CS-235
Computational Geometry

Subhash Suri

Computer Science Department
UC Santa Barbara

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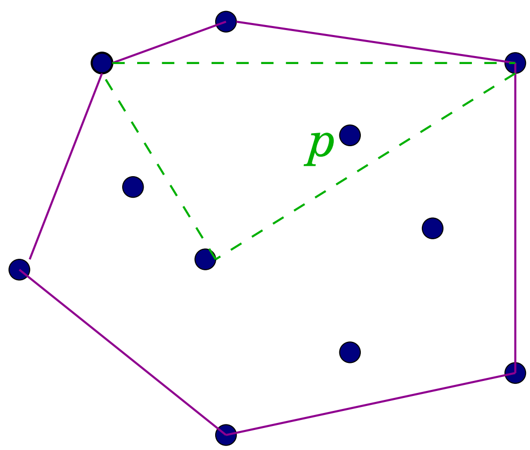
Convex Hulls

1. **Convex hulls are to CG what sorting is to discrete algorithms.**
2. **First order shape approximation.**
   Invariant under rotation and translation.



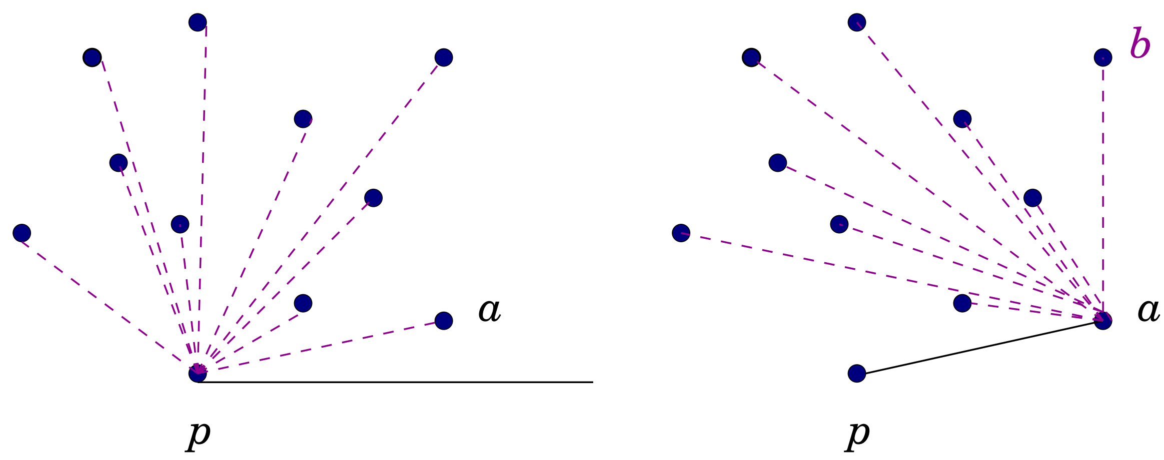
3. **Rubber-band analogy.**
4. **Many applications in robotics, shape analysis, line fitting etc.**
5. **Example: if $CH(P_1) \cap CH(P_2) = \emptyset$, then objects P_1 and P_2 do not intersect.**
6. **Convex Hull Problem:**
   Given a finite set of points S, compute its convex hull $CH(S)$. (Ordered vertex list.)

Classical Convexity

1. **Given points $p_1, p_2, \ldots, p_k$, the point $\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_k p_k$ is their convex combination if $\alpha_i \geq 0$ and $\sum_{i=1}^{k} \alpha_i = 1$.**
2. **$CH(S)$ is union of all convex combinations of S.**
3. **S convex iff for all $x, y \in S$, $\overline{xy} \in S$.**
4. **$CH(S)$ is intersection of all convex sets containing S.**
5. **$CH(S)$ is intersection of all halfspaces containing S.**
6. **$CH(S)$ is smallest convex set containing S.**
7. **In R^2, $CH(S)$ is smallest area (perimeter) convex polygon containing S.**
8. **In R^2, $CH(S)$ is union of all triangles formed by triples of S.**
9. **These descriptions do not yield efficient algorithms. At best $O(N^3)$.**

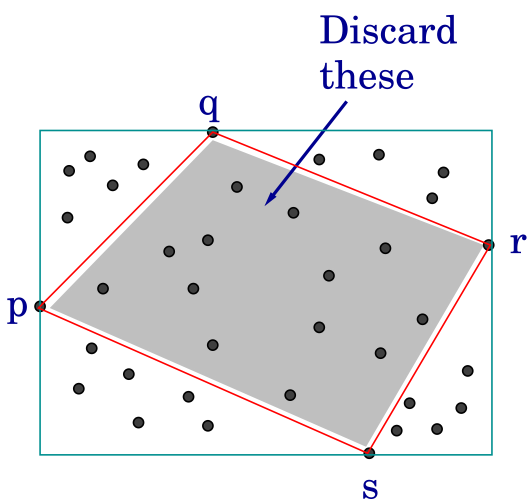
Efficient CH Algorithms

Gift Wrapping: [Jarvis '73; Chand-Kapur '70]

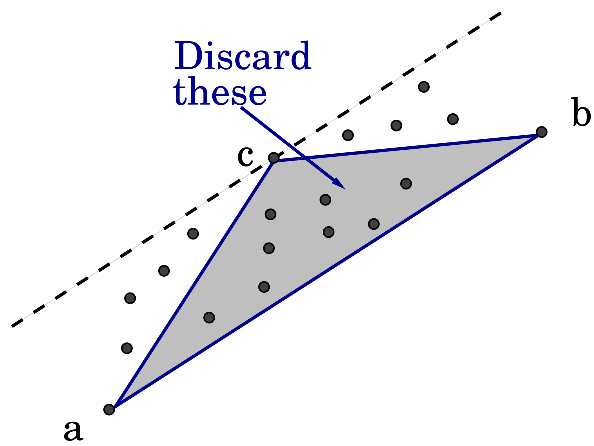


1. **Start with bottom point p.**
2. **Angularly sort all points around p.**
3. **Point a with smallest angle is on CH.**
4. **Repeat algorithm at a.**
5. **Complexity $O(Nh)$; $3 \leq h = |CH| \leq N$.**
   Worst case $O(N^2)$.

