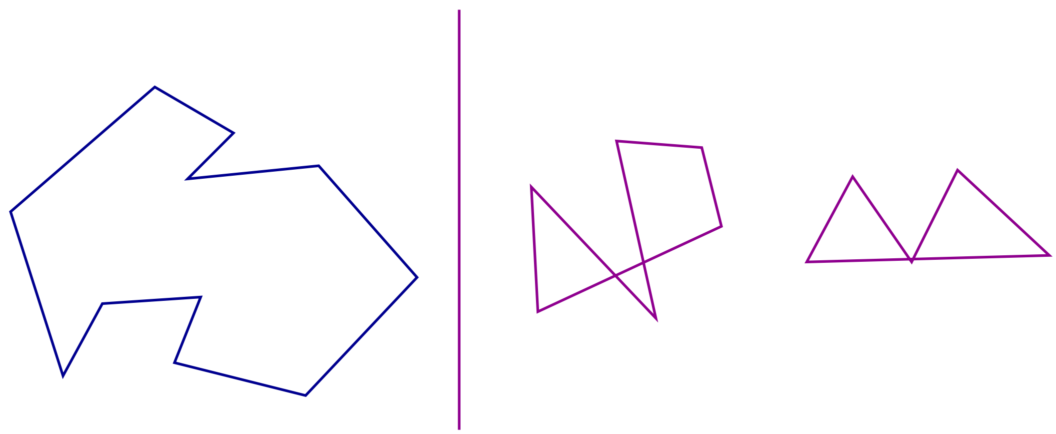
Polygon Triangulation

- **A polygonal curve is a finite chain of line segments.**
- **Line segments called edges, their endpoints called vertices.**
- **A simple polygon is a closed polygonal curve without self-intersection.**

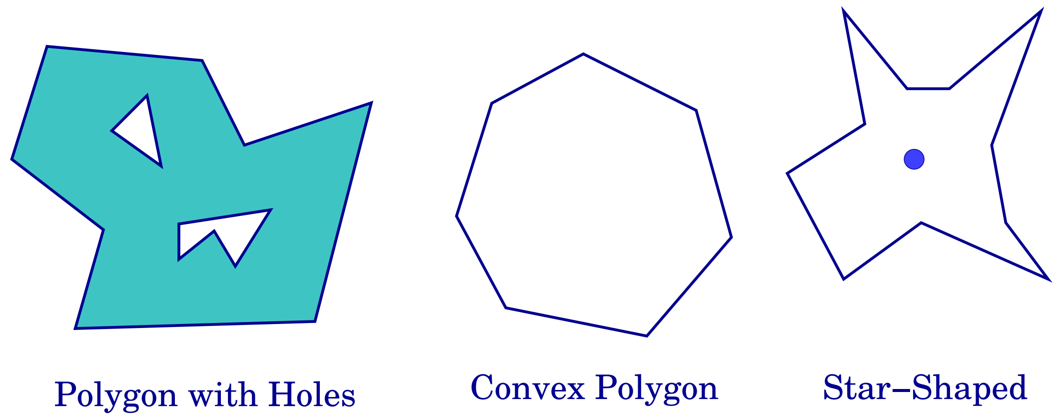


Simple Polygon   Non–Simple Polygons

- **By Jordan Theorem, a polygon divides the plane into interior, exterior, and boundary.**
- **We use polygon both for boundary and its interior; the context will make the usage clear.**

Polygons

- **Polygons with holes** are topologically different; **two paths may not be homeomorphic.**



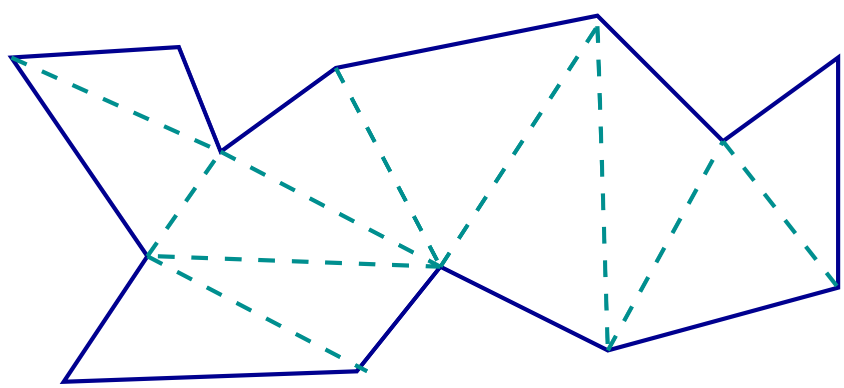
- **Other common classes of polygons are convex, star-shaped, monotone.**

- **Polygons are basic buliding blocks in most geometric applications.**

  – **Flexible:** model arbitrarily complex shapes.
  – **Efficient:** admit simple algorithms and algebraic representation/manipulation.
  – Thus, **significantly more powerful**, say, than rectangles as building blocks.

