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THE SEARCHLIGHT SCHEDULING PROBLEM*

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Abstract. The problem of searching for a mobile robber in a simple polygon by a number of searchlights is considered. A searchlight is a stationary point which emits a single ray that cannot penetrate the boundary of the polygon. The direction of the ray can be changed continuously, and a point is detected by a searchlight at a given time if and only if it is on the ray. A robber is a point that can move continuously with unbounded speed. First, it is shown that the problem of obtaining a search schedule for an instance having at least one searchlight on the polygon boundary can be reduced to that for instances having no searchlight on the polygon boundary. The reduction is achieved by a recursive search strategy called the one-way sweep strategy. Then various sufficient conditions for the existence of a search schedule are presented by using the concept of a searchlight visibility graph. Finally, a simple necessary and sufficient condition for the existence of a search schedule for instances having exactly two searchlights in the interior is presented.

Key words. geometry, searchlight, visibility

AMS(MOS) subject classification. 68E99

1. Introduction. We consider the problem of searching for a mobile robber in a simple polygon by a number of searchlights. A searchlight is a stationary point which emits a single ray. The ray cannot penetrate the boundary of the polygon, but its direction can be changed continuously. A point is detected at a given time if and only if it is on the ray of a searchlight. A robber is a point which can move continuously with unbounded speed. We refer to this problem as the *searchlight scheduling problem.* The objective is to decide whether there exists a search schedule for detecting a robber regardless of its movement, for a given instance. A possible application of the searchlight scheduling problem is security enforcement in industrial plants where searchlights or TV cameras are used to find an intruder.

In the searchlight scheduling problem, the locations of searchlights are given as part of a problem instance. Obviously, there exists a search schedule for an instance only if every point in the given polygon is visible from at least one searchlight. The problem of obtaining a set of locations of searchlights having this property is known as the art gallery problem [2]-[6].

First, we present a recursive search strategy called the one-way sweep strategy, and show that this strategy can be used to reduce the problem of obtaining a search schedule for an instance having at least one searchlight on the polygon boundary to that for instances having no searchlight on the polygon boundary. Next, we give a number of sufficient conditions for the existence of a search schedule by using the concept of a searchlight visibility graph which represents the visibility relations among searchlights. Finally, we consider the case in which no searchlight is located on the polygon boundary, and present a simple necessary and sufficient condition for the

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existence of a search schedule for instances having exactly two searchlights in the interior.

It is not a goal of this paper to investigate the computational complexity of the problem. We also note that to our knowledge, the searchlight scheduling problem has not been addressed in the literature.

The problem is stated formally in § 2. The one-way sweep strategy is described in § 3. Searchlight visibility graphs and a number of sufficient conditions for the existence of a search schedule are discussed in § 4. Instances having two searchlights in the interior are considered in § 5. Concluding remarks are found in § 6.

2. Problem formulation. We denote by $b(R)$ the boundary of a two-dimensional region R. The term *simple polygon* is used to denote the union of a closed simple polygonal chain and its interior. For a simple polygon P and points $a, b \in b(P)$, $[a, b]_{b(P)}$ (or $(a, b)_{b(P)}$) denotes the closed (or open) continuous segment of $b(P)$ from a to b taken in the counterclockwise direction.

An instance of the searchlight scheduling problem is a pair $S = (P, L)$, where P is a simple polygon and L is a set of distinct points $l \in P$ called *searchlights.* A point x is said to be *visible* from a searchlight l if and only if $\overline{lx} \subseteq P$. Note that a searchlight does not block visibility from other searchlights. We denote by V_l the set of points visible from l.

DEFINITION 1. A *schedule* of a searchlight $l \in L$ is a continuous function $f_l : [0, T] \to \mathcal{R}$, where $[0, T]$ is an interval of real time and $\mathcal{R}$ is the set of real numbers. The *ray* of l at time $t \in [0, T]$ is the intersection of V_l and the semi-infinite ray with direction $f_l(t)$ emanating from l.[1] We say that l is *aimed* at a point $x \in P$ at time t if x is on the ray of l. A point $x \in P$ is said to be *illuminated* at time t if there exists a searchlight which is aimed at x.

DEFINITION 2. Two points in P are said to be *separable* at time $t \in [0, T]$ if every path between them within P contains an illuminated point; otherwise they are said to be *nonseparable.*

DEFINITION 3. Let $x \in P$ be any point.

(1) At time zero, x is *contaminated* if and only if x is not illuminated.

(2) At time $0 < t \leqq T$, x is *contaminated* if and only if there exists a point $y \in P$ such that (1) y is contaminated at some $0 \leqq t' < t$, (2) y is not illuminated at any time in the interval $[t', t]$, and (3) x and y are nonseparable at t.

A point which is not contaminated is said to be *clear.* A region $R \subseteq P$ is said to be *contaminated* if it contains a contaminated point; otherwise it is *clear.*

It is easy to see that $x \in P$ is contaminated at $t \in [0, T]$ if and only if a robber who has not been detected in the interval $[0, t]$ can be located at x at t, where a robber is detected only when it is illuminated. Definition 3 is based on the assumption that a robber can move continuously with unbounded speed.

By definition, an illuminated point is clear, and a contaminated point remains contaminated until it is illuminated. The following lemma is immediate from the definition.

LEMMA 1. *At time $t \in [0, T]$, if two points x and $y \in P$ are nonseparable, then x is contaminated if and only if y is contaminated.*

By Lemma 1, a maximal contaminated region is a nonempty connected open region not containing any illuminated point, and hence it cannot consist only of points on the boundary of P. Therefore we have Lemma 2.

[1] The value of $f_l(t)$ is taken in radian. Directions are measured counterclockwise from the positive x-axis.

LEMMA 2. *Any maximal contaminated subregion of P contains a point in the interior of P.*

Our objective is to detect a robber in P regardless of the movement. Thus we have Definition 4.

DEFINITION 4. $F=\{f_l \mid f_l:[0, T]\rightarrow \mathcal{R}$ is a schedule of $l \in L\}$ is a *search schedule* for S if P is clear at T.

In the following, we describe a schedule of a searchlight l by using expressions such as "aim l at a point x" and "turn l clockwise," instead of specifying a function f_l explicitly.

Example 1. Consider the instance shown in Fig. 1. Searchlights l_1 and l_2 are aimed at point a at time zero. $(b, d)_{b(P)}$ is a maximal open segment of $b(P)$ not visible from l_1. If we turn l_1 counterclockwise from a to b without turning l_2, then the shaded region determined by segment $[a, b]_{b(P)}$ and the rays of l_1 and l_2 becomes clear. Since triangle bcd is still contaminated, the clear region becomes contaminated if l_1 is turned counterclockwise any further.

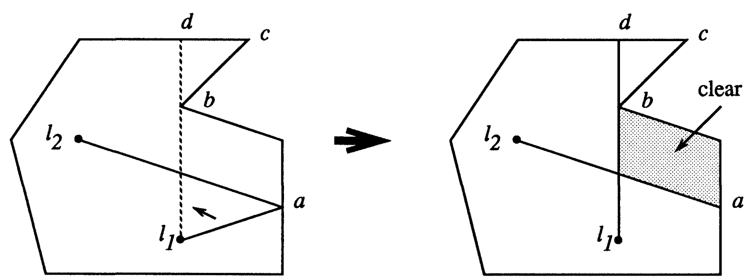

FIG. 1. *Illustration for Example* 1.

Example 2. The following is a search schedule for the instance shown in Fig. 2(a). Clear regions are shown shaded in Fig. 2.