Quick Hull Algorithm



Initialization



Recursive Elimination

1. **Form initial quadrilateral Q, with left, right, top, bottom. Discard points inside Q.**

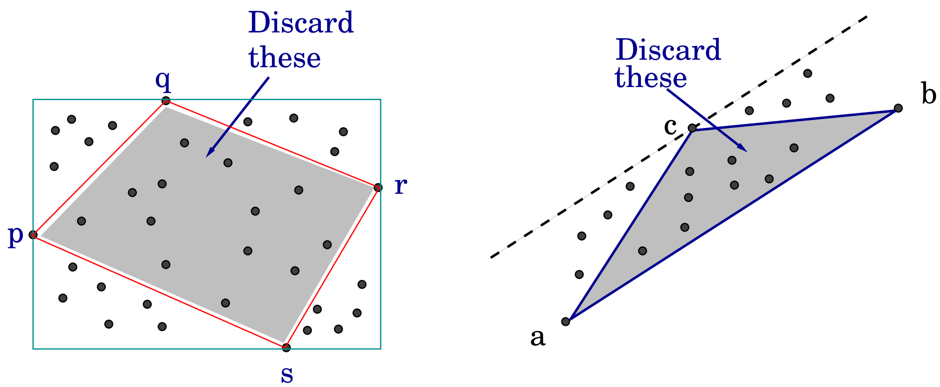
2. **Recursively, a convex polygon, with some points "outside" each edge.**

3. **For an edge ab, find the farthest outside point c. Discard points inside triangle abc.**

4. **Split remaining points into "outside" points for ac and bc.**

5. **Edge ab on CH when no point outside.**

Complexity of QuickHull



Initialization

Recursive Elimination

1. **Initial quadrilateral phase takes $O(n)$ time.**

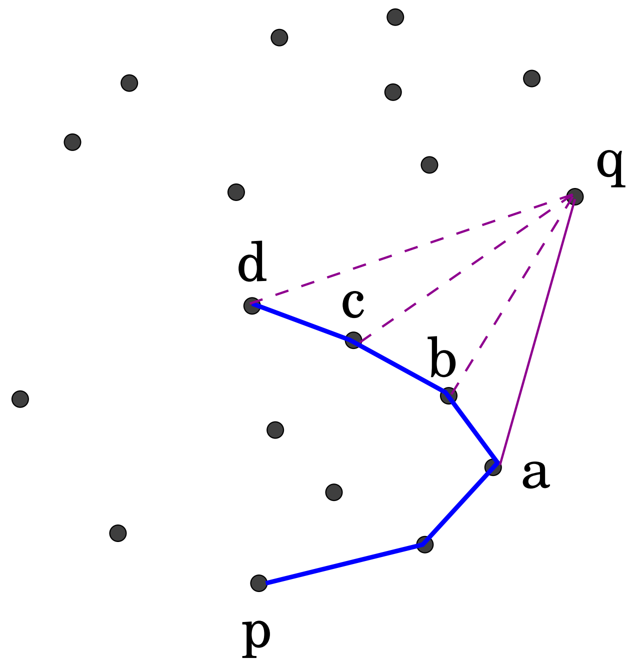
2. **$T(n)$: time to solve the problem for an edge with n points outside.**

3. **Let n_1, n_2 be sizes of subproblems. Then,**

$$T(n) = \begin{cases} 1 & \textbf{if } n = 1 \\ n + T(n_1) + T(n_2) & \textbf{where } n_1 + n_2 \leq n \end{cases}$$

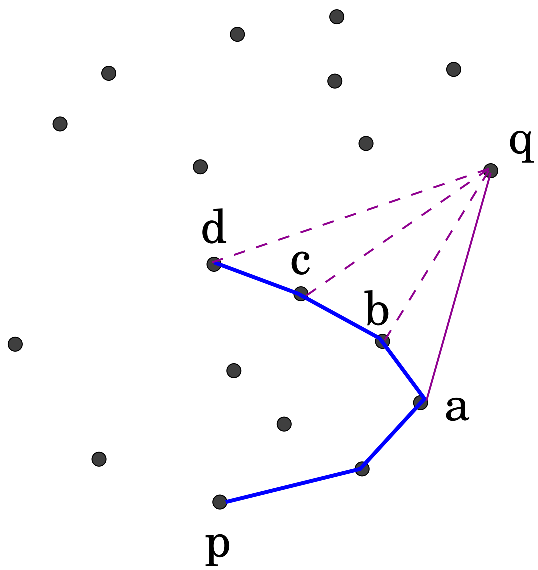
4. **Like QuickSort, this has expected running time $O(n \log n)$, but worst-case time $O(n^2)$.**

Graham Scan



1. **Sort by Y-order;** $p_1, p_2, \ldots, p_n$.
2. **Initialize. push** $(p_i, stack)$, $i = 1, 2$.
3. **for** $i = 3$ **to** n **do**
   while $\angle$ **next, top,** $p_i \neq$ **Left-Turn**
   pop $(stack)$
   push $(p_i, stack)$.
4. **return** $stack$.
5. **Invented by R. Graham '73. (Left and Right convex hull chains separately.)**

Analysis of Graham Scan



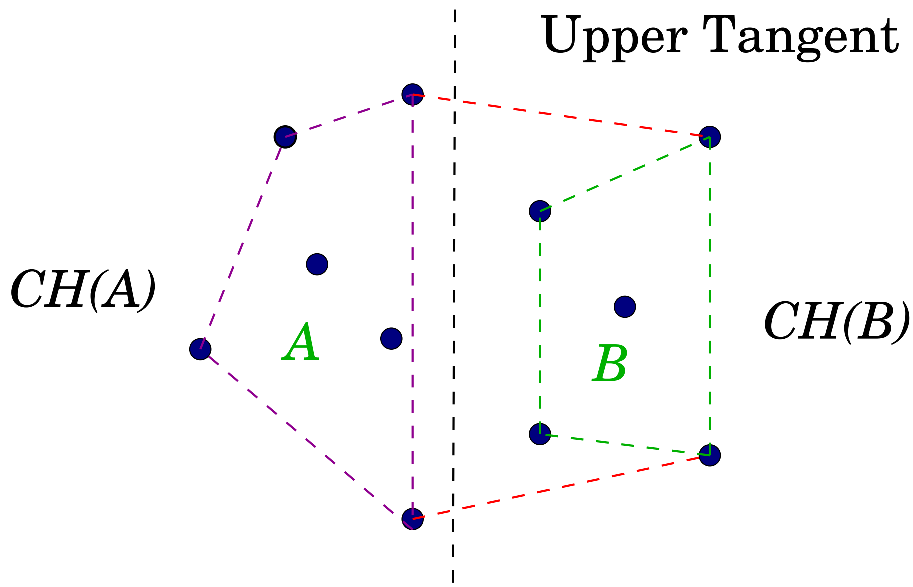
1. **Invariant:** $\langle p_1, \ldots, top(stack) \rangle$ **is convex. On termination, points in** $stack$ **are on** CH**.**

2. **Orientation Test:** $D = \begin{vmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{vmatrix}$

