1$ or $\text{inv}_{\text{flip}}(P) = \text{inv}(w(P)) + 2$.

We now list all geodesics P from x to y , and show how each term in (8) arises. In [Table 3](#), $w(P)$ is the block word with 1 encoding the flips in I_1 and 2 encoding the flips in I_2 .

Every geodesic P from x to y is obtained by shuffling the fixed sequence $(7, 8)$ with one of the two admissible internal orders on $I_1 = \{2, 3, 4\}$, namely $(3, 2, 4)$ or $(3, 4, 2)$. There are $\binom{5}{3} = 10$ such shuffles for each choice, yielding 20 geodesics in total. The statistic $\text{inv}_{\text{flip}}(P)$ splits as the sum of the shuffle inversion $\text{inv}(w(P))$ and the internal contribution $\text{inv}(\text{std}(\tau_1(P))) \in \{1, 2\}$.

Reading off the last column, we obtain exactly one geodesic with $\text{inv}_{\text{flip}}(P) = 1$, two with $\text{inv}_{\text{flip}}(P) = 2$, three with $\text{inv}_{\text{flip}}(P) = 3$, and so on, producing the coefficients in (8).

Table 3
Geodesics for the two-block example (with the flip-time statistic).

$\pi(P)$	$w(P)$	$\text{inv}(w(P))$	$\text{inv}(\text{std}(\tau_1(P)))$	$\text{inv}_{\text{flip}}(P)$
(3, 2, 4, 7, 8)	11 122	0	1	1
(3, 2, 7, 4, 8)	11 212	1	1	2
(3, 4, 2, 7, 8)	11 122	0	2	2
(3, 2, 7, 8, 4)	11 221	2	1	3
(3, 7, 2, 4, 8)	12 112	2	1	3
(3, 4, 7, 2, 8)	11 212	1	2	3
(3, 7, 2, 8, 4)	12 121	3	1	4
(7, 3, 2, 4, 8)	21 112	3	1	4
(3, 4, 7, 8, 2)	11 221	2	2	4
(3, 7, 4, 2, 8)	12 112	2	2	4
(3, 7, 8, 2, 4)	12 211	4	1	5
(7, 3, 2, 8, 4)	21 121	4	1	5
(3, 7, 4, 8, 2)	12 121	3	2	5
(7, 3, 4, 2, 8)	21 112	3	2	5
(7, 3, 8, 2, 4)	21 211	5	1	6
(3, 7, 8, 4, 2)	12 211	4	2	6
(7, 3, 4, 8, 2)	21 121	4	2	6
(7, 8, 3, 2, 4)	22 111	6	1	7
(7, 3, 8, 4, 2)	21 211	5	2	7
(7, 8, 3, 4, 2)	22 111	6	2	8

9.2. A three-block example

Let x and y be the following vertices in Γ_{12} .

$$y = 010100010010,$$

$$x = 001000100000.$$

Then

$$D(x, y) = \{2, 3, 4, 7, 8, 11\} = I_1 \sqcup I_2 \sqcup I_3,$$

with

$$I_1 = \{2, 3, 4\}, \quad I_2 = \{7, 8\}, \quad I_3 = \{11\},$$

so $m_1 = 3$, $m_2 = 2$, $m_3 = 1$ and $m = 6$. Here $x_2 = x_{11} = 0$, therefore I_1 and I_3 are oriented alternating, while $x_7 = 1$ implies that I_2 is oriented reverse-alternating.

By [Corollary 8.2](#)

$$\sum_{P \in \text{Geo}(x, y)} q^{\text{inv}_{\text{flip}}(P)} = \binom{6}{3, 2, 1}_q q^{\binom{2}{2}} E_3(q) E_2(q^{-1}) E_1(q).$$

Since $E_3(q) = q + q^2$, $E_2(q) = q$, and $E_1(q) = 1$, we have $q^{\binom{2}{2}} E_2(q^{-1}) = q \cdot q^{-1} = 1$, and hence

$$\sum_{P \in \text{Geo}(x, y)} q^{\text{inv}_{\text{flip}}(P)} = (q + q^2) \binom{6}{3, 2, 1}_q.$$

Using [Lemma 6.3](#) and expanding,

$$\begin{aligned} \sum_{P \in \text{Geo}(x, y)} q^{\text{inv}_{\text{flip}}(P)} &= q + 3q^2 + 6q^3 + 10q^4 + 14q^5 + 17q^6 + 18q^7 + \\ &\quad 17q^8 + 14q^9 + 10q^{10} + 6q^{11} + 3q^{12} + q^{13}. \end{aligned}$$

The coefficient of q^2 is 3, so there must be exactly three geodesics P with $\text{inv}_{\text{flip}}(P) = 2$. We now verify this directly using the flip-time statistic.