(1) Aim l_2 at a.
(2) Aim l_3 at a and turn it counterclockwise until it is aimed at b (Fig. 2(b)).
(3) Aim l_1 at b and turn it counterclockwise until it is aimed at c (Fig. 2(c)).
(4) Turn l_3 counterclockwise until it is aimed at d (Fig. 2(d)).
(5) Aim l_1 at g.
(6) Turn l_2 clockwise until it is aimed at h (Fig. 2(e)).
(7) Turn l_1 counterclockwise until it is aimed at h (Fig. 2(f)).
(8) Turn l_1 clockwise until it is aimed at g (Fig. 2(g)).
(9) Turn l_3 counterclockwise until it is aimed at e (Fig. 2(h)).
(10) Aim l_2 at e and turn it counterclockwise until it is aimed at f.
(11) Turn l_3 counterclockwise until it is aimed at g (Fig. 2(i)).

An instance for which there exists no search schedule is given in Example 4 at the end of § 5.

Throughout this paper we assume that any given instance $S=(P, L)$ satisfies the following conditions (P1) and (P2), since obviously, otherwise there cannot exist any search schedule.

(P1) $P=\bigcup_{l\in L} V_l$. (Every point in P is visible from at least one searchlight.)

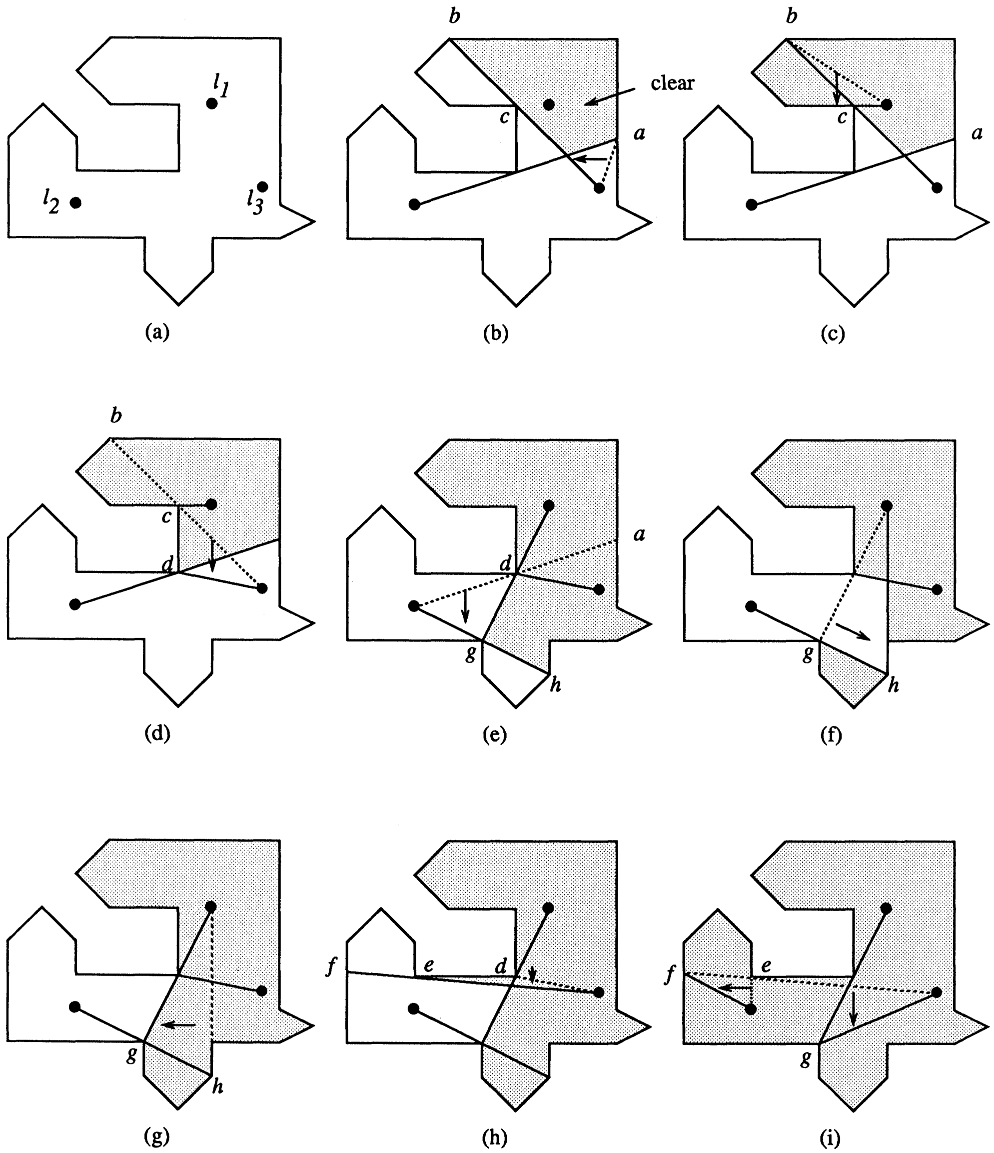

FIG. 2. *A search schedule for an instance of the searchlight scheduling problem.*

(P2) For each $l \in L$, either $l \in b(P)$ or $l \in V_{l'}$ for some $l' \in L - \{l\}$. (Every searchlight is either on the boundary of P or visible from another searchlight.)

3. One-way sweep strategy. In this section we show that the problem of obtaining a search schedule for an instance having at least one searchlight on the polygon boundary can be reduced to that for instances having no searchlight on the polygon boundary. The reduction is achieved by a recursive search strategy called the one-way sweep strategy.

It is convenient to describe the one-way sweep strategy as a method for clearing a subregion of P determined by the rays of searchlights. For this reason, we begin the discussion with the following definition.

DEFINITION 5. Let $S=(P, L)$ be an instance. *Semiconvex subpolygons* of *P supported* by a set of searchlights at a given time are defined recursively as follows.

(1) P is a semiconvex subpolygon of P supported by $\varnothing$ at any time $t \geqq 0$.

(2) Let $R \subseteq P$ be a semiconvex subpolygon of P supported by $K \subset L$ at time $t \geqq 0$. For an arbitrary searchlight $l \in L-K$ and an arbitrary maximal open segment $(a, b)_{b(P)}$ of $b(P)$ not visible from l, let Q be the closed simple region whose boundary is $[a, b]_{b(P)} \cup \overline{ba}$. If (1) $R \cap Q \neq \varnothing$ and (2) l is aimed at a and b at t, then $R \cap Q$ is a semiconvex subpolygon of P supported by $K \cup \{l\}$ at t.

In Fig. 3, the boundary of a semiconvex subpolygon R supported by $K=\{l_1, l_2\}$ is shown in thick lines.

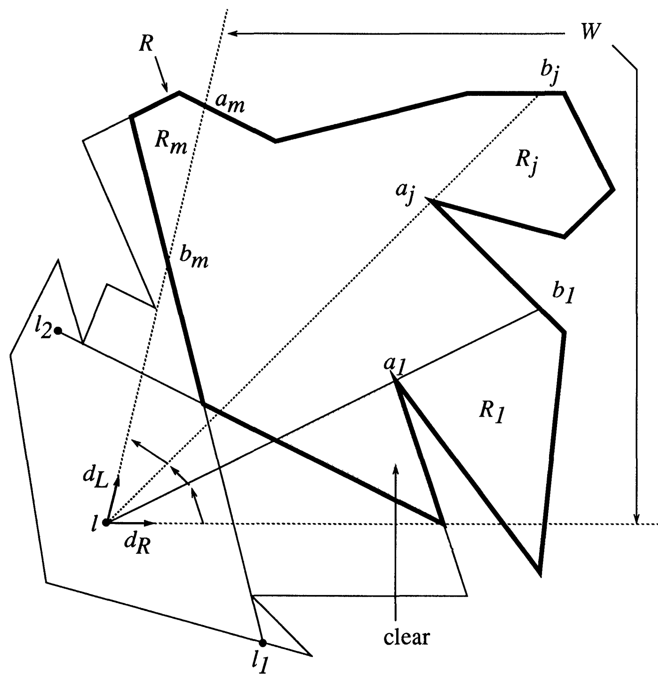

FIG. 3. *The one-way sweep strategy* OWSS (R, K, l), *where* $K=\{l_1, l_2\}$.

