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# Delaunay Graphs Are Almost as Good as Complete Graphs*

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**Abstract.** Let $S$ be any set of $N$ points in the plane and let $\mathrm{DT}(S)$ be the graph of the Delaunay triangulation of $S$. For all points $a$ and $b$ of $S$, let $d(a, b)$ be the Euclidean distance from $a$ to $b$ and let $\mathrm{DT}(a, b)$ be the length of the shortest path in $\mathrm{DT}(S)$ from $a$ to $b$. We show that there is a constant $c$ $(\leq((1+\sqrt{5})/2)\pi \approx 5.08)$ independent of $S$ and $N$ such that

$$\frac{\mathrm{DT}(a, b)}{d(a, b)} < c.$$

## 1. Introduction

Let $\mathrm{DL}_i(S)$ be the Delaunay triangulation of $S$ in the $L_i$ norm $(i=1, 2)$. Chew [Ch] shows that there exists a constant $c_1$ such that the ratio of shortest distances in $\mathrm{DL}_1(S)$ to straight line (i.e., $L_2$) distances is bounded above by $c_1$ where $c_1 = \sqrt{10} \approx 3.16\,228$. We extend this result here demonstrating a constant $c_2$ such that the ratio of shortest distances in $\mathrm{DL}_2(S)$ to straight line distances is bounded above by $c_2 = ((1+\sqrt{5})/2)\pi \approx 5.08$. The best-known lower bound on $c_2$ is $\pi/2$ and is also due to Chew.

In his paper, Chew describes applications of his (and our) result to problems of motion planning, polygon visibility, and extensions of Voronoi diagrams/Delaunay triangulations. Our focus is the derivation of $c_2$ and potential extensions to other problems involving distances in the plane.

In what follows, we provide the definitions and lemmas necessary to prove our main result in Section 2; Section 3 contains the proofs. We conclude with some open problems.

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## 2. The Main Result

We begin with (informal) definitions of the Voronoi diagram and the Delaunay triangulation. The *Voronoi diagram* for a set $S$ of $N$ points in the plane is a partition of the plane into regions, each containing exactly one point in $S$, such that, for each point $p \in S$, every point within its corresponding region (denoted $\mathrm{Vor}(p)$) is closer to $p$ than to any other point of $S$. The boundaries of these regions form a planar graph. The *Delaunay triangulation* of $S$ is the straight-line dual of the Voronoi diagram for $S$; that is, we connect a pair of points in $S$ if and only if they share a Voronoi boundary. Under the standard assumption that no four points of $S$ are cocircular, the Delaunay triangulation is indeed a triangulation [PS]; we denote its corresponding graph by $\mathrm{DT}(S)$.

For the remainder of this section, fix points $a, b \in S$; we will construct a path in $\mathrm{DT}(S)$ that is not too long in relation to $d(a, b)$. Assume for simplicity that $a$ and $b$ lie on the $x$-axis, with $x(a) < x(b)$ (we denote the coordinates of a point $q$ in the plane by $x(q)$ and $y(q)$, respectively). We refer to members of $S$ alternatively as points or vertices, and to edges of $\mathrm{DT}(S)$ as edges or line segments, as the context indicates.

Our original idea for the path was simply to use the vertices $a = b_0$, $b_1, \ldots, b_{m-1}, b_m = b$ corresponding to the sequence of Voronoi regions traversed by walking from $a$ to $b$ along the $x$-axis, as illustrated in Fig. 1, where $m = 4$ (in the case in which a Voronoi edge happens to lie on the $x$-axis somewhere between $a$ and $b$, we—arbitrarily—choose that Voronoi region lying above, rather than