   $\angle p, q, r$ **is LEFT if** $D > 0$**, RIGHT if** $D < 0$**, and straight if** $D = 0$**.**

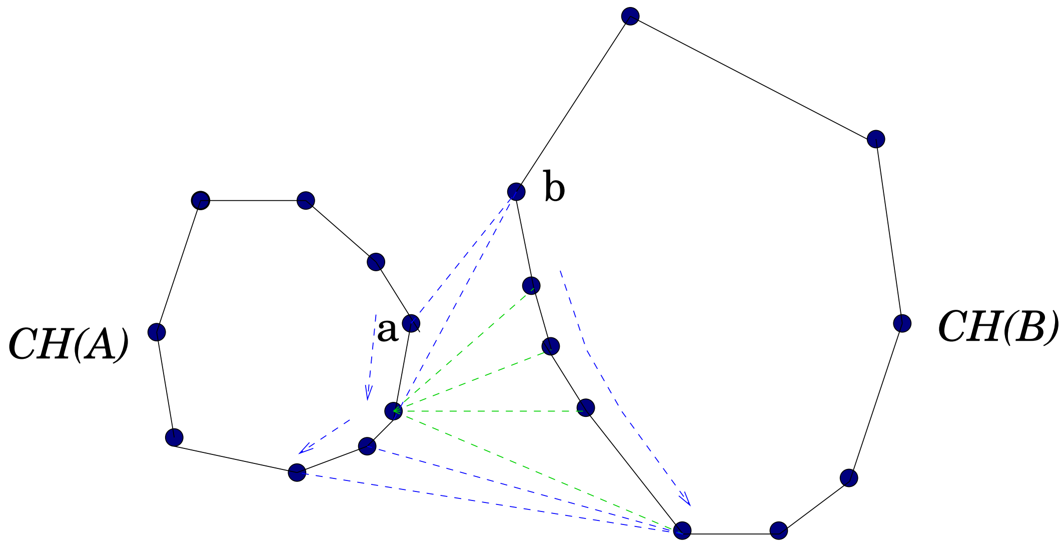
3. **After sorting, the scan takes** $O(n)$ **time: in each step, either a point is deleted, or added to stack.**

Divide and Conquer



- **Sort points by X-coordinates.**
- **Let A be the set of $n/2$ leftmost points, and B the set of $n/2$ rightmost points.**
- **Recursively compute $CH(A)$ and $CH(B)$.**
- **Merge $CH(A)$ and $CH(B)$ to obtain $CH(S)$.**

Merging Convex Hulls



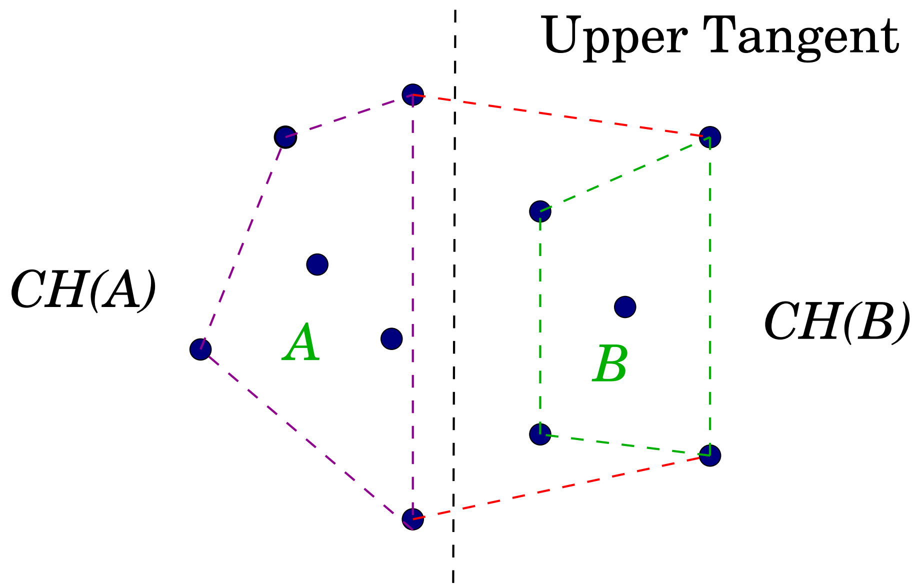
Lower Tangent

- $a =$ **rightmost point of** $CH(A)$.
- $b =$ **leftmost point of** $CH(B)$.
- **while** ab **not lower tangent of** $CH(A)$ **and** $CH(B)$ **do**

  1. **while** ab **not lower tangent to** $CH(A)$ **set** $a = a - 1$ **(move** a **CW);**
  2. **while** ab **not lower tangent to** $CH(B)$ **set** $b = b + 1$ **(move** b **CCW);**

- **Return** ab

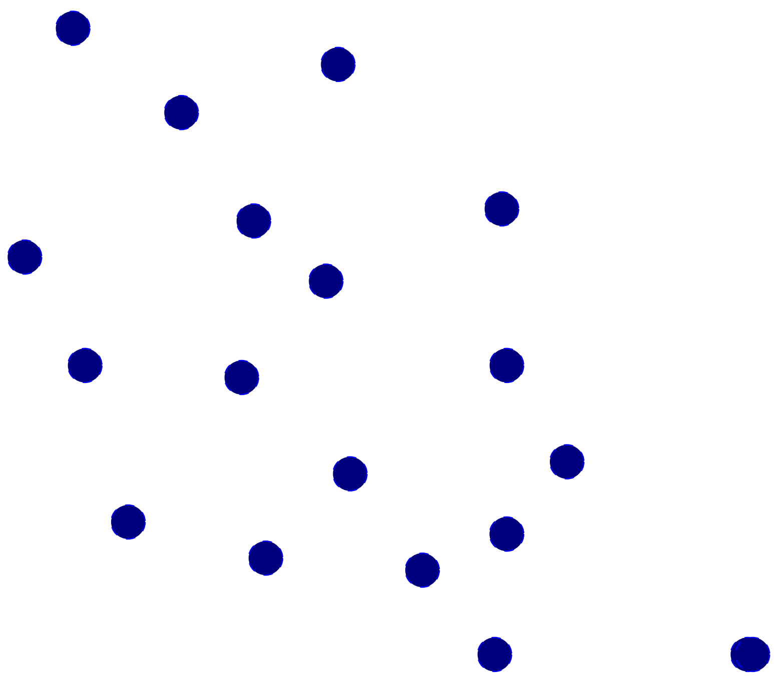
Analysis of D&C



- **Initial sorting takes $O(N \log N)$ time.**
- **Recurrence for divide and conquer** $T(N) = 2T(N/2) + O(N)$
- $O(N)$ **for merging (computing tangents).**
- **Recurrence solves to $T(N) = O(N \log N)$.**