If R is supported by K at time t, then (1) it is "enclosed" by a segment of $b(P)$ and the rays of (some of) the searchlights in K, and (2) the interior of R is not visible from any searchlight in K. In the following, the qualifier "at time t" may be omitted when it is understood from the context. The term "semiconvex" is due to the following fact which is straightforward from definition: any reflex vertex of R is a vertex of P.

Let $S=(P, L)$ be an instance. Let R be a semiconvex subpolygon of P supported by $K \subset L$. Suppose that there exists a searchlight $l \in L-K$ such that

(1) $l \notin R-b(R)$ (l is either on the boundary of R or external to R), and

(2) $(R-b(R)) \cap V_l \neq \varnothing$ (at least one point in the interior of R is visible from l).

Let W be the smallest wedge with apex l such that $R \cap V_l \subseteq W$. Let d_R and d_L be the bounding semi-infinite rays of W where the interior of W lies to the left of d_R and to the right of d_L. Let $(a_j, b_j)_{b(R)} \subseteq b(R)-V_l$, $1 \leqq j \leqq m$, be the maximal open segments of $b(R)$ not visible from l, where the line segments $\overline{a_1 b_1}, \overline{a_2 b_2}, \cdots, \overline{a_m b_m}$ appear in counterclockwise order within W when viewed from l. Let R_j be the closed simple region whose boundary is $[a_j, b_j]_{b(R)} \cup \overline{b_j a_j}$ (see Fig. 3). Then the *one-way sweep strategy* OWSS (R, K, l) for R (with respect to K and l) is the following.

OWSS (R, K, l)

1. Aim l in the direction of d_R.
2. **for** $j = 1$ **to** m **do**
   2.1. Turn l counterclockwise until it is aimed at a_j and b_j.
   2.2. If there exists a searchlight $l' \in L - (K \cup \{l\})$ such that $l' \notin R_j - b(R_j)$ (l' is either on the boundary of R_j or external to R_j) and $(R_j - b(R_j)) \cap V_{l'} \neq \emptyset$ (at least one point in the interior of R_j is visible from l'), then execute OWSS $(R_j, K \cup \{l\}, l')$. Otherwise, if there exists a search schedule for the instance $S_{R_j} = (R_j, L \cap R_j)$, then execute it; otherwise output **failure** and halt.
3. Turn l counterclockwise until it is aimed in the direction of d_L.

In OWSS (R, K, l), we clear R by sweeping it by l in one direction, in such a way that every region R_j not visible from l is cleared in step 2.2 (if possible) without turning any searchlight in $K \cup \{l\}$. Since R is supported by K, it is easy to see that if each R_j can be cleared without turning any searchlight in $K \cup \{l\}$, then R becomes clear when step 3 is completed.

In step 2.2, to clear R_j we apply the one-way sweep strategy recursively if there exists a searchlight $l' \in L - (K \cup \{l\})$ which is not in the interior of R_j and from which at least one point in the interior of R_j is visible. Note that the idea of applying the strategy to R_j is valid, since R_j is a semiconvex subpolygon of P supported by $K \cup \{l\}$ when l is aimed at a_j and b_j. If there exists no such l', then the interior of R_j is visible only from the searchlights in the interior of R_j (and hence there exists no searchlight on the boundary of R_j, since at least one point in the interior of R_j would be visible from any searchlight on the boundary of R_j). In this case we regard $S_{R_j} = (R_j, L \cap R_j)$ as a separate instance and clear R_j by executing a search schedule for S_{R_j}, if such a search schedule exists. If there exists no search schedule for S_{R_j}, then the strategy outputs **failure** and halts.

THEOREM 1. *Let $S = (P, L)$ be an instance. Let R be a semi-convex subpolygon of P supported by $K \subset L$. Suppose that there exists a searchlight $l \in L - K$ such that $l \notin R - b(R)$ (l is either on the boundary of R or external to R) and $(R - b(R)) \cap V_l \neq \emptyset$ (at least one point in the interior of R is visible from l). Then R can be cleared without turning any searchlight in K if and only if there exists a search schedule for the instance $S_Q = (Q, L \cap Q)$ for every semiconvex subpolygon Q of R found during the execution of* OWSS (R, K, l) *to which the strategy cannot be applied recursively.*

Proof. (If) Execute OWSS (R, K, l). As is discussed above, R becomes clear when the execution terminates, since (1) R is supported by K and (2) every semiconvex subpolygon Q of R found during the execution of OWSS (R, K, l) can be cleared either by a recursive application of the one-way sweep strategy or the execution of a search schedule for the instance $S_Q = (Q, L \cap Q)$.

(Only if) Let Q be a semiconvex subpolygon Q of R found during the execution of OWSS (R, K, l) to which the strategy cannot be applied recursively. Let $F = \{f_l : [0, T] \to \mathcal{R} \mid l \in L - K\}$ be a collection of schedules which clears R without turning any searchlight in K starting from the state in which R is supported by K. Suppose that there exists no search schedule for $S_Q = (Q, L \cap Q)$. Then Q is contaminated when the execution of $F_Q = \{f_l \in F \mid l \in Q\}$ terminates at T, and hence by Lemma 2 there exists a contaminated point x in the interior of Q. Here, since the interior of Q is visible only from the searchlights in the interior of Q, for any $0 \leqq t \leqq T$, a point in the interior of Q is illuminated at t during the execution of F_Q if and only if it is illuminated at t during the execution of F. This, together with Lemma 2, implies that x is contaminated when the execution of F terminates at T. This contradicts the assumption that F clears R. □

Let $S=(P, L)$ be an instance having at least one searchlight on the boundary of P, and let $l \in L \cap b(P)$ be an arbitrary searchlight on the boundary of P. Since P is a semiconvex subpolygon of P supported by $\emptyset$ and at least one point in the interior of P is visible from l, we can execute OWSS $(P, \emptyset, l)$. Then by Theorem 1, there exists a search schedule for S if and only if there exists a search schedule for the instance $S_Q=(Q, L \cap Q)$ for every semiconvex subpolygon Q of P found during the execution of OWSS $(P, \emptyset, l)$ to which the strategy cannot be applied recursively. Since there exists no searchlight on the boundary of such Q, the problem of finding a search schedule for an instance having at least one searchlight on the polygon boundary has been reduced to that for instances having no searchlight on the polygon boundary.

Example 3. Consider the instance $S=(P, \{l_1, l_2, l_3, l_4\})$ shown in Fig. 4. It is easy to see that the one-way sweep strategy can be recursively applied to every semiconvex subpolygon of P found during the execution of OWSS $(P, \emptyset, l_1)$, and hence by Theorem 1 there exists a search schedule for S.