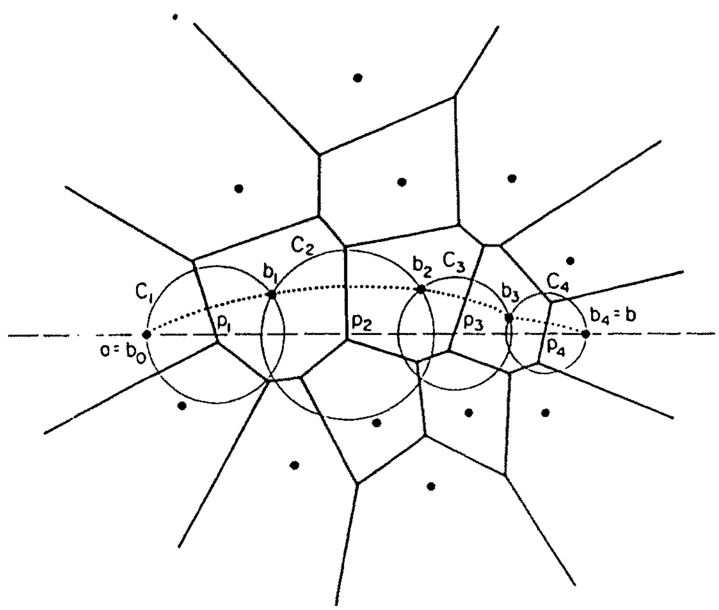


**Fig. 1.** The Voronoi diagram is shown in solid line, and the direct DT path between $a$ and $b$ in dotted line.

below, the $x$-axis). In general, we refer to the DT path constructed in this way between some $z$ and $z'$ in $S$ as the *direct DT path* from $z$ to $z'$. Let $p_i$ denote the point on the $x$-axis that also lies on the boundary between $\text{Vor}(b_{i-1})$ and $\text{Vor}(b_i)$, for $i = 1, 2, \ldots, m$. The definition of the Voronoi diagram immediately gives that $p_i$ is the center of a circle $C_i$ passing through $b_{i-1}$ and $b_i$ but containing no points of $S$ in its interior.

Two simple properties of direct DT paths are:

**Lemma 1.** $x(b_0) \leq x(b_1) \leq \cdots \leq x(b_m)$.

**Lemma 2.** *For all $i$, $0 \leq i \leq m$, $b_i$ is contained within, or on the boundary of,* circle$(a, b)$ *(by which we denote the circle with a and b diametrically opposed).*

Note in Fig. 1 that all the $b_i$ happen to be in the same half-plane defined by the line connecting $a$ and $b$ (i.e., $y(b_i) \geq 0$ for all $0 \leq i \leq m$). In such cases, we say that the direct path between the two points is *one-sided.* One-sided paths are fortuitous for our purposes, because the ratio of the path length to the Euclidean distance is at most $\pi/2$; this is a simple consequence of Lemma 1 above and the following:

**Lemma 3.** *Let $D_1, D_2, \ldots, D_k$ be circles all centered on the x-axis such that $D = \bigcup_{1 \leq i \leq k} D_i$ is connected. Then* boundary$(D)$ *has length at most $\pi \cdot (x_r - x_l)$, where $x_l$ and $x_r$ are the least and greatest x-coordinates of D, respectively (see Fig.* 2).

Lemma 3 applies to the one-sided paths because the half of boundary$(C)$ (where $C$ is defined as $\bigcup_{1 \leq k \leq m} C_k$) that lies above the $x$-axis has length at least as great as the path itself (because the $b_i$ are monotonic in $x$).

The trouble with this approach is that the path is not necessarily even close to being one-sided; the path may zig-zag across the $x$-axis (as is illustrated in Fig. 3) $\Theta(N)$ times.

Our modified approach, then, is to try to stay above the $x$-axis. Should the direct path dip below the $x$-axis, we determine how costly the dip will be. If dipping below is not too expensive (in a sense defined below) then we follow the direct path below the $x$-axis and then back up. Otherwise, we construct a shortcut between the two points above the $x$-axis. Most of the proof consists of showing that the shortcut is not too long. The exact path we take is made more precise in the proof of the following:

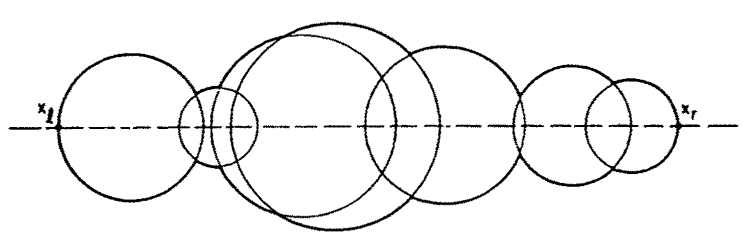


**Fig. 2.** Illustration for Lemma 3.

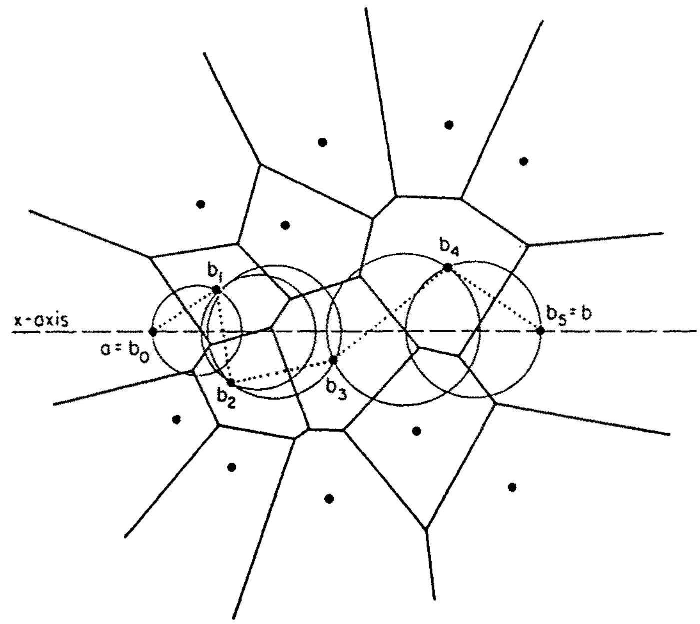


**Fig. 3.** A direct DT path that is *not* one-sided.

**Theorem.** *There exists a DT path from a to b of length*

$$\leq((1+\sqrt{5})/2)\pi \cdot d(a, b).$$

*Proof.* We present an algorithm for constructing a DT path from $a = b_0$ to $b = b_m$, and then analyze the length of the path it produces. Assume that the path so far has brought us to some $b_i$ such that (1) $y(b_i) \geq 0$ (initially, $i = 0$), (2) $i < m$ (meaning we are not finished), and (3) $y(b_{i+1}) < 0$. Thus the direct path would dip below the $x$-axis for a while after $b_i$. Let $j$ be the least number greater than $i$ such that $y(b_j) \geq 0$ (e.g., in Fig. 4, if $i = 2$ then $j = 4$). Let $T$ denote the path along the boundary of $C$ clockwise from $b_i$ to $b_j$. Let $w$ denote the length of the projection of $T$ onto the $x$-axis (thus $w = x(b_j) - x(b_i)$). Define $h = \min\{y(q)\colon q \text{ lies on } T\}$. Now if $h \leq w/4$ then continue along the direct path to $b_j$

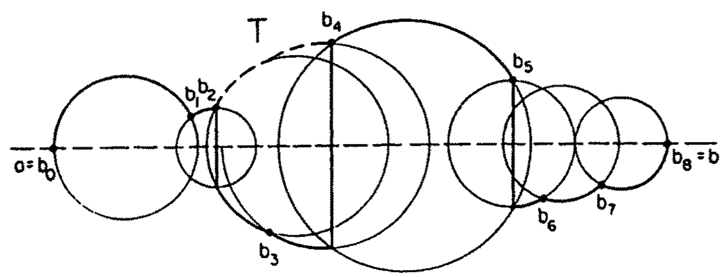


**Fig. 4.** An upper bound on the length of the direct DT path.

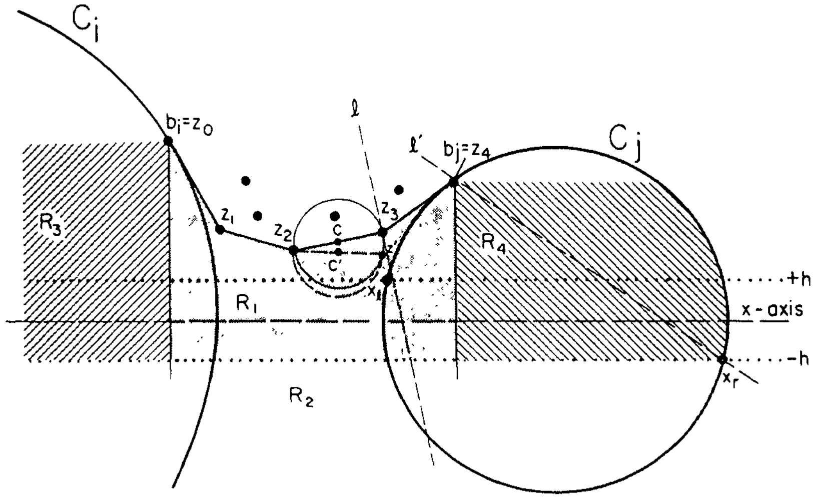


**Fig. 5.** The shortcut from $b_i$ to $b_j$. Here $k = 2$.

(i.e., use edges $b_i b_{i+1}, b_{i+1} b_{i+2}, \ldots, b_{j-1} b_j$). Otherwise we take a shortcut as follows. Construct the lower convex hull $b_i = z_0, z_1, z_2, \ldots, z_n = b_j$ of the set

$$\{q \in S: x(b_i) \le x(q) \le x(b_j) \text{ and } y(q) \ge 0 \text{ and } q \text{ lies under } b_i b_j\}$$

