Delaunay Triangulation

Subhash Suri

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1 Delaunay Triangulation

- The Voronoi diagram of n sites in the plane is a *planar subdivision*, which is the embedding of a planar graph. (Use a vertex at infinity as terminus for all half-rays.)
- We now consider another important structure related to VoD, called Delaunay Triangulation. (Assume general position, meaning no four points are cocircular and no three collinear.)
- Define the *graph dual* of VoD, as follows:
  1. For each face of the primal graph (VoD), we create a vertex, and then we add an edge between two such vertices if their faces are adjacent in VoD.
  2. Since each face of the VoD corresponds to a site, say, p_i, we conveniently use p_i as the "vertex" dual of $V(p_i)$.
  3. Two vertices p_i and p_j are joined by an edge (drawn as straight line segment) if $V(p_i)$ and $V(p_j)$ share a voronoi edge.
  4. Observe that each Voronoi vertex corresponds to a 'face' of the dual, which will be a triangle.
  5. (One can use the empty circle property to show that in this straightline dual construction, no two edges cross, so DT is a legal embedding of a triangulation.)
- We observe that the dual graph is a "triangulation" of the input point set P. This follows because each 'face' of the dual corresponds to a Voronoi vertex, which has degree 3 and so the face corresponding to this voronoi vertex has three edges.
- If the sites are not in general position, then the dual graph may not be a triangulation—face dual to a vertex with degree ≥ 4 will be a polygon with 4 or more sides. In that case, one can either arbitrarily triangulate each of those faces, or simulate general position using symbolic perturbation.

- If P has n points, of which k lie on the convex hull of P. Then, Delaunay triangulation of P (in fact, every triangulation) has $(2n - 2 - k)$ triangles and $(3n - 3 - k)$ edges.
- Proof by induction. The k CH vertices create $k - 2$ triangles. Each of the remaining $(n - k)$ points destroys 1 and adds 3 new triangles, giving 2 additional triangles. The total is $(k - 2) + 2(n - k) = (2n - 2 - k)$.

Properties and Applications

- Delaunay triangulations have many nice and surprising geometric properties, which make them a worthy topic of research on their own, not just an after thought as Voronoi duals. The triangles and edges of $DT(P)$ have some nice property.
- **Empty Circle Property of Triangles:** *the circumcircle of* $\triangle pqr$ *does not contain any other site of* P.
- A priori, the existence such a triangulation seems too good to be true: *every point set be triangulated so that each of its triangles has the Empty Circle property!*
- Proof follows from the duality: $\triangle p_i p_j p_k$ is a triangle of DT if the voronoi regions $V(p_i), V(p_j), V(p_k)$ are pairwise neighbors, meaning they share a voronoi vertex. The three closest neighbors of this voronoi vertex v are p_i, p_j, p_k, and so the circle centered at v and passing through p_i, p_j, p_k is empty.
- **Empty Circle Property of Edges:** A pair (p_i, p_j) is an edge of DT *if and only if* there exists an *empty circle* passing through p_i, p_j.
- **Proof.** To prove this, we show that p_i, p_j satisfies the empty circle condition if and only if $V(p_i) \cap V(p_j) \neq \emptyset$.
    1. First, if $V(p_i) \cap V(p_j) \neq \emptyset$, then pick any point x on the shared edge $e_{ij} = V(p_i) \cap V(p_j)$. By property of the Voronoi diagram, we have $d(x, p_i) = d(x, p_j) < d(x, p_k)$, for any $k \neq i, j$. Therefore, the circle with center at x and radius $d(x, p_i)$ satisfies the empty circle claim.
    2. On the other hand, if C is an empty circle passing through p_i, p_j, then let x be its center. Since $d(x, p_i) = d(x, p_j)$, we must have $x \in V(p_i) \cap V(p_j)$. Since P is a finite point set in non-degenerate position, we can move x infinitesimally without violating the empty circle condition. This shows that x lies on an edge that is on the common boundary of $V(p_i)$ and $V(p_j)$.
- **Closest Pair Property:** Given a point set P, if p_i, p_j are the two closest pair of points, then (p_i, p_j) is an edge of DT.

- Proof. The circle with diameter p_i, p_j cannot contain any other point inside, since otherwise p_i, p_j cannot be closest, and so the center of this circle is on a Voronoi edge common to $V(p_i)$ and $V(p_j)$.

- **Largest Empty Circle**: Given a set of n points in the plane, find the largest empty circle, with center inside the convex hull. Applications: dump site, location of a new store, etc.