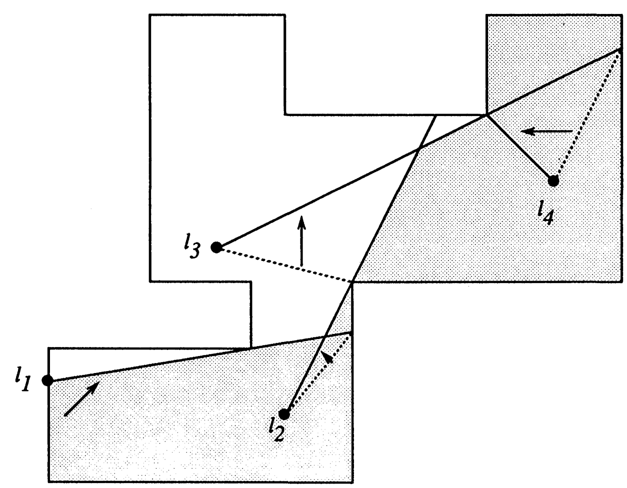

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FIG. 4. *An instance having a search schedule.*

4. Searchlight visibility graphs. In this section we present a number of simple sufficient conditions for the existence of a search schedule. The conditions are stated by using the concept of a searchlight visibility graph introduced below.

DEFINITION 6. Let $S=(P, L)$ be an instance. The *searchlight visibility graph* of S is an undirected graph SVG $(S)=(L, E)$ with vertex set L and edge set E such that for any l and $l' \in L$, $(l, l') \in E$ if and only if $l \neq l'$ and $l \in V_{l'}$.

THEOREM 2. *Let $S=(P, L)$ be an instance. There exists a search schedule for S if for every connected component $G_i=(L_i, E_i)$ of* SVG (S), *there exists at least one searchlight $l \in L_i$ such that $l \in b(P)$.*

Proof. Suppose that we execute OWSS $(P, \emptyset, l)$, where $l \in L \cap b(P)$ is an arbitrary searchlight on the boundary of P. By Theorem 1, it suffices to show that the one-way sweep strategy can be applied recursively to any semiconvex subpolygon Q of P found during the execution of OWSS $(P, \emptyset, l)$. Suppose that the strategy cannot be applied to some Q. Consider the instance $S_Q = (Q, L \cap Q)$. Note that the interior of Q is visible only from the searchlights in the interior of Q and there exists no searchlight on the boundary of Q. This observation, together with condition (P1), implies that (1) there exists at least one searchlight in the interior of Q, (2) any connected component of SVG (S_Q) is a connected component of SVG (S), and (3) $L_i \cap b(P) = \emptyset$ for any connected component $G_i = (L_i, E_i)$ of SVG (S_Q). This contradicts the assumption. □

LEMMA 3. *Let $S = (P, L)$ be an instance. For an arbitrary searchlight $l \in L$, let $(a, b)_{b(P)} \subseteq b(P) - V_l$ be a maximal open segment of $b(P)$ not visible from l, and let R be the closed simple region whose boundary is $[a, b]_{b(P)} \cup \overline{ba}$. If SVG (S) is connected, then R can be cleared while l is kept aimed at a and b.*

Proof. Aim l at a and b (Fig. 5). Then R is a semiconvex subpolygon of P supported by $\{l\}$. By condition (P1) and the connectedness of SVG (S), there exists a searchlight l' such that $l' \notin R - b(R)$ and $(R - b(R)) \cap V_{l'} \neq \emptyset$. Thus we can execute OWSS $(R, \{l\}, l')$. By Theorem 1, it suffices to show that the one-way sweep strategy can be applied recursively to any semiconvex subpolygon Q of R found during the execution of OWSS $(R, \{l\}, l')$. Suppose that the strategy cannot be applied recursively to some Q. By condition (P1) and the fact that the interior of Q is visible only from the searchlights in the interior of Q, there exists at least one searchlight in the interior of Q. But then the searchlights in the interior of Q are not visible from any searchlight outside of Q, and thus SVG (S) cannot be connected. □

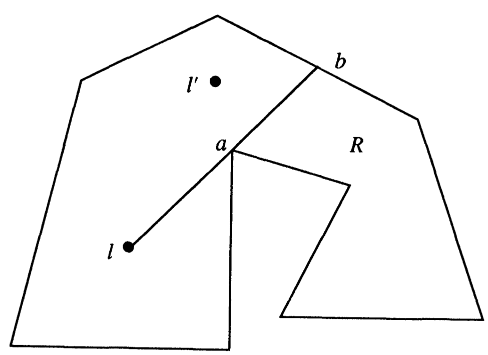

FIG. 5. *Illustration for Lemma* 3; *l is aimed at a and b.*

THEOREM 3. *Let $S = (P, L)$ be an instance. If SVG (S) is connected, then there exists a search schedule for the instance $S' = (P, L \cup \{l'\})$, where $l' \in P$ is an arbitrary searchlight not in L.*

Proof. By condition (P1), l' is visible from some searchlight $l \in L$. Let p be the first intersection of $b(P)$ and the ray emanating from l in the direction from l' to l (Fig. 6). Aim l and l' at p, and then turn l counterclockwise through a rotation of 2π. During this rotation, whenever l is aimed at points a and $b \in b(P)$ such that $(a, b)_{b(P)} \subseteq b(P) - V_l$ is a maximal open segment of $b(P)$ not visible from l, clear the closed region R whose boundary is $[a, b]_{b(P)} \cup \overline{ba}$ without turning l. This is possible by Lemma 3, since SVG (S) is connected and R is a semiconvex subpolygon of P supported by $\{l\}$ when l is aimed at a and b. Since l' need not be turned while R is being cleared, P becomes clear when the rotation of l is completed. □

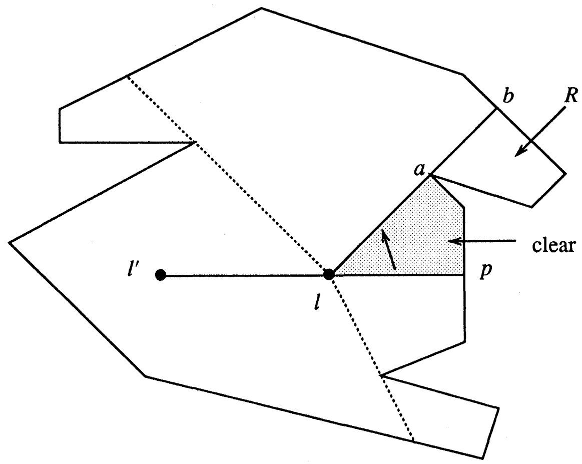

FIG. 6. *An additional searchlight l'.*

THEOREM 4. *Let $S=(P, L)$ be an instance. If* SVG (S) *is connected and there exist two searchlights l and $l' \in L$ such that $V_l \cap V_{l'} = \emptyset$, then there exists a search schedule for P.*

Proof. Let $(a, b)_{b(P)} \subseteq b(P) - V_l$ be the maximal open segment of $b(P)$ not visible from l such that $l' \in R$, where R is the closed simple region whose boundary is $[a, b]_{b(P)} \cup \overline{ba}$. Similarly, let $(a', b')_{b(P)} \subseteq b(P) - V_{l'}$ be the maximal open segment of $b(P)$ not visible from l' such that $l \in R'$, where R' is the closed simple region whose boundary is $[a', b']_{b(P)} \cup \overline{b'a'}$ (Fig. 7). Since SVG (S) is connected, by Lemma 3 we can aim l at a and b and then clear R without turning l. At this state $P - R'$ is clear, since $V_l \cap V_{l'} = \emptyset$. Next, we aim l' at a' and b' and clear R' without turning l'. Again, this is possible by Lemma 3. Then P becomes clear. □

5. Instances having two interior searchlights. In this section we present a simple necessary and sufficient condition for the existence of a search schedule for instances having exactly two searchlights in the interior.