(see Fig. 5). Note that these convex hull edges are certainly not on the direct DT path from $a$ to $b$. Now the shortcut consists of taking the direct DT path from $z_k$ to $z_{k+1}$ for each $0 \le k \le n-1$. The key fact (proved in Section 3) is:

**Lemma 4.** *Let $z_k z_{k+1}$ be an edge of the lower convex hull described above. Then the direct DT path from $z_k$ to $z_{k+1}$ is one-sided.*

Next we analyze the length of the path produced by this algorithm. When proceeding from $b_i$ to $b_j$, let $t$ denote the length of $T$. If $h \le w/4$ then let $q_0$ be the point of $T$ with least $y$-value (see Fig. 6), let $t_i$ denote the length of the portion of $T$ from $b_i$ to $q_0$, and $t_j$ the length of the portion of $T$ from $q_0$ to $b_j$ (thus $t_i + t_j = t$). Let $w_i$ and $w_j$ denote the lengths of the projections of those two portions of $T$, respectively (thus $w_i + w_j = w$). Then the path we take (i.e., no shortcuts) has length at most

$$\begin{aligned} t + 2(y(b_i) + y(b_j)) &= t + 2(2h + (y(b_i) - h) + (y(b_j) - h)) \\ &\le t + 2\left(\frac{w}{2} + (y(b_i) - h) + (y(b_j) - h)\right) \\ &= t + 2\left(\frac{w_i}{2} + (y(b_i) - h) + \frac{w_j}{2} + (y(b_j) - h)\right) \\ &\le t + 2\left(\frac{\sqrt{5}}{2} t_i + \frac{\sqrt{5}}{2} t_j\right) = t(1 + \sqrt{5}). \end{aligned}$$

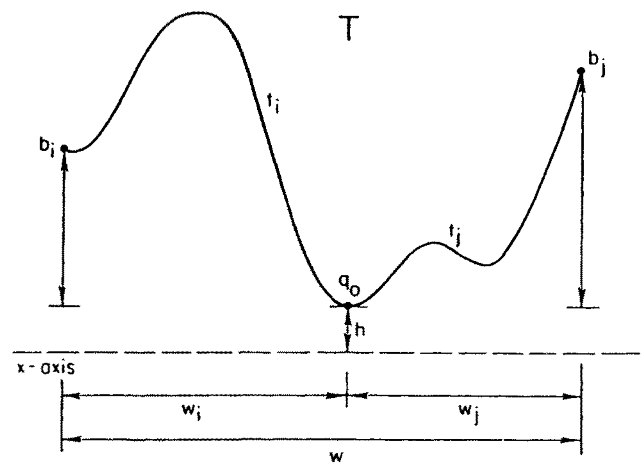


**Fig. 6.** Analyzing the path length when the shortcut is *not* taken.

The last inequality follows from the (easily proved) fact that

$$\frac{a}{2}+b\leq\frac{\sqrt{5}}{2}c$$

whenever $a$ and $b$ are the legs of a right triangle with hypotenuse $c$.

On the other hand, if $h > w/4$ then we take the shortcut, which has length at most

$$\sum_{k=0}^{n-1} \text{length of one-sided path from } z_k \text{ to } z_{k+1}$$

(by Lemma 4) which is $\leq\sum_{k=0}^{n-1} d(z_k, z_{k-1})\pi/2\leq t\pi/2$ (by Lemma 3). Hence in either case, the distance we travel in getting from $b_i$ to $b_j$ is at most $(1+\sqrt{5})t$. Therefore summing over all such trips $b_i$ to $b_j$ as well as the trips (for which we travel at most $t$ units) where the direct DT path from $a$ to $b$ stays completely above the $x$-axis, we get (by Lemma 3) a total path length of at most $d(a, b)((1+\sqrt{5})/2)\pi$. □