- One can show that the center is either a vertex of the Voronoi diagram, or lies where a Voronoi edges meets the convex hull.

- **Minimum Spanning Tree.** Another nice property of DT is that the *minimum spanning tree* of the sites is a subgraph of DT.

- The set of n sites induces an Euclidean Graph whose edges are the $\binom{n}{2}$ undirected pairs of distinct points, and each edge's weight is its Euclidean length. The Euclidean MST of this graph is the connected spanning subgraph with minimum total length.

- We could compute the EMST using Kruskal's or Prim's algorithm but since the input graph has $O(n^2)$ edges, the time complexity will be $O(n^2 \log n)$.

- If $EMST \subseteq DT$, then we could build EMST in $O(n \log n)$ time because DT has only $O(n)$ edges and can be computed in $O(n \log n)$ time.

- **MST Theorem:** The MST of a set of n points P (in any dimension) is a subgraph of the DT.

  - **Proof.** Let T be the MST of P, and let $w(T)$ be its weight.
  - Let a, b be two sites such that $ab \in MST$ but $ab \notin DT$.
  - Then, there is no empty circle passing through a, b; in particular, the circle with diameter ab is not empty, and contains another site c.
  - Delete ab from T, which disconnects it into two subtrees T_a, T_b. Assume, wlog, that $c \in T_a$.
  - Let T' be the tree $T - \{ab\} + \{bc\}$, which is also a spanning tree, and whose weight satisfies:

    $$w(T') \ = \ w(T) + \|bc\| - \|ab\| \ < \ w(T)$$

    because ab is the diameter of the circle, and c lies strictly inside, and therefore bc is shorter than ab.
  - This contradicts the hypothesis that T is the MST, and the proof is complete.

- **Minimum Weight Triangulations.** The nearest neighbor property of DT suggests another question: *Among all triangulations of P, does DT minimize the total edge length?*
  - It was claimed (without proof) in a famous paper on DT, and one still hears it quoted occasionally. *The claim, however, is false. There is a simple* 4*-point counterexample, if you want to try.*
  - The complexity of MWT was an open problem for many years, dating back to the original development of NP-completeness in 1970s. Only recently (2008), the problem was shown to be NP-hard; complicated, computer-assisted proof (to verify some of the constructions used).

- **Geometric Spanners.**
  - Suppose we have an undirected graph $G = (V, E, w)$ with non-negative edge weights. A subgraph $H = (V, E', w)$ is called an t-spanner of G, if

$$d_H(u, v) \;\le\; t \cdot d_G(u, v), \quad \forall\, u, v \in V$$

  - That is, pairwise distances in the subgraph approximate the distances of the original graph, within a factor of t. The spanners are useful when G is *dense* and we want a much sparser graph.
  - In our geometric setting, suppose P is a set of cities and we want to build a road network, with roads connecting city-pairs by straightline segments. The only way to achieve minimum distance between all city pairs is to construct $\binom{n}{2}$ roads, one for each pair. Logistically that is too expensive so a natural question is whether there is a sparse subgraph, say, with only $O(n)$ road segments that approximates the shortest distances nicely. Specifically, is there a sparse graph on the point set P such that

$$d_G(u, v) \;\le\; t\|uv\|, \quad \forall\, u, v \in V$$

  - If $t = 1$, G must be the complete graph. The question is if there is a graph with $O(n)$ edges that is a spanner for some small value of t.
  - **Spanner Theorem.** Delaunay triang. is a spanner with $t = 4\pi\sqrt{3}/9 \approx 2.418$.
  - It has been conjectured for many years that DT was a $(\pi/2)$-spanner, where $\pi/2 \approx 1.5708$, but this was disproved in 2009, showing a lower bound of 1.5846
  - Open Problem: Narrow the gap between upper and lower bounds.

- **Maximizing the Minimum Angle.** In many applications, "thin" (small angle) triangles are undesirable—e.g. linear interpolation, finite element method, etc.

- We can, therefore, ask the following question: given a set of points P, find a triangulation of P for which the *smallest angle* is as large as possible. That is, maximize the minimum angle.

- A stronger demand can be to maximize the *angle sequence*. Take any triangulation T of the point set P, and order all the angles of T into the increasing sequence $A(T) = (\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m)$. (Observe that all triangulations of P have the same number of triangles, so this sequence has the same length.)

- Find the triangulation that has the *lexicographically largest* angle sequence $A(T)$.