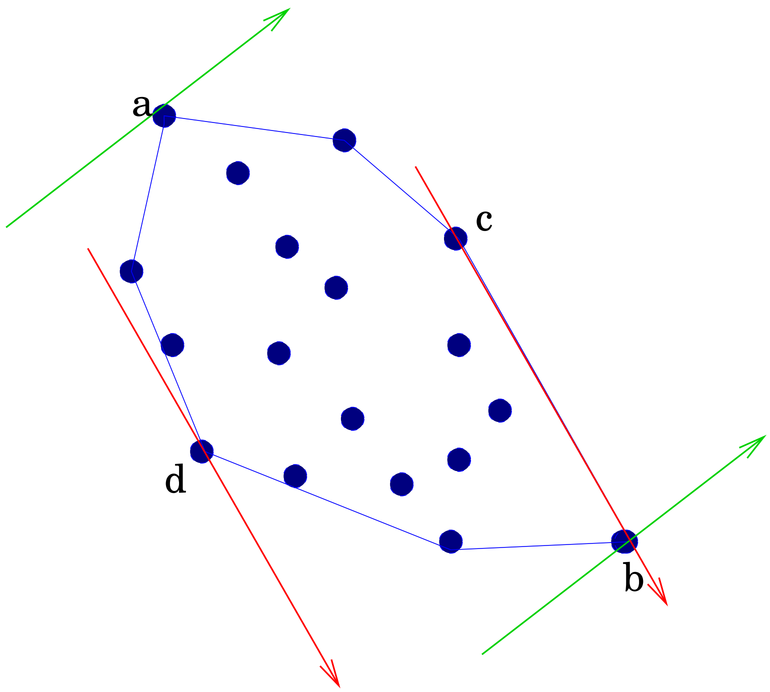
Applications of CH

A problem in statistics



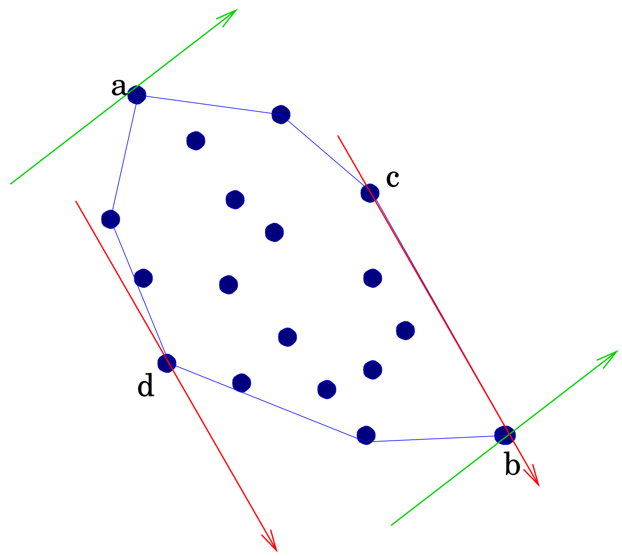
- Given a set of N data points in R^2, fit a line that minimizes the maximum error.
- A data point's error is its L_2 norm distance to the line.

Line Fitting



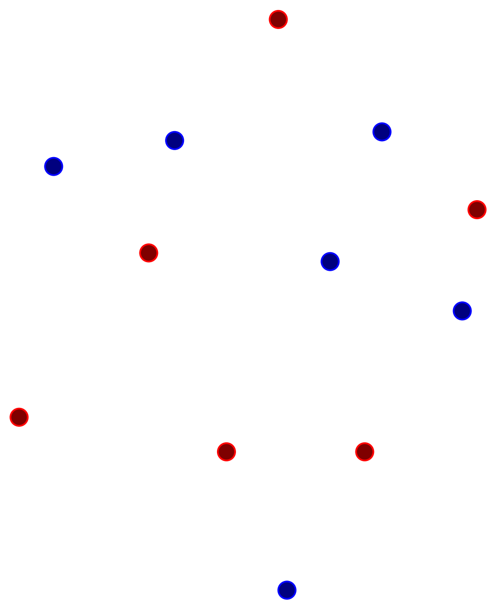
- Minimizing max error = parallel lines of support with Min separation.
- Max error D implies parallel lines of support with separation $2D$, and vice versa.
- Min separation between parallel support lines is also called width of S.

Algorithm for Width



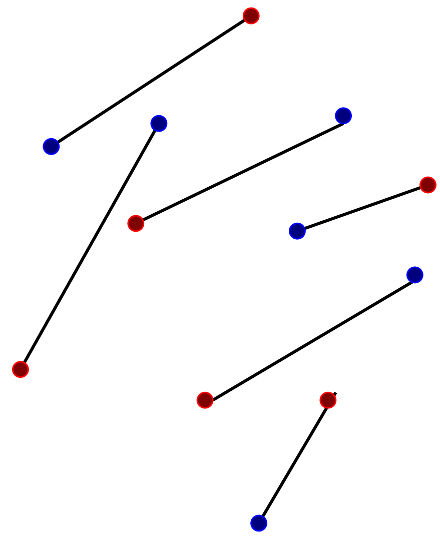
- **Call a, b antipodal pair if they admit parallel lines of support.**
- **In R^2, only $O(N)$ antipodal pairs.**
- **If L_1, L_2 are parallel support lines, with minimum separation, then at least one of the lines contains an edge of $CH(S)$.**
- **We can enumerate all antipodal pairs by a linear march around CH.**

Noncrossing Matching

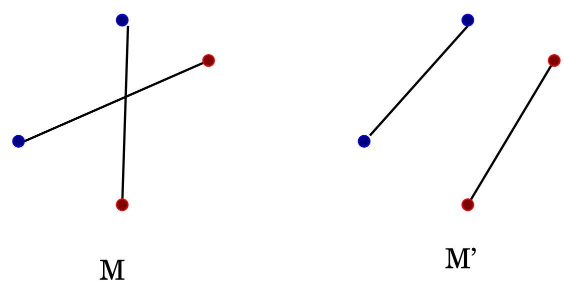


- **Given N red and N blue points in the plane (no three collinear), compute a red-blue non-crossing matching.**
- **Does such a matching always exist?**
- **Find if one exists.**