The block decomposition is

$$I_1 = \{2, 3, 4\}, \quad I_2 = \{7, 8\}, \quad I_3 = \{11\},$$

with I_1 alternating, I_2 reverse-alternating, and I_3 trivial. For any geodesic P , let $\tau_P : D(x, y) \rightarrow [6]$ be the flip-time map, i.e., $\tau_P(i) = t$ if and only if coordinate i is flipped at step t .

Table 4

The flip sequence, block word, and the decomposition of $\text{inv}_{\text{flip}}(P) = 2$ for the three geodesics from $x = 001000100000$ to $y = 010100010010$ in Γ_{12} .

$\pi(P)$	$w(P)$	$\text{inv}(w(P))$	$\text{inv}(\text{std}(\tau_1(P)))$	$\text{inv}_{\text{flip}}(P)$
(3, 4, 2, 7, 8, 11)	111223	0	2	2
(3, 2, 7, 4, 8, 11)	112123	1	1	2
(3, 2, 4, 7, 11, 8)	111232	1	1	2

Table 5

Vertices along the three geodesics from $x = 001000100000$ (bottom) to $y = 010100010010$ (top) with $\text{inv}_{\text{flip}}(P) = 2$.

Step	$P_1 : (3, 4, 2, 7, 8, 11)$	$P_2 : (3, 2, 7, 4, 8, 11)$	$P_3 : (3, 2, 4, 7, 11, 8)$
$y = v_6$	010100010010	010100010010	010100010010
v_5	010100010000	010100010000	010100000010
v_4	010100000000	010100000000	010100000000
v_3	010100100000	010000000000	010100100000
v_2	000100100000	010000100000	010000100000
v_1	000000100000	000000100000	000000100000
$x = v_0$	001000100000	001000100000	001000100000

For $I_2 = \{7, 8\}$, reverse-alternating orientation forces the flip at 7 (to change $1 \rightarrow 0$) to occur before the flip at 8 (to change $0 \rightarrow 1$), hence

$$\tau(7) < \tau(8), \quad \text{std}(\tau_2(P)) = 12, \quad \text{inv}(\text{std}(\tau_2(P))) = 0.$$

For $I_3 = \{11\}$ we trivially have $\text{inv}(\text{std}(\tau_3(P))) = 0$. For $I_1 = \{2, 3, 4\}$ with alternating orientation, the admissible internal orders are precisely those whose flip-time standardizations lie in $\text{Alt}(3)$. Equivalently, the admissible internal coordinate orders are

$$(3, 2, 4) \quad \text{and} \quad (3, 4, 2).$$

Reading flip times in index order (2, 3, 4), these give

$$\text{std}(\tau_1(P)) = 213 \quad (\text{inv} = 1), \quad \text{std}(\tau_1(P)) = 312 \quad (\text{inv} = 2).$$

Thus, for every geodesic P ,

$$\text{inv}_{\text{flip}}(P) = \text{inv}(w(P)) + \text{inv}(\text{std}(\tau_1(P))).$$

We now list the three geodesics with $\text{inv}_{\text{flip}}(P) = 2$ (see Table 4).

For completeness, Table 5 records the vertices along these three geodesics, with x at the bottom and y at the top. These three paths account exactly for the term $3q^2$ in the q -geodesic enumerator.

10. Conclusions

We have constructed a refinement of the enumeration of geodesics between a pair of vertices x, y in the Fibonacci cube Γ_n . The statistic used is an inversion statistic derived from the flip-time map of a geodesic, which records the relative order in which the differing coordinates are flipped. The resulting generating function factors as a product in which a q -multinomial coefficient accounts for inversions created by the interleaving of distinct blocks, while blockwise q -Euler polynomials record inversions arising internally within each block according to its orientation. The blocks are the maximal consecutive sequences of indices in the set of coordinates on which x and y differ. Specializing to $q = 1$ recovers the multinomial-Euler formula for the total number of geodesics.

Data availability

No data was used for the research described in the article.

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