THEOREM 5. *Let $S=(P, \{l_1, l_2\})$ be an instance such that $l_1, l_2 \notin b(P)$. Let p (or q) be the first intersection of the boundary of P and the extension of $\overline{l_1 l_2}$ in the direction from l_2 to l_1 (or from l_1 to l_2). Let $W_u = [p, q]_{b(P)}$ and $W_l = [q, p]_{b(P)}$. There exists a search schedule for P if and only if one of the following conditions holds.*

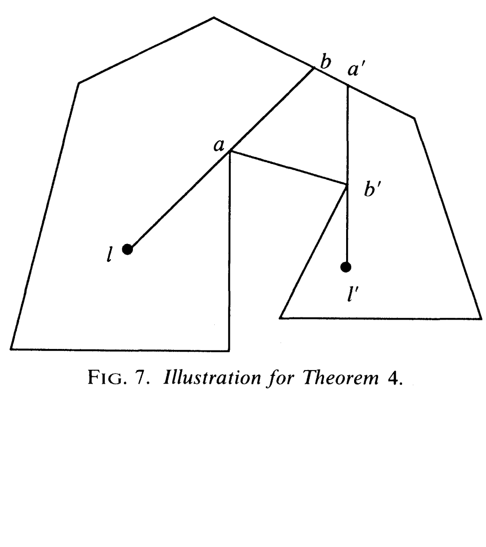

FIG. 7. *Illustration for Theorem 4.*

(1) *There exist points* $c_u \in W_u$ *and* $c_l \in W_l$ *such that* $[c_u, c_l]_{b(P)} \subseteq V_{l_1}$ *and* $[c_l, c_u]_{b(P)} \subseteq V_{l_2}$.

(2) $\overline{l_1 l_2} \cap W_u \neq \emptyset$ *and* $\overline{l_1 l_2} \cap W_l \neq \emptyset$.

(3) $\overline{l_1 l_2} \cap W_u \neq \emptyset$ *and either* $W_l \subseteq V_{l_1}$ *or* $W_l \subseteq V_{l_2}$.

(4) $\overline{l_1 l_2} \cap W_l \neq \emptyset$ *and either* $W_u \subseteq V_{l_1}$ *or* $W_u \subseteq V_{l_2}$.

Note that S is assumed to satisfy conditions (P1) and (P2) given at the end of § 2. Since $\overline{l_1 l_2} \cap b(P) = \emptyset$ holds if there exist points $c_u \in W_u$ and $c_l \in W_l$ such that $[c_u, c_l]_{b(P)} \subseteq V_{l_1}$ and $[c_l, c_u]_{b(P)} \subseteq V_{l_2}$, Theorem 5 follows from Lemmas 4 and 5 given below.

LEMMA 4. *If* $\overline{l_1 l_2} \cap b(P) = \emptyset$, *then there exists a search schedule for P if and only if there exist points* $c_u \in W_u$ *and* $c_l \in W_l$ *such that* $[c_u, c_l]_{b(P)} \subseteq V_{l_1}$ *and* $[c_l, c_u]_{b(P)} \subseteq V_{l_2}$.

Proof. (If) The following is a search schedule for P (Fig. 8).

(1) Aim l_1 at c_u.

(2) Aim l_2 at c_u.

(3) Turn l_1 counterclockwise until it is aimed at q.

(4) Turn l_2 clockwise until it is aimed at p.

(5) Turn l_1 counterclockwise until it is aimed at c_l.

(6) Turn l_2 clockwise until it is aimed at c_l.

(Only if) Assume that such $c_u \in W_u$ and $c_l \in W_l$ do not exist. We consider the case in which there exist maximal open segments $(a_1, b_1)_{b(P)} \subseteq W_u - V_{l_2}$ and $(a_2, b_2)_{b(P)} \subseteq W_u - V_{l_1}$ not visible from l_2 and l_1, respectively, such that a_1, b_1, a_2, and b_2 appear in counterclockwise order in W_u (Fig. 9). The argument for the case in which there exist similar open segments in W_l is basically the same. By $\overline{l_1 l_2} \cap b(P) = \emptyset$ and condition (P1), we have $a_1, b_1, a_2, b_2 \notin \overline{pq}$. We may assume that a_1, b_1, a_2, and b_2 have been chosen so that $[p, a_1]_{b(P)} \subseteq V_{l_2}$ and $[b_2, q]_{b(P)} \subseteq V_{l_1}$. For $i = 1, 2$, let R_i be the closed simple region whose boundary is $[a_i, b_i]_{b(P)} \cup \overline{b_i a_i}$. Let R_0 be the closed region whose boundary is $W_l \cup \overline{pq}$.

Before we proceed, we prove the following proposition.

PROPOSITION 1. *In any search schedule for P, if* R_1 *is changed from contaminated to clear at time t, then there exists some* $\delta > 0$ *such that in the interval* $[t-\delta, t)$, l_1 *is aimed at a point in* $(a_1, b_1)_{b(P)}$ *and* l_2 *is aimed at a point in* $[p, a_1]_{b(P)}$.

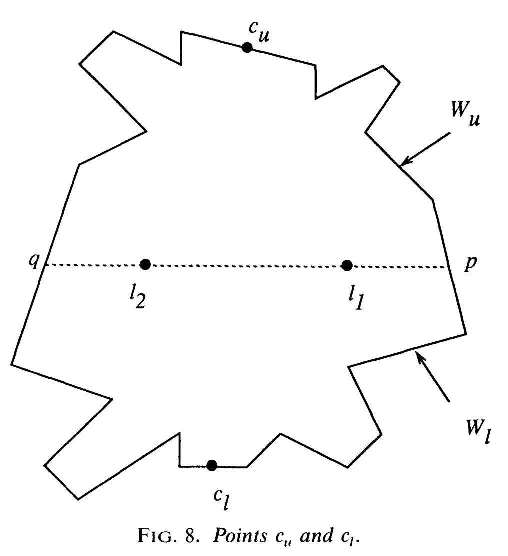

FIG. 8. *Points* c_u *and* c_l.

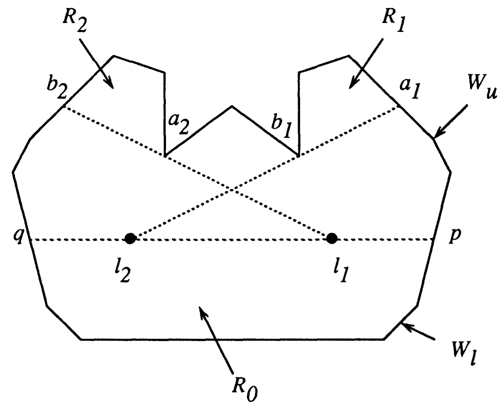

FIG. 9. *Illustration for the proof of Lemma* 4.

Proof. Let $\delta > 0$ be any value such that R_1 is contaminated in $[t-\delta, t)$. Suppose that in $[t-\delta, t)$, either l_1 is not aimed at any point in $(a_1, b_1)_{b(P)}$ or l_2 is not aimed at any point in $[p, a_1]_{b(P)}$ (Fig. 10). At any time in $[t-\delta, t)$, since R_1 is contaminated and any two points in R_1 which are not illuminated are nonseparable, by Lemma 1 any point in R_1 which is not illuminated is contaminated. Then it is impossible to change R_1 from contaminated to clear at t, since contaminated points remain contaminated until they are illuminated. □

The proof of the following proposition is basically the same as that of Proposition 1 and is thus omitted.