Noncrossing Matching

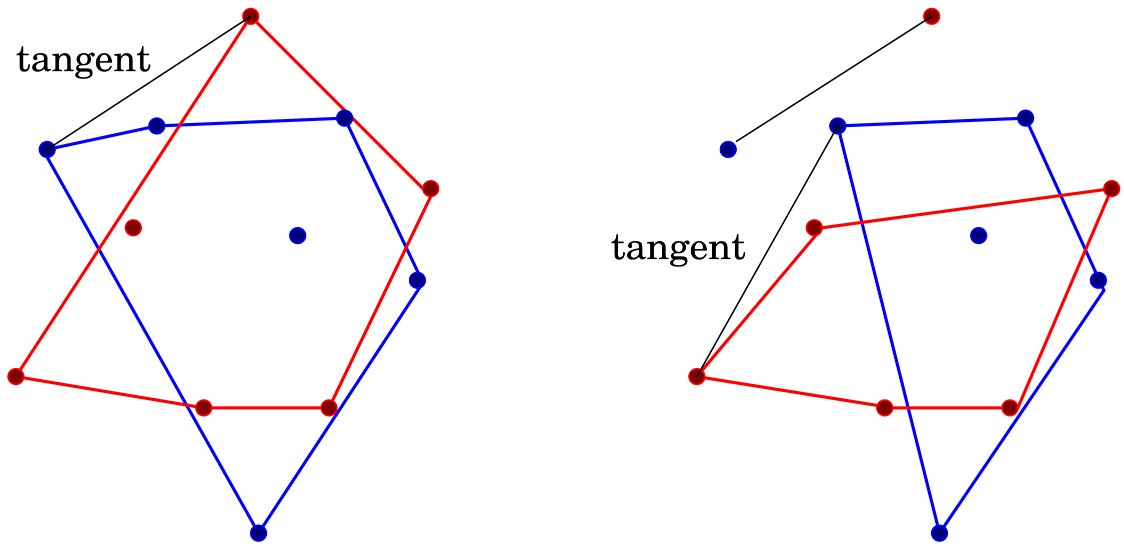


- A non-crossing matching always exists.
- (Non-constructive:) Matching of minimum total length must be non-crossing.



- But how about an algorithm?

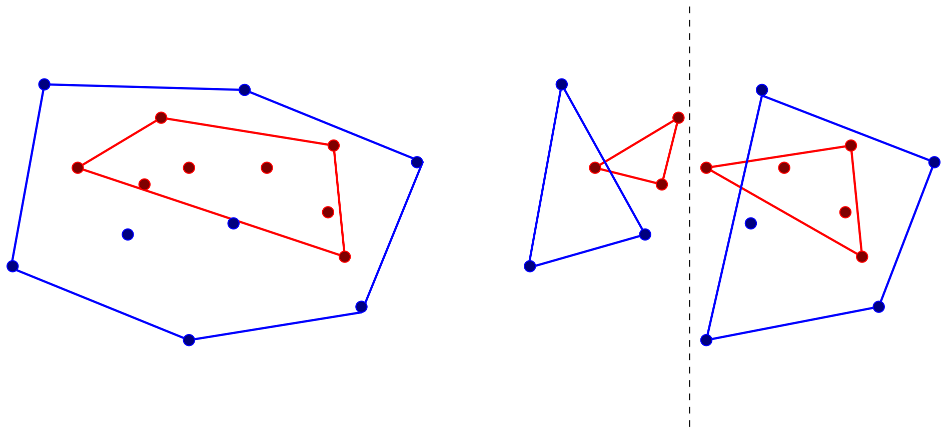
Algorithm



- **Compute $CH(R)$ and $CH(B)$.**
- **Compute a common tangent, say, rb.**
- **Output rb as a matching edge; remove r, b, update convex hulls and iterate.**

When CH Nest?

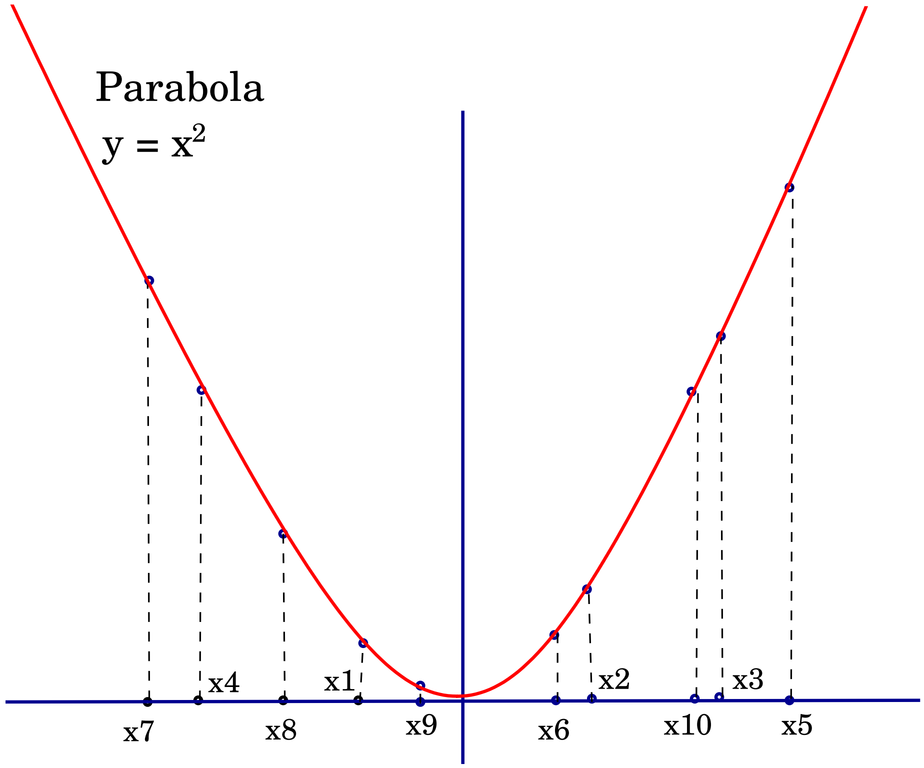
- **Algorithm fails if $CH(R)$ and $CH(B)$ nest.**



- **Split by a vertical line, creating two smaller, hull-intersecting problems.**

- **[Hershberger-Suri '92] gives optimal $O(N \log N)$ solution.**

Lower Bounds



- **Reduce sorting to convex hull.**
- **List of numbers to sort $\{x_1, x_2, \ldots, x_n\}$.**
- **Create point $p_i = (x_i, x_i^2)$, for each i.**
- **Convex hull of $\{p_1, p_2, \ldots, p_n\}$ has points in sorted x-order. $\Rightarrow$ CH of n points must take $\Omega(n \log n)$ in worst-case time.**
- **More refined lower bound is $\Omega(n \log h)$. LB holds even for identifying the CH vertices.**

Output-Sensitive CH

1. **Kirkpatrick-Seidel (1986) describe an $O(n \log h)$ worst-case algorithm. Always optimal—linear when $h = O(1)$ and $O(n \log n)$ when $h = \Omega(n)$.**

