Linking Voronoi Diagram, Delaunay Triangulation and Convex Hulls through the Lifting Transform

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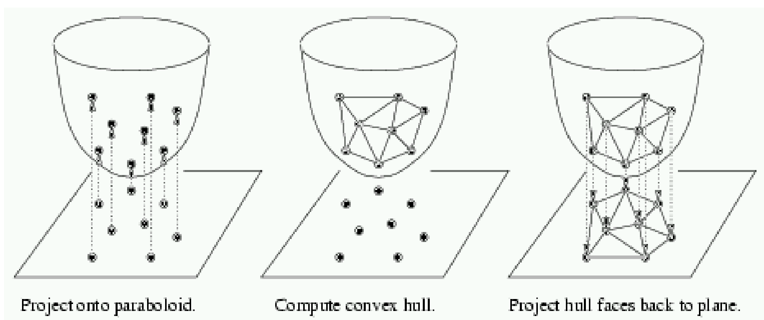
The Lifting Transform

- There is a fascinating relationship between Voronoi diagrams and Delaunay triangulations of 2-dimensional points and the convex hulls of a particular set of 3-dimensional points.
- At first, these structures appear to be quite different. For instance, the Voronoi diagram uses metric properties (distances) while the convex hull depends on affine properties (sided-ness, halfspaces).
- The connection between the two is through a *Lifting Transform*, which maps a set of d-dim points to a set of objects (points or hyperplanes) in dimension $d + 1$. We will demonstrate the connection in dimension 2.
- The basis of the transform is the paraboloid $z = x^2 + y^2$, which defines a surface whose vertical cross sections (constant x or constant y) are parabolas, and whose horizontal cross sections (constant z) are circles.
- For each point (x, y) in the plane, the vertical projection of this point onto this paraboloid is $(x, y, x^2 + y^2)$ in 3 space.
- Given a set of points S in the plane, let S_0 denote the projection of the points in S onto this paraboloid.
- Now, consider the *lower convex hull* of S_0. This is the portion of the convex hull of S_0 which is visible to a viewer standing at $z = -1$.

- We claim that if we take the lower convex hull of S_0, and project it back onto the plane, then we get the Delaunay triangulation of S.
- In particular, let (p, q, r) be elements of S, and let p_0, q_0, r_0 denote the projections of these points onto the paraboloid.
- Then $p_0q_0r_0$ define a face of the lower convex hull of S_0 if and only if pqr is a triangle of the Delaunay triangulation of S. The process is illustrated in the following figure.



Delaunay Triangulation The question is, why does this work? To see why, we need to establish the connection between the triangles of the Delaunay triangulation and the faces of the convex hull of transformed points. In particular, recall that

- [**Delaunay condition:**] Three points p, q, r, in S form a Delaunay triangle if and only if the circumcircle of these points contains no other point of S.
- [**Convex hull condition:**] Three points p_0, q_0, r_0 in S_0 form a face of the convex hull of S_0 if and only if the plane passing through p_0, q_0, and r_0 has all the points of S_0 lying to one side.

Clearly, the connection we need to establish is between the emptiness of circumcircles in the plane and the emptiness of halfspaces in 3 space. We will prove the following claim.

Lemma 1. *Consider 4 distinct points p, q, r, s in the plane, and let p_0, q_0, r_0, s_0 be their respective projections onto the paraboloid, $z = x^2 + y^2$. The point s lies within the circumcircle of p, q, r if and only if s_0 lies on the lower side of the plane passing through p_0, q_0, r_0.*

- To prove the lemma, first consider an arbitrary (nonvertical) plane in 3 space, which we assume is tangent to the paraboloid above some point (a, b) in the plane.

- What is the equation of this tangent plane? We determine the 'slopes' of the plane by taking the derivatives of $z = x^2 + y^2$ with respect to x and y, namely, $dz/dx = 2x$ and $dz/dy = 2y$, and evaluating them at the point $(a, b, a^2 + b^2)$. These evaluate to $2a$ and $2b$. Therefore, the plane passing through these point has the form

$$z = 2ax + 2by + k$$

- To solve for k we use the fact that the plane passes through (a, b, a^2+b^2), and so we can eliminate z by setting:

$$a^2 + b^2 = 2a^2 + 2b^2 + k,$$

  which gives $k = -(a^2 + b^2)$.