## 3. Proofs of the Lemmas

*Proof of Lemma 1.* The perpendicular bisector of $b_i$ and $b_{i+1}$ contains $p_i$. Point $b_{i+1}$ lies to the right of this bisector, and $b_i$ lies to the left; hence $x(b_i)\leq x(b_{i+1})$. □

*Proof of Lemma 2.* Let $c$ denote the midpoint of segment $ab$; let $k$ be such that $c$ lies in the Voronoi region of $b_k$. Then

$$d(b_0, c)\geq d(b_1, c)\geq\cdots\geq d(b_k, c)$$

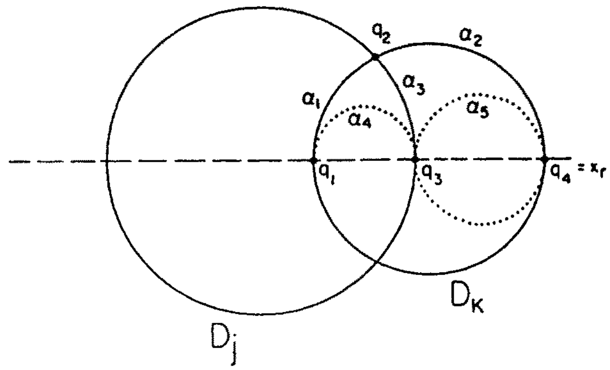


**Fig. 7.** Illustration for the proof of Lemma 3.

and

$$d(b_k, c) \le d(b_{k+1}, c) \le \cdots \le d(b_m, c). \qquad \square$$

*Proof of Lemma 3.* By induction on $k$. The claim is easy if $k=1$; so let $k \ge 2$ and assume it for $k-1$. Let $q_1$ and $q_4$ denote the leftmost and rightmost points of $D_k$, respectively (see Fig. 7), and assume without loss of generality that $q_4 = x_r$. Let $q_2$ be the rightmost point at which $D_k$ intersects another circle $D_j$ (thus $j < k$); let $q_3$ be the rightmost point of $D_j$. We can assume that $D_k$ does not entirely contain any circle $D_i$ ($i \ne k$), since otherwise $D_i$ would not contribute to boundary(D) and hence the induction would be trivial. Denote by $\alpha_1$ ($\alpha_2$) the length of the arc on circle $D_k$ clockwise from $q_1$ to $q_2$ (resp. $q_2$ to $q_4$). Let $\alpha_3$ be the length of the arc on circle $D_j$ clockwise from $q_2$ to $q_3$. Finally, let $\alpha_4 = (\pi/2)(x(q_3) - x(q_1))$ and let $\alpha_5 = (\pi/2)(x(q_4) - x(q_3))$. Then a simple convexity argument shows that

$$\alpha_1 + \alpha_3 \ge \alpha_4.$$

Also, we have

$$\alpha_4 + \alpha_5 = \alpha_1 + \alpha_2.$$

Hence

$$\alpha_1 + \alpha_3 + \alpha_5 \ge \alpha_4 + \alpha_5 = \alpha_1 + \alpha_2,$$

implying $\alpha_3 + \alpha_5 \ge \alpha_2$. Therefore, denoting the length of the boundary of $D$ by bd($D$), we have

$$\begin{aligned}
\mathrm{bd}(D) &\le \mathrm{bd}\left(\mathrm{circle}(q_3, q_4) \cup \bigcup_{1 \le i \le k-1} D_i\right) \\
&\le \mathrm{bd}(\mathrm{circle}(q_3, q_4)) + \mathrm{bd}\left(\bigcup_{1 \le i \le k-1} D_i\right) \\
&\le \pi(x_r - x(q_3)) + \pi(x(q_3) - x_l) \qquad \text{(by the inductive hypothesis)} \\
&\le \pi(x_r - x_l). \qquad \square
\end{aligned}$$

*Proof of Lemma 4.* By Lemma 2, the direct DT path from $z_k$ to $z_{k+1}$ lies entirely within circle$(z_k, z_{k+1})$. We now show that there are no points of $S$ within the lower semicircle of circle$(z_k, z_{k+1})$, so the path must be one-sided.