2. **T. Chan (1996) achieved the same result with a much simpler algorithm.**

3. **Remarkably, Chan's algorithm combines two slower algorithms (Jarvis and Graham) to get the faster algorithm.**

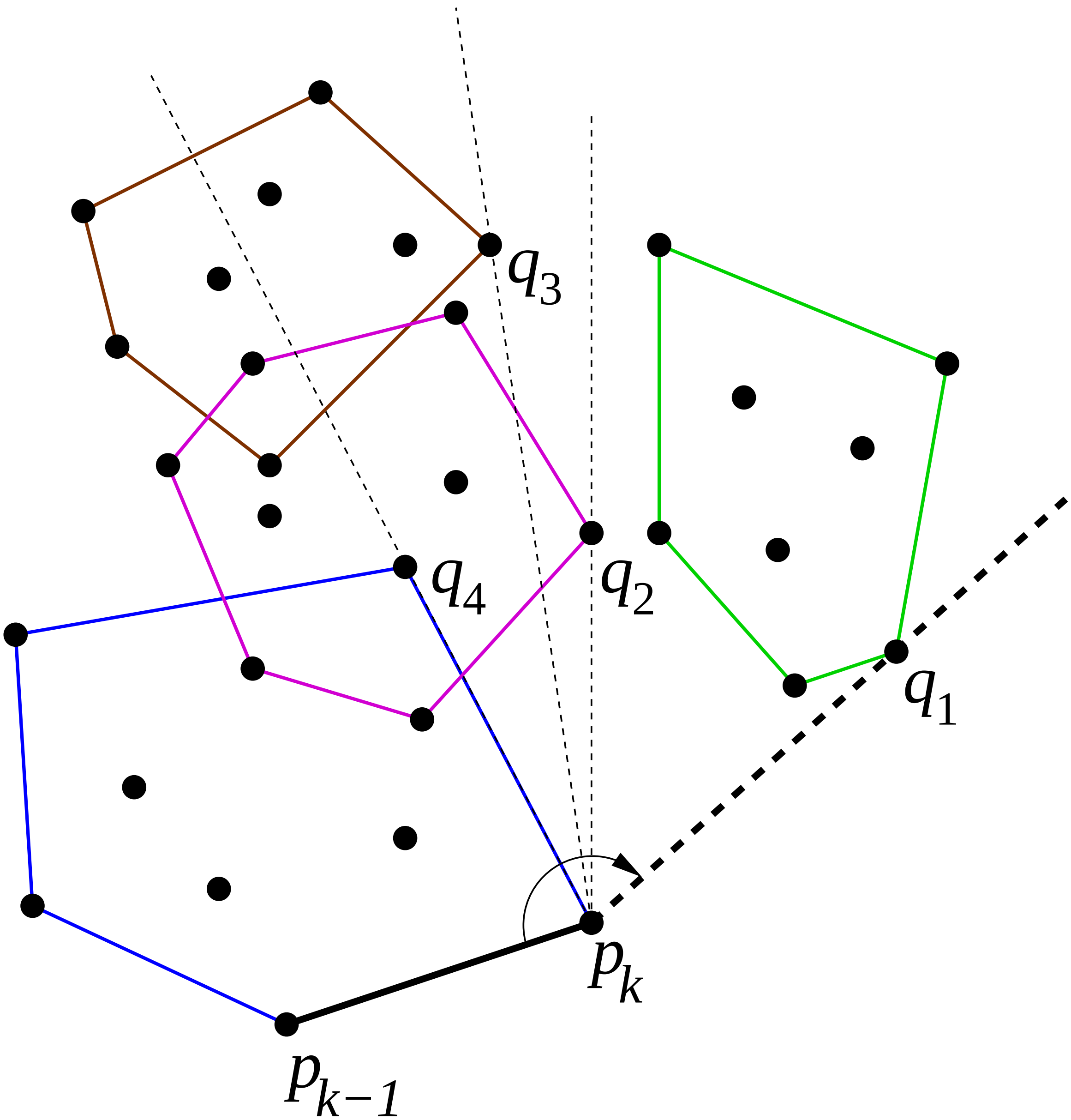
4. **Key idea of Chan is as follows.**

   (a) **Partition the n points into groups of size m; number of groups is $r = \lceil n/m \rceil$.**
   (b) **Compute hull of each group with Graham's scan.**
   (c) **Next, run Jarvis on the groups.**

Chan's Algorithm

1. **The algorithm requires knowledge of CH size h.**
2. **Use m as proxy for h. For the moment, assume we know $m = h$.**
3. **Partition P into r groups of m each.**
4. **Compute $\mathrm{Hull}(P_i)$ using Graham scan,** $i = 1, 2, \ldots, r$.
5. $p_0 = (-\infty, 0)$**;** p_1 **bottom point of P.**
6. **For $k = 1$ to m do**
   - **Find $q_i \in P_i$ that maximizes the angle** $\angle p_{k-1} p_k q_i$.
   - **Let p_{k+1} be the point among q_i that maximizes the angle $\angle p_{k-1} p_k q$.**
   - **If $p_{k+1} = p_1$ then return $\langle p_1, \ldots, p_k \rangle$.**
7. **Return "m was too small, try again."**

Illustration

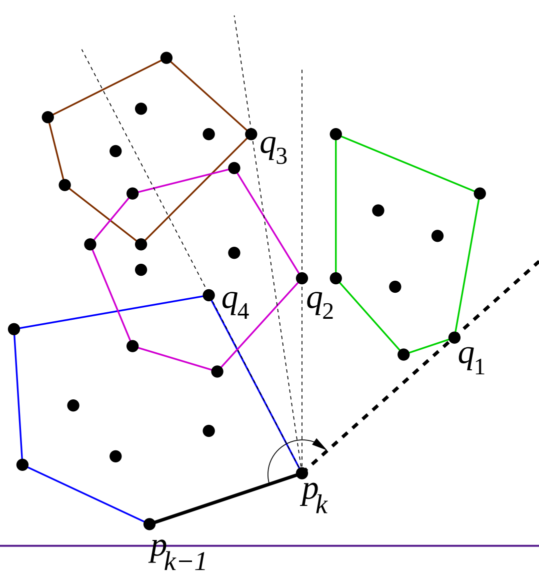


Time Complexity

- **Graham Scan:** $O(rm \log m) = O(n \log m)$**.**
- **Finding tangent from a point to a convex hull in** $O(\log n)$ **time.**
- **Cost of Jarvis on** r **convex hulls: Each step takes** $O(r \log m)$ **time; total** $O(hr \log m) = ((hn/m) \log m)$ **time.**
- **Thus, total complexity**

$$O\left(\left(n + \frac{hn}{m}\right) \log m\right)$$

- **If** $m = h$**, this gives** $O(n \log h)$ **bound.**
- **Problem: We don't know** h**.**