PROPOSITION 2. *In any search schedule for P, if R_2 is changed from contaminated to clear at time t, then there exists some $\delta > 0$ such that in the interval $[t-\delta, t)$, l_2 is aimed at a point in $(a_2, b_2)_{b(P)}$ and l_1 is aimed at a point in $[b_2, q]_{b(P)}$.*

We return to the proof of Lemma 4. Assume that there exists a search schedule for P. Let F be a search schedule in which the total number of times R_1 and R_2 are changed from contaminated to clear is smallest among all search schedules. Suppose that during the execution of F, R_1 is changed from contaminated to clear at t_1 and R_1 remains clear after t_1. Since R_1 and R_2 cannot be changed from contaminated to clear simultaneously by Propositions 1 and 2, without loss of generality assume that R_2 is contaminated at t_1 or at some time after t_1. Let $t_2 > t_1$ be the first time after t_1 at which R_2 is changed from contaminated to clear.

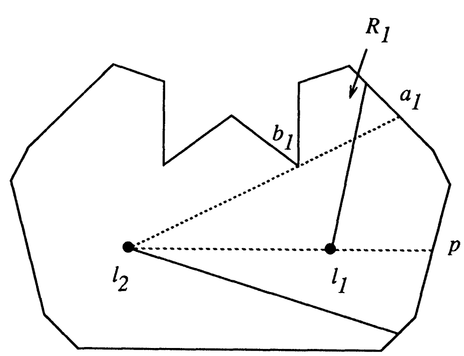

FIG. 10. *Illustration for the proof of Proposition* 1; *any two points in* R_1 *which are not illuminated are nonseparable.*

First, we show that both R_0 and R_2 are contaminated at t_1. Let $\delta_1 > 0$ be a value satisfying the conditions of Proposition 1 with respect to t_1, that is, l_1 is aimed at a point in $(a_1, b_1)_{b(P)}$ and l_2 is aimed at a point in $[p, a_1]_{b(P)}$ in $[t_1 - \delta_1, t_1)$. Then by the assumption that $\overline{l_1 l_2} \cap W_l = \emptyset$, in $[t_1 - \delta_1, t_1)$ any two points in $R_0 \cup R_2$ which are not illuminated are nonseparable, and hence by Lemma 1 either R_0 and R_2 are both clear or both contaminated. Suppose that R_0 and R_2 are clear in $[t_1 - \delta_1, t_1)$, and hence by Lemma 1 the points in $R_0 \cup R_2$ are separable from any contaminated point. Since $[p, a_1]_{b(P)} \subseteq V_{l_2}$ and $\overline{l_1 l_2} \cap W_l = \emptyset$, there are only two possibilities at any time in $[t_1 - \delta_1, t_1)$.

Case 1. l_2 is not aimed at a_1, and the region determined by some segment of $[p, b_1)_{b(P)}$ and the rays of l_1 and l_2 is the only contaminated region (Fig. 11).

Case 2. l_2 is aimed at a_1, and some of the regions determined by some segments of $[a_1, a_2]_{b(P)}$ and the rays of l_1 and l_2 are the only contaminated regions (Fig. 12). (Without the assumption that $[p, a_1]_{b(P)} \subseteq V_{l_2}$, there may exist a contaminated region determined by the ray of l_2 and some segments of $[p, a_1]_{b(P)}$. Also, if $\overline{l_1 l_2} \cap W_l \neq \emptyset$, then there may exist a contaminated region determined by the ray of l_2 and some segments of W_l.) In Case 1, P can be cleared by turning l_1 and l_2 to a_1 clockwise and counterclockwise, respectively. In Case 2, the contaminated regions are visible from l_1 by condition (P1), and thus P can be cleared without changing any of R_1 and R_2 from clear to contaminated after t_1. In either case, there exists a search schedule for S in which the number of times R_1 and R_2 are changed from contaminated to clear is smaller than that in F. Since this contradicts the assumption on F, it cannot be the case that R_0 and R_2 are clear in $[t_1 - \delta_1, t_1)$. Thus both R_0 and R_2 are contaminated in $[t_1 - \delta_1, t_1)$. Then, since by Proposition 1 it is impossible to change either of R_0 and

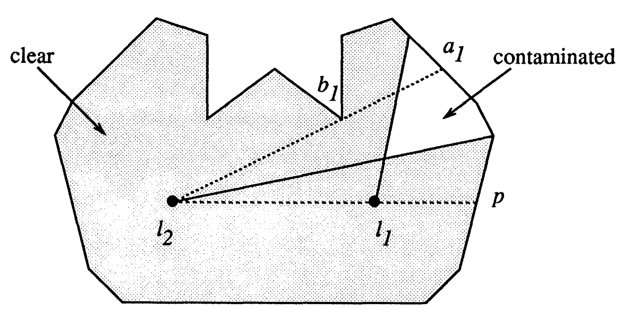

FIG. 11. *Case* 1 *in* $[t_1 - \delta_1, t_1)$ *in the proof of Lemma* 4.

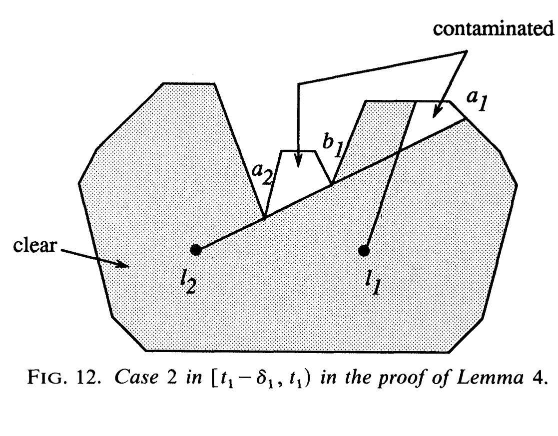

FIG. 12. *Case* 2 *in* $[t_1 - \delta_1, t_1)$ *in the proof of Lemma* 4.

R_2 from contaminated to clear at t_1, both R_0 and R_2 are contaminated at t_1. Also, note that by the argument given above, q is contaminated at t_1 since $a_2, b_2 \notin \overline{pq}$.

Let $\delta_2 > 0$ be a value satisfying the conditions of Proposition 2 with respect to t_2, that is, l_2 is aimed at a point in $(a_2, b_2)_{b(P)}$ and l_1 is aimed at a point in $[b_2, q]_{b(P)}$ in $[t_2 - \delta_2, t_2)$. Then by the assumption that $\overline{l_1 l_2} \cap W_l = \emptyset$, in $[t_2 - \delta_2, t_2)$ any two points in $R_0 \cup R_1$ which are not illuminated are nonseparable. Thus by Lemma 1 and the assumption that R_1 remains clear after t_1, R_0 is clear in $[t_2 - \delta_2, t_2)$.

In summary, we have found that R_1 is clear in $[t_1, t_2]$, R_2 is contaminated in $[t_1, t_2)$, R_0 is contaminated at t_1, and R_0 is changed from contaminated to clear in $[t_1, t_2)$. In the following we show that at least one of p and q is contaminated at any time in $[t_1, t_2)$, and hence R_0 cannot become clear in $[t_1, t_2)$.

Since in $[t_1, t_2)$ R_1 is clear and R_2 is contaminated, by Lemma 1 the points in R_1 should be separable from any contaminated point in R_2. Thus we have Proposition 3.

PROPOSITION 3. *In the interval* $[t_1, t_2)$,

(1) *Whenever* l_1 *is aimed at* p, l_2 *is aimed at* a_1 *and* b_1 (*Fig.* 13), *and*

(2) *Whenever,* l_2 *is aimed at* q, l_1 *is aimed at* a_2 *and* b_2.

Also, by Lemma 1, $a_1, b_1, a_2, b_2 \notin \overline{pq}$ and the condition on R_1 and R_2, we have Proposition 4.