- **Lex Order Theorem.** Among all triangulations, $DT(P)$ has the *lexicographically largest* angle sequence. In particular, it maximizes the minimum angle.

- **Proof.** We will show that if a triangulation is non-Delaunay, and therefore violates empty-circle property for at least one of its triangles, then we can perform a local operation, called *edge flip*, which increases the lex order of the angle sequence.

- The edge flip a key step in many Delaunay triangulation algorithms. Given two adjacent triangles $\triangle abc$ and $\triangle abd$ whose union is a *convex quadrilateral*, the edge flip operation swaps diagonal ab with cd. (Note that it can only be performed when the the quad $abcd$ is convex.)

Lawson's Flip Algorithm and Local vs. Global Delaunay.

- Let T be a triangulation of P. We say an edge $ab \in T$ is *locally Delaunay* if
  - either ab is an edge of the convex hull, or
  - the apex of each triangle incident to ab lies outside the circumcircle of the other.

- That is, if the triangles incident to ab are $\triangle abc$ and $\triangle abd$, then d must lie outside the circle defined by abc, and vice versa.

- **Globally Delaunay Definition:** Triangulation T is *globally Delaunay* if the circumcircle of each of its triangles is empty of other sites.

- The important point is that the locally Delaunay condition *only checks for empty-circle property against neighboring triangles, and is applied to individual edges,* while DT is a global property. For instance, all edges of T may pass the local Delaunay condition but a triangle may still contain other (non-neighboring) sites.

- But surprisingly the following theorem holds.
- **Theorem:** If all edges of T are locally Delaunay, then T is globally Delaunay.
- We skip the proof, which uses power distances of circle geometry. But we show that, assuming this theorem, we can reach DT through a sequence of flip moves.

Lawson Flip Algorithm

- Start with an arbitrary triangulation T of P, and push all edges of T onto a Stack, and *mark* them.
- **while Stack non-empty, do**
  - Pop the top edge ab and *unmark* it
  - If ab is not locally Delaunay, then swap it with the other diagonal
  - If any of the four edges in $\{ac, ad, bd, bc\}$ is unmarked and no longer locally Delaunay, *mark* and push onto the Stack.
- We show that the algorithm does not get stuck: *flipping* is always possible as long as some edge is non-locally Delaunay.
- In addition, we show that each flip also increases the lexical order of the angle sequence, and so at termination DT must have the largest possible lex order of angle sequence.
- We recall a property from Euclidean geometry (called Thales' Theorem):

  Suppose ab is a chord in a circle, and p, q, r, s are four points lying on the same side of ab, with p, q on, r inside the circle, and s outside the circle. Then, the angles formed by ab at them have the following ordering:

$$\angle r \;>\; \angle p \;=\; \angle q \;>\; \angle s$$

- Recall also that opposite pairs of interior angles of an inscribed (cyclic) quadrilateral sum to 180^o.
- First, we show that "flipping is always possible as long there is an illegal edge." Specifically, if one diagonal is not locally Delaunay, then the other one is.

- Suppose ab is not locally Delaunay, and the circumcircle of abc contains point d.
- Let x_1, x_2, x_3 be the angles of the $\triangle abc$ at a, c and b. Similarly, let y_1, y_2, y_3 be the angles of the $\triangle abd$ at a, d and b.
- By triangle rule, we have $x_1 + x_2 + x_3 = \pi$ and $y_1 + y_2 + y_3 = \pi$.
- By Facts 1 and 2, we observe that $x_2 + y_2 > \pi$. (If d were on the circle, the two angles would have summed to π.)
- Therefore, we have $(x_1 + y_1) + (x_3 + y_3) < \pi$.
- Now consider the circumcircle for $\triangle acd$. The opposite apex b must be outside the circle since the angles at a and b sum to $(x_1 + y_1) + (x_3 + y_3) < \pi$. (Apply Thales theorem for the chord cd!)
- Thus, upon termination, the Lawson algorithm has a triangulation that is globally Delaunay.
- How long does it take?
- **Theorem:** Lawson's flip algorithm terminates in $O(n^2)$ steps.
- Proof is non-trivial. We will later establish it using a duality transform.
- Finally, we need to argue that each flip improves the lexical angle sequence, which then implies that at termination DT has the max angle sequence.

- We just show that the smallest angle after each flip improves.

Computing DT

- DT can be recovered from the Voronoi diagram in linear time and so can be computed in $O(nn \log n)$ time in the plane.
- There are direct flip-based algorithms, most notably randomized incremental constructions, which also run in expected time $O(n \log n)$.