Finishing Chan

Hull(P)

- **for** $t = 1, 2, \ldots$ **do**

  1. **Let** $m = \min(2^{2^t}, n)$**.**
  2. **Run Chan with** m**, output to** L**.**
  3. **If** $L \neq$ **"try again" then return** L**.**

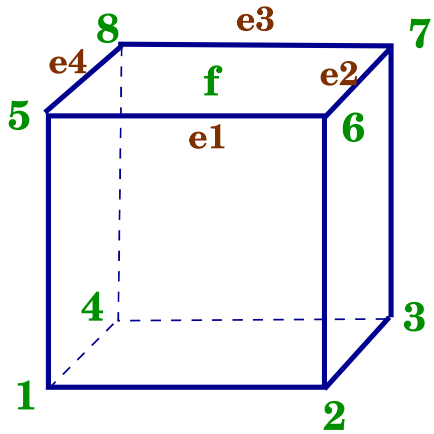
1. **Iteration** t **takes time** $O(n \log 2^{2^t}) = O(n 2^t)$**.**
2. **Max value of** $t = \log\log h$**, since we succeed as soon as** $2^{2^t} > h$**.**
3. **Running time (ignoring constant factors)**

$$\sum_{t=1}^{\lg\lg h} n 2^t = n \sum_{t=1}^{\lg\lg h} 2^t \leq n 2^{1+\lg\lg h} = 2n \lg h$$

4. **2D convex hull computed in** $O(n \log h)$ **time.**

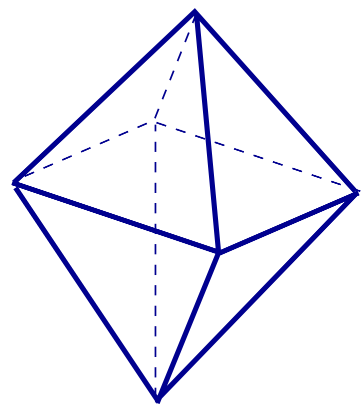
Convex Hulls in d-Space

- New and unexpected phenomena occur in higher dimensions.



cube

V = 8, F = 6



cross polytope

V = 6, F = 8

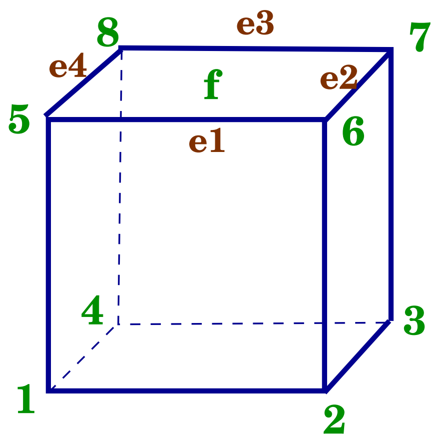
- Number of vertices, faces, and edges not the same.
- How to represent the convex hull? Vertices alone may not contain sufficient information.

Faces

- In d-dimensions, a face can have any dimension k, where $k = 0, 1, \ldots, d-1$.

- Special names: Vertices (dim 0), Edges (dim 1), and Facets (dim $d-1$).

- In general, a k-dim face.



cube

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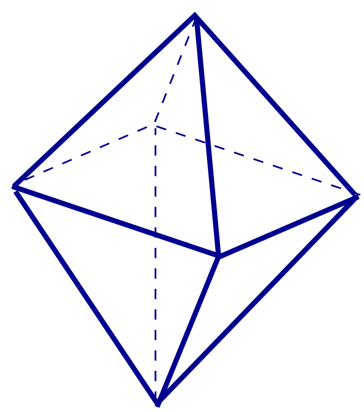
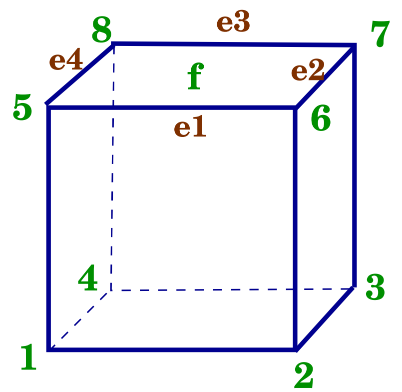


cross polytope

V = 6, F = 8

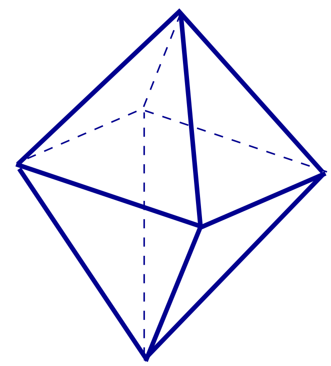
- In 4-dimension, faces are 3d subspace, 2d faces, edges and vertices.

Facial Lattice



cube

V = 8, F = 6

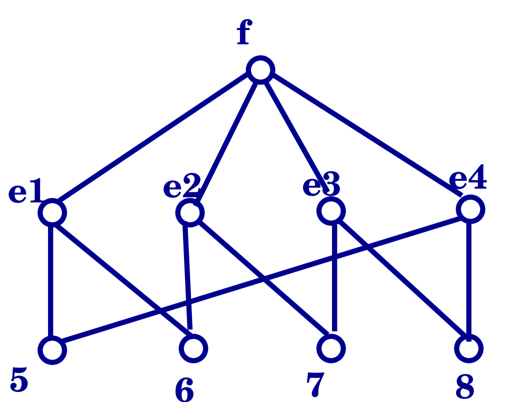


cross polytope

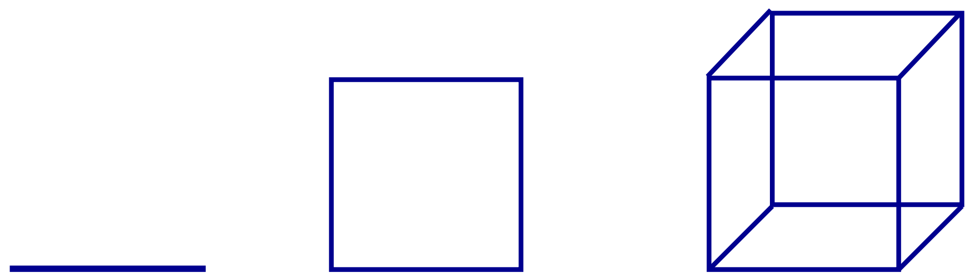
V = 6, F = 8

- **Complete description of how faces of various dimension are incident to each other.**

Face lattice of f



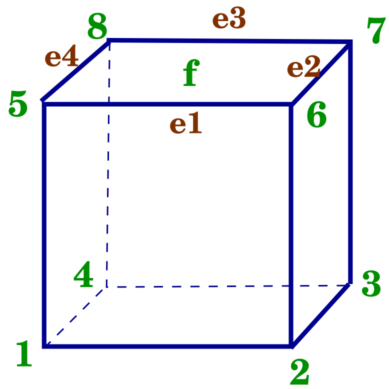
Complexity



Cubes of dim 1, 2, 3....