PROPOSITION 4. *In the interval* $[t_1, t_2)$,

(1) *Whenever* l_1 *is aimed at* q, l_2 *is aimed at a point in* $[b_1, a_2]$ (*Fig.* 14), *and*

(2) *Whenever* l_2 *is aimed at* p, l_1 *is aimed at a point in* $[b_1, a_2]$.

Furthermore, since $a_1, b_1, a_2, b_2 \notin \overline{pq}$, we have Proposition 5.

PROPOSITION 5. *At any time, if neither* p *nor* q *is illuminated, then* p *and* q *are nonseparable.*

By $a_1, b_1, a_2, b_2 \notin \overline{pq}$ and Propositions 3 and 4, (1) at most one of p and q is illuminated at any time in $[t_1, t_2)$, and (2) if p and q (or q and p) are illuminated at

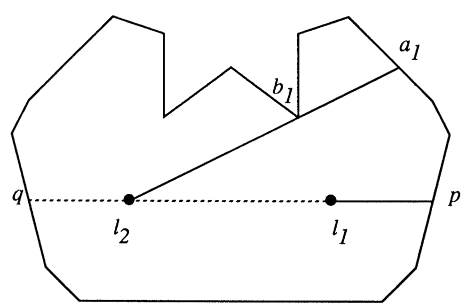

FIG. 13. *Illustration for Proposition* 3; l_1 *is aimed at* p.

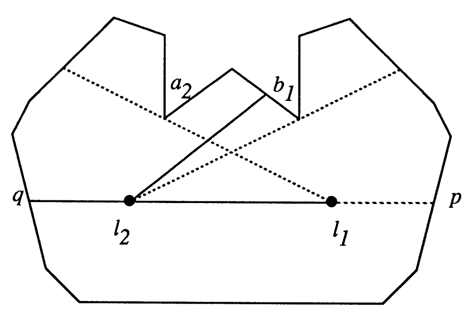

FIG. 14. *Illustration for Proposition* 4; l_1 *is aimed at* q.

s_1 and s_2 for some $t_1 \leqq s_1 < s_2 < t_2$, respectively, then there exists some $s_1 < t < s_2$ such that neither p nor q is illuminated at t. This observation, together with Proposition 5, Lemma 1, and the fact that q is contaminated at t_1, implies that p and q cannot be clear simultaneously in $[t_1, t_2)$. Thus R_0 cannot be clear in $[t_1, t_2)$. This is a contradiction. □

LEMMA 5. *If $\overline{l_1 l_2} \cap b(P) \neq \emptyset$, then there exists a search schedule for P if and only if one of the following conditions holds*:

(1) $\overline{l_1 l_2} \cap W_u \neq \emptyset$ *and* $\overline{l_1 l_2} \cap W_l \neq \emptyset$.

(2) $\overline{l_1 l_2} \cap W_u \neq \emptyset$ *and either* $W_l \subseteq V_{l_1}$ *or* $W_l \subseteq V_{l_2}$.

(3) $\overline{l_1 l_2} \cap W_l \neq \emptyset$ *and either* $W_u \subseteq V_{l_1}$ *or* $W_u \subseteq V_{l_2}$.

Proof. (If) Note that $\overline{l_1 l_2} \subseteq P$ by condition (P2). The following is a search schedule for P if $\overline{l_1 l_2} \cap W_u \neq \emptyset$ and $\overline{l_1 l_2} \cap W_l \neq \emptyset$ (Fig. 15).

(1) Aim l_1 at q.
(2) Aim l_2 at p.
(3) Turn l_1 counterclockwise through a rotation of 2π.
(4) Turn l_2 counterclockwise through a rotation of 2π.

If $\overline{l_1 l_2} \cap W_u \neq \emptyset$ and $W_l \subseteq V_{l_1}$, then P can be cleared by the following (Fig. 16).

(1) Aim l_1 at q.
(2) Aim l_2 at p.
(3) Turn l_1 clockwise through a rotation of 2π.
(4) Turn l_2 counterclockwise through a rotation of π.

Search schedules for other cases are similar and are thus omitted.

(Only if) Since the argument is similar to that in the (only if) part of Lemma 4, we only give an outline. Consider the case in which $\overline{l_1 l_2} \cap W_u \neq \emptyset$, $\overline{l_1 l_2} \cap W_l = \emptyset$, and

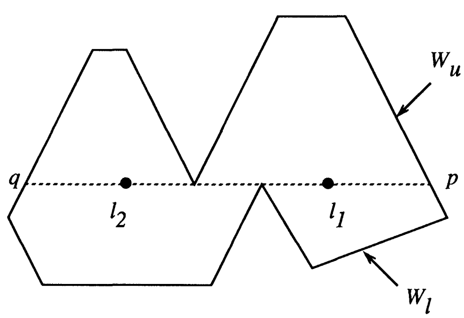

FIG. 15. $\overline{l_1 l_2} \cap W_u \neq \emptyset$ *and* $\overline{l_1 l_2} \cap W_l \neq \emptyset$.

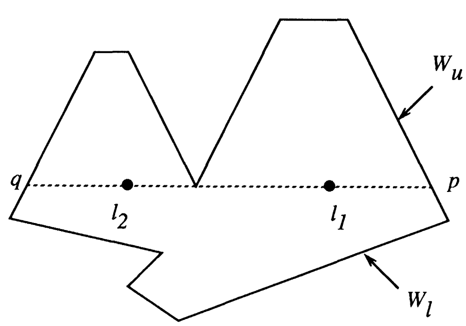

FIG. 16. $\overline{l_1 l_2} \cap W_u \neq \emptyset$ *and* $W_l \subseteq V_{l_1}$.

$W_l \not\subseteq V_{l_i}$ for $i=1,2$. The argument for the other case ($\overline{l_1 l_2} \cap W_u = \emptyset$, $\overline{l_1 l_2} \cap W_l \neq \emptyset$, and $W_u \not\subseteq V_{l_i}$ for $i=1,2$) is similar and is thus omitted.

Since $l_1, l_2 \notin b(P)$ and $\overline{l_1 l_2} \cap W_u \neq \emptyset$, there exist maximal open segments $(a_1, b_1)_{b(P)} \subseteq W_u - V_{l_2}$ and $(a_2, b_2)_{b(P)} \subseteq W_u - V_{l_1}$ not visible from l_2 and l_1, respectively, such that a_1, b_1, a_2, and b_2 appear in counterclockwise order in W_u (Fig. 17). By condition (P1), if $b_1 \neq a_2$ then $\overline{b_1 a_2} \subseteq b(P)$. For $i=1,2$, let R_i be the closed simple region whose boundary is $[a_i, b_i]_{b(P)} \cup \overline{b_i a_i}$. Let R_0 be the closed region whose boundary is $W_l \cup \overline{pq}$.

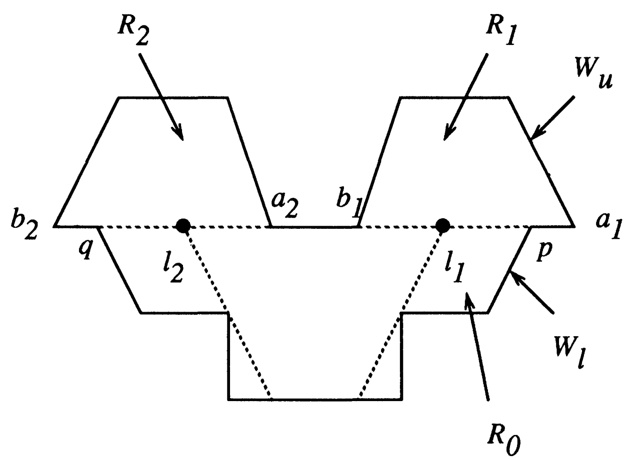

FIG. 17. *Illustration for the proof of Lemma* 5.