- Thus the plane equation is:

$$z = 2ax + 2by - (a^2 + b^2)$$

- Next, if we shift the plane upwards by some positive amount ρ^2 we get the plane

$$z = 2ax + 2by - (a^2 + b^2) + \rho^2$$

- How does this plane intersect the paraboloid? Since the paraboloid is defined by: $z = x^2 + y^2$, we can eliminate z, giving

$$x^2 + y^2 = 2ax + 2by - (a^2 + b^2) + \rho^2$$

  which after some simple rearrangements is equal to

$$(x - a)^2 + (y - b)^2 = \rho^2$$

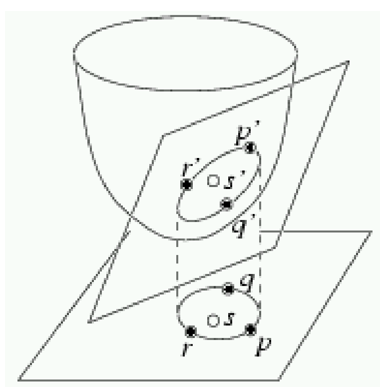
- This is just a circle. Thus, the intersection of a plane with the paraboloid produces a space curve (which turns out to be an ellipse), which when projected back onto the (x, y) coordinate plane is a circle centered at (a, b).

- Furthermore, the squared radius of the circle equals the vertical distance between the projection of the (a, b) onto the paraboloid and its projection onto the plane.

- Thus, we conclude that the intersection of an arbitrary lower halfspace with the paraboloid, when projected onto the (x, y) plane is the interior of a circle.

- Going back to the lemma, when we project the points p, q, r onto the paraboloid, the projected points p_0, q_0, r_0 define a plane. Since p_0, q_0, r_0, lie at the intersection of the plane and paraboloid, the original points p, q, r lie on the projected circle.

- Thus this circle is the (unique) circumcircle passing through these p, q, r. The point s lies within this circumcircle, if and only if its projection s_0 onto the paraboloid lies within the lower halfspace of the plane passing through.



Voronoi Diagram

- Given a point $p = (a, b)$, the hyperplane $H(p)$ that is tangent to p's lifting, namely, $(a, b, a^2 + b^2)$, has the equation

$$z = 2ax + 2by - (a^2 + b^2)$$

- Now, consider an arbitrary point $q = (\alpha, \beta)$ in the plane. What is the *vertical distance* from q to the paraboloid? Just $(\alpha^2 + \beta^2)$.

- What is the vertical distance from q to plane $H(p)$? It is $2a\alpha + 2b\beta - (a^2 + b^2)$.

- Let $\Delta(p, q)$ denote the *difference* between these two vertical distance, namely, the additional distance that q' projection on $H(p)$ has to travel to reach the paraboloid. We get

$$\Delta(p, q) = \alpha^2 + \beta^2 - 2(a\alpha + b\beta) + a^2 + b^2 = (a - \alpha)^2 + (b - \beta)^2$$

- That is, $\Delta(p, q)$ equals precisely the *two-dimensional distance* between p and q in their ambient space.

- Now, consider two points p_1 and p_2 in the plane $z = 0$.

- We claim that q is closer to p_1 *if and only if* at the position $q = (\alpha, \beta)$, the plane $H(p_1)$ lies *above* (closer to the paraboloid) than $H(p_2)$. It simply follows from the vertical distance formula.

- We, therefore, have the following lemma.

Lemma 2. *Let $p_1, p_2, \ldots, p_n$ be a set of points in the plane $z = 0$. Then, a point q belongs to the Voronoi cell of the point p_i if and only if $H(p_i)$ is the highest plane (seen from $z = +\infty$) at q.*

Therefore, the Voronoi diagram of $p_1, p_2, \ldots, p_n$ is simply the vertical projection, down to plane $z = 0$, of the point-wise maxima of the downward facings halfspaces $H(p_i)$. Or, equivalently, is the uppermost face of the arrangement defined by these planes.

Order k Voronoi Diagrams

- We can define order k Voronoi diagram as a partition of the plane into convex regions where each region has the same set of k nearest neighbors.

- The ordinary Voronoi diagram is the order 1. (The relative order of neighbors may change, but the *set* is the same.)

- By the vertical distance argument it is also clear that if we consider the k^{th} *level* in the arrangement formed by the hyperplanes $H(p_i)$, where the topmost level is level 1, then we have the property that for any point on the level k, the same k planes lie above (and thus are closest to) the point on the projected space.