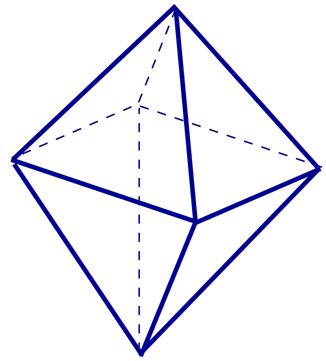
- **How many vertices does d-dim cube have?**
- **How many facets does d-dim cube have?**
- **So, already as a function of d, there is exponential difference between V and F.**
- **But, for a fixed dimension d, how large can the face lattice be as a function of n, the number of vertices?**

3 Dimensions



cube

V = 8, F = 6



cross polytope

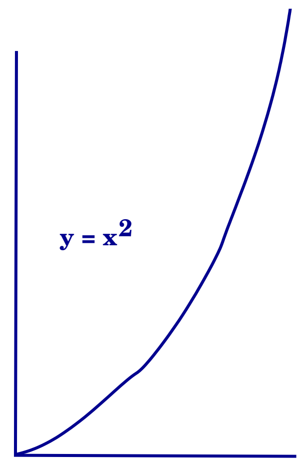
V = 6, F = 8

- **Steinitz: The facial lattice of a 3-d convex polytope is isomorphic to a 3-connected planar graph and vice versa.**
- **By Euler's formula,** $V - E + F = 2$**.**
- **Verify this for cube:** $V = 8, E = 12, F = 6.$
- **In 3D,** E **and** F **are linear in** n**.**
  $E \leq 3n - 6$**, and** $F \leq 2n - 4$**.**

Higher Dimensions

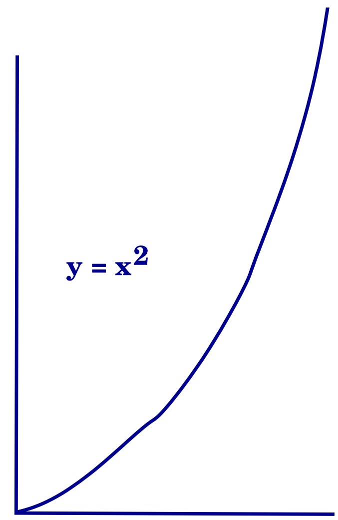
- **Convex polytopes in higher dimensions can exhibit strange and unexpected behavior.**
- **In 4D, there are n points in general position so that the edge joining every pair of points is on the convex hull!**
- **That is, a 4D convex hull of n points can have $\Theta(n^2)$ edges!**
- **In d dimensions, the number of facets can be $n^{\lfloor d/2 \rfloor}$.**
- **Thus, explicit representation of convex hulls is not very practical in higher dimensions.**
- **But this does not mean they are useless: after all linear programming is optimization over convex polytopes.**

Cyclic Polytopes



- **Discovered in 1900's, their importance comes from the Upper Bound Theorem by McMullen and Shephard 1971).**
- **Moment curve:** $\gamma = \{(t, t^2, \ldots, t^d) \mid t \in R\}$.
- **A point** $p = (u, u^2, \ldots, u^d)$ **is given by the single parameter** u.
- **Consider** n **values** $u_1 < u_2 < \cdots < u_n$. **Let** $p_1, p_2, \ldots, p_n$ **be the corresponding points on the moment curve.**
- **Then, any** k**-tuple of points, where** $k \leq d/2$**, is a face of their convex hull.**

4D Example



- **Moment curve is** $\gamma = \{(t, t^2, t^3, t^4)\}$**.**
- **Fix any two** i, j**. Consider the polynomial**

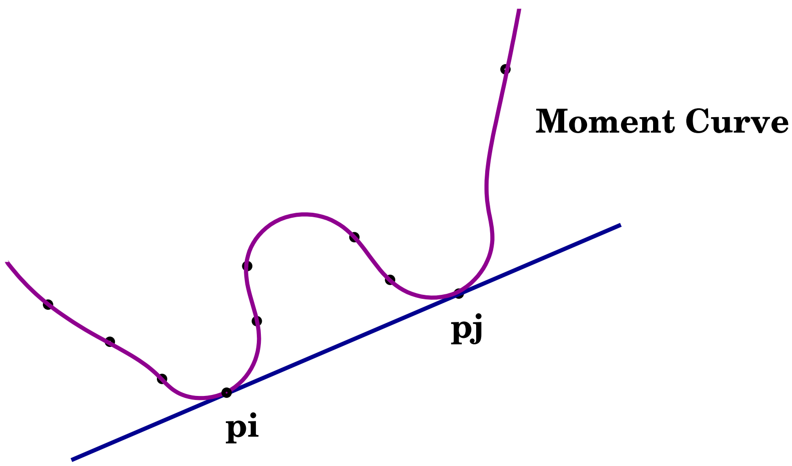
$$P(t) = (t - u_i)^2 (t - u_j)^2$$

- **This polynomial can be written as:**

$$P(t) = t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

- **Clearly,** $P(t) \geq 0$**, for all** t**. Furthermore, the only zeros of the polynomial occur at** $t = u_i$ **and** $t = u_j$**.**

4D Example



- **But $x_4 + a_3x_3 + a_2x_2 + a_1x_1 + a_0 = 0$ is the equation of a hyperplane. This evaluates to zero when $x = p_i$ or p_j.**
- **Since for all other points, the polynomial evaluates to ≥ 0, it means that the moment curves lies on the same side of this plane.**
- **Thus, this plane is the witness that p_ip_j is on the convex hull.**
- **We chose i, j arbitrarily, so all pairs are on the convex hull.**

Upper Bound Theorem

- **Among all d-dim convex polytopes with n vertices, the cyclic polytope has the maximum number of faces of each dimension.**
- **A d-dim convex polytope with n vertices has at most $2\binom{n}{d/2}$ facets and at most $2^{d+1}\binom{n}{d/2}$ faces in total.**
- **Thus, asymptotically, a d-dim convex polytope has $\Theta(n^{\lfloor d/2 \rfloor})$ faces.**
- **A worst-case optimal algorithmn of this complexity is by Chazelle [1993].**