Assume that there exists a search schedule for P, and let F be a search schedule in which the total number of times R_1 and R_2 are changed from contaminated to clear is smallest among all search schedules. First, as we did in the proof of Lemma 4, we can show that R_1 and R_2 cannot be cleared simultaneously. Thus without loss of generality we can assume that R_1 is changed from contaminated to clear at t_1, R_1 remains clear after t_1, and R_2 is contaminated at t_1 or at some time after t_1. Let $t_2 > t_1$ be the first time after t_1 at which R_2 is changed from contaminated to clear. Then by $\overline{l_1 l_2} \cap W_l = \emptyset$, the assumption on F and an argument similar to that in the proof of Lemma 4, we can show that R_0 and R_2 are contaminated at t_1 (more specifically, any point in $R_0 \cup R_2$ which is not illuminated is contaminated at t_1). Next, by using the assumption that $\overline{l_1 l_2} \cap W_l = \emptyset$, we can show that R_0 must be clear at $t_2 - \delta$ for some $\delta > 0$.

In summary, R_1 is clear in $[t_1, t_2]$, R_2 is contaminated in $[t_1, t_2)$, R_0 is contaminated at t_1, and R_0 is changed from contaminated to clear in $[t_1, t_2)$. Since by assumption $W_l \not\subseteq V_{l_i}$, for $i=1,2$, R_0 cannot be cleared unless each of l_1 and l_2 is aimed at the points in W_l not visible from the other searchlight. Here, since R_1 is clear and R_2 is contaminated in $[t_1, t_2)$, l_2 must be aimed at a_1 and b_1 whenever l_1 is aimed at a point W_l not visible from l_2 (Fig. 18), and l_1 must be aimed at a_2 and b_2 whenever l_2 is aimed at a point in W_l not visible from l_1 (Fig. 19). Thus (1) at any time in $[t_1, t_2)$ at most one of l_1 and l_2 can be aimed at a point in W_l not visible from the other searchlight, and (2) if l_1 and l_2 (or l_2 and l_1) are aimed at a point in W_l not visible from the other searchlight at s_1 and s_2 for some $t_1 \leqq s_1 < s_2 < t_2$, respectively, then there exists some $s_1 < t < s_2$ such that any two points in W_l visible from only one of l_1 and l_2 are nonseparable at t. This observation, together with Lemma 1 and the fact that any point in R_0 which is not illuminated is contaminated at t_1, implies that R_0 contains a

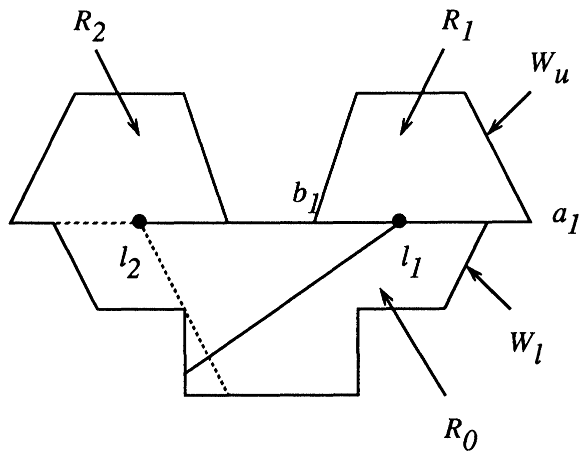

FIG. 18. *Illustration for the proof of Lemma* 5; l_1 *is aimed at a point in* W_l *not visible from* l_2.

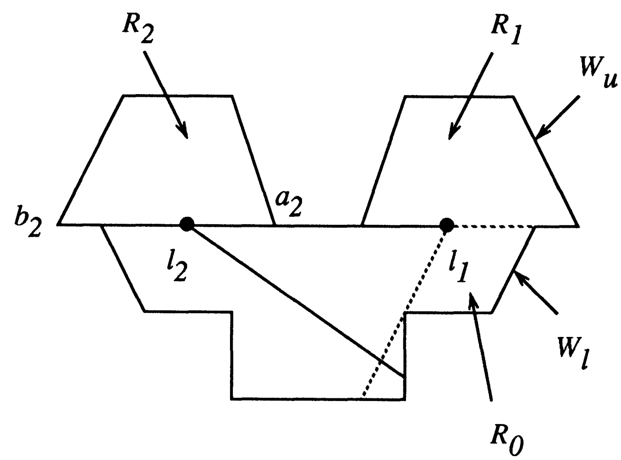

FIG. 19. *Illustration for the proof of Lemma* 5; l_2 *is aimed at a point in* W_l *not visible from* l_1.

contaminated point at any time in $[t_1, t_2)$, and hence R_0 cannot be clear in $[t_1, t_2)$. This is a contradiction. □

Example 4. Consider the instance $S=(P, \{l_1, l_2, l_3\})$ shown in Fig. 20. When the one-way sweep strategy is applied to S, we obtain a semiconvex subpolygon Q of P supported by $\{l_1\}$ containing two searchlights l_2 and l_3 in the interior. Note that the strategy cannot be applied to Q, since the interior of Q is visible only from l_2 and l_3. Also, the instance $S_Q=(Q, \{l_2, l_3\})$ does not satisfy any of the conditions of Theorem 5, and hence there exists no search schedule for S_Q. Thus by Theorem 1, there exists no search schedule for S.

6. Concluding remarks. We have posed the searchlight scheduling problem and presented various conditions for the existence of a search schedule. In particular, we have shown that the problem of obtaining a search schedule for an instance having at least one searchlight on the polygon boundary can be reduced to that for instances having no searchlight on the polygon boundary, and then presented a simple necessary and sufficient condition for the existence of a search schedule for instances having exactly two searchlights in the interior. Some preliminary results for the case in which

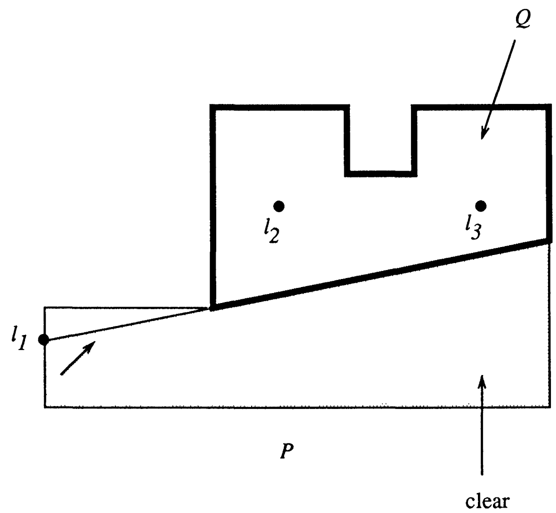

FIG. 20. *An instance having no search schedule.*

there are three searchlights in the interior have been reported in [8], but obtaining a necessary and sufficient condition for this case remains as a challenging open problem.

As a final note, we remark that given an n-sided simple polygon P we can compute, in $O(n \log\log n)$ time, a set L of searchlights such that (1) $|L| = \lfloor n/3 \rfloor$ and (2) the instance $S = (P, L)$ has a search schedule. This is an immediate corollary of Theorem 2 and a linear time coloring algorithm (see [1], [6, Chap. 1]) for computing, given a triangulation of P, a subset L of the vertices of P such that $L = \lfloor n/3 \rfloor$ and every point in the interior of P is visible from at least one vertex in L. It is known that a triangulation of an n-sided polygon can be computed in $O(n \log\log n)$ time [9]. If P is rectilinear, then a set L with the desired property such that $|L| = \lfloor n/4 \rfloor$ can be computed in $O(n \log\log n)$ time [7].

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