Let $q$ be an arbitrary point in this lower semicircle; we must show $q \notin S$. If $x(b_i) \le x(q) \le x(b_j)$ and $y(q) \ge -h$ (i.e., $q$ lies in region $R_1$ in Fig. 5) then we claim $q \notin S$. To see this, note that if $y(q) \ge h$ then it lies outside the lower convex hull; whereas if $-h < y(q) < h$ then $q$ lies in the interior of $\bigcup_{i \le k \le j} C_k$.

We next show that $y(q) > -h$ (that is, $q \notin R_2$). Assume without loss of generality that $y(z_k) \le y(z_{k+1})$. Since $z_k \in S$ it must lie directly above some point of $T$, since the area below $T$ and above the $x$-axis is contained in $C$ and therefore contains no members of $S$. Therefore $y(z_k) \ge h > w/4$. Let $z'$ be the point with coordinates $(x(z_{k+1}), y(z_k))$. Let $c$ and $c'$ denote the midpoints of segments $z_k z_{k+1}$ and $z_k z'$, respectively. Then $y(c') > w/4$. That $q \in \text{circle}(z_k, z')$ follows from $q \in \text{circle}(z_k, z_{k+1})$ and $y(q) \le y(z_k) = y(z')$. Furthermore, $x(z_{k+1}) - x(z_k) \le w$, since by extending $z_k z_{k+1}$ on both sides we encounter points on $T$ and since $T$ is connected (and hence the projection of $T$ onto the $x$-axis is at least as long as the projection of $z_k z_{k+1}$ onto the $x$-axis). Therefore radius(circle$(z_k, z')) \le w/2$. Hence

$$y(q) \ge y(c') - \text{radius}(\text{circle}(z_k, z')) > w/4 - w/2 = -w/4.$$

Note that $x(q) \ge x(b_i)$ (that is $q \notin R_3$), because of our assumption $y(z_k) \le y(z_{k+1})$.

Finally, we assume $x(q) > x(b_j)$ (hence $q \in R_4$). We show that $q$ lies in the interior of $C_j$, implying $q \notin S$. Let $x_l$ be the leftmost point of intersection of circle $C_j$ with the line $y = h$. Let $x_r$ be the rightmost point of intersection of $C_j$ with the line $y = -h$. Let $l$ denote the line that passes through $z_{k+1}$ perpendicular to segment $z_k z_{k+1}$, and let $l'$ be the line containing $b_j$ and $x_r$. Note that both $l$ and $l'$ must have negative slopes. Clearly, the entire circle $(z_k, z_{k+1})$ lies below $l$ and in particular so does $q$. We claim that this implies that $q$ lies below $l'$ as well. To see this, first note that our assumption $y(z_k) \le y(z_{k+1})$ implies $y(z_{k+1}) \le y(b_j)$, and hence line $l$ intersects the line $x = x(b_j)$ below $b_j$. Therefore it suffices to show that slope$(l) \le$ slope$(l')$ (recall that both are negative). The monotonicity of slopes in the lower convex hull gives slope$(z_k z_{k+1}) \le$ slope$(x_l b_j)$. Therefore since $l$ and $l'$ are perpendicular to $z_k z_{k+1}$ and $x_l b_j$, respectively (the latter is because $x_l$ and $x_r$ are diametrically opposed on $C_j$), we have slope$(l) \le$ slope$(l')$. Thus $q$ indeed lies below $l'$; hence since $q$ is in $R_4$ it must also be in $C_j$ and therefore not in $S$. □

## 4. Related Problems

There are many interesting problems related to that solved here. For example, Raghavan [Ra] suggests that our results extend to a special case problem in 3-space. He conjectures that if $S$ is a set of points on the unit sphere, there is a constant $c$ such that

$$\frac{d_H(a, b)}{d(a, b)} < c,$$

where $d_H$ is the distance along edges of the convex hull and $d$ is the (three-dimensional) Euclidean distance.

The generalization of our result to arbitrary point sets in 3-space and their Delaunay graphs remains open.

In another direction, Feder and others [Fe] have shown that for each $k \geq 7$ there is a constant $c$ such that, for each finite set $S$ of points in the plane, there is a graph $G$ with vertices corresponding to these points, and the following properties:

(1) Each vertex in $G$ has degree at most $k$.
(2) $d_G(a, b)/d(a, b) < c$, where $d_G$ is the distance along edges of $G$.

Extensions to the cases $k = 5$ and 6 have been proposed by others. It is not difficult to show that no such constant exists for $k = 2$. What is the minimum $k$ for which such a result is possible?

## Acknowledgment

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