Triangulation

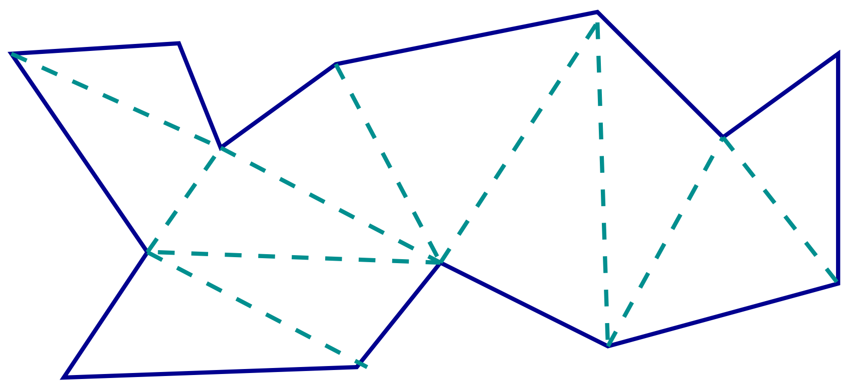
- **Partition polygon P into non-overlapping triangles using diagonals only.**
- **Is this always possible for any simple polygon? If not, which polygons are triangulable.**
- **Does the number of triangles depend on the choice of triangulation? How many triangles?**



- **Triangulation reduces complex shapes to collection of simpler shapes. First step of many advanced algorithms.**
- **Many applications: visibility, robotics, mesh generation, point location etc.**

Triangulation Theorem

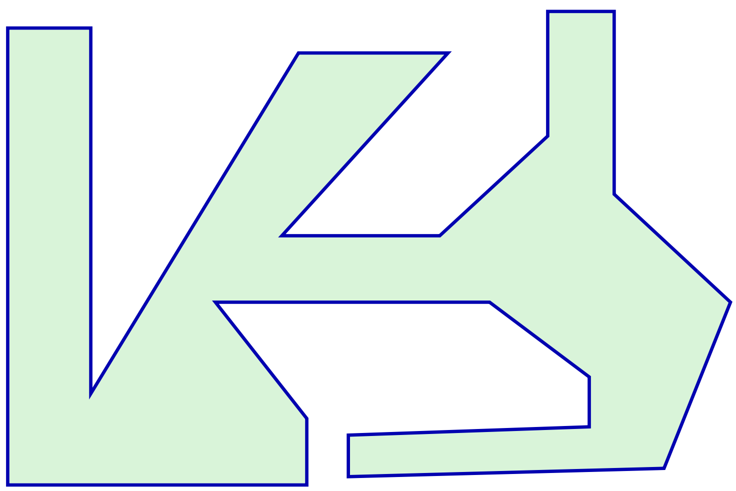
1. **Every simple polygon admits a triangulation.**
2. **Every triangulation of an n-gon has exactly $n - 2$ triangles.**



3. **Polygon in picture has $n = 13$, and 11 triangles.**
4. **Before proving the theorem and developing algorithms, consider a cute puzzle that uses triangulation: Art Gallery Theorem.**

Art Gallery Theorem

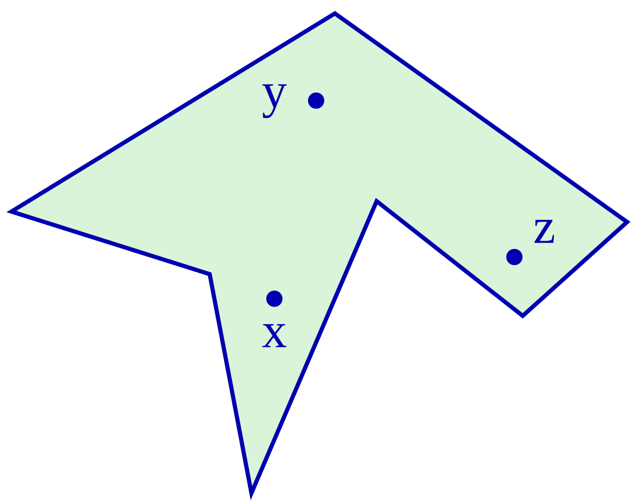
- **The floor plan of an art gallery modeled as a simple polygon with n vertices.**
- **How many guards needed to see the whole room?**
- **Each guard is stationed at a fixed point, has 360^o vision, and cannot see through the walls.**



- **Story:** **Problem posed to Vasek Chvatal by Victor Klee at a math conference in 1973. Chvatal solved it quickly with a complicated proof, which has since been simplified significantly using triangulation.**

Formulation

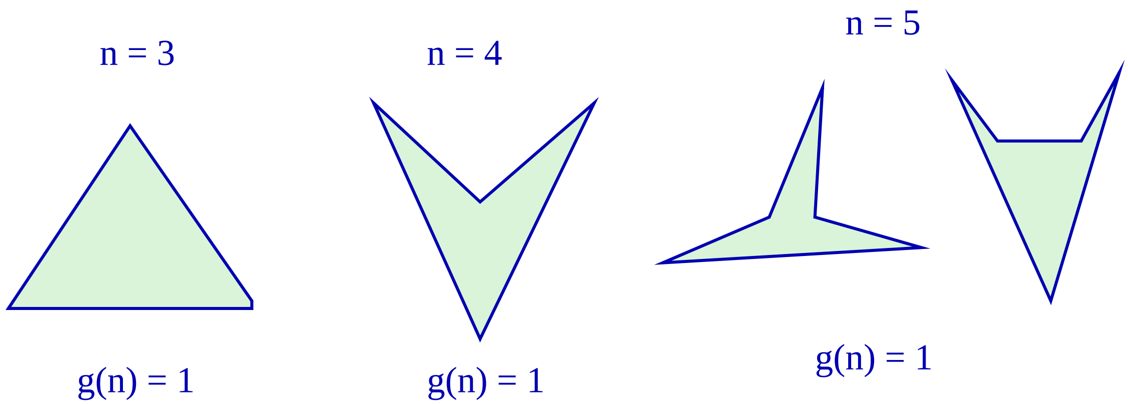
- **Visibility:** p, q **visible if** $pq \in P$.
- y **is visible from** x **and** z**. But** x **and** z **not visible to each other.**



- $g(P) =$ **min. number of guards to see** P
- $g(n) = \max_{|P|=n} g(P)$
- **Art Gallery Theorem asks for bounds on function** $g(n)$**: what is the smallest** $g(n)$ **that always works for any** n**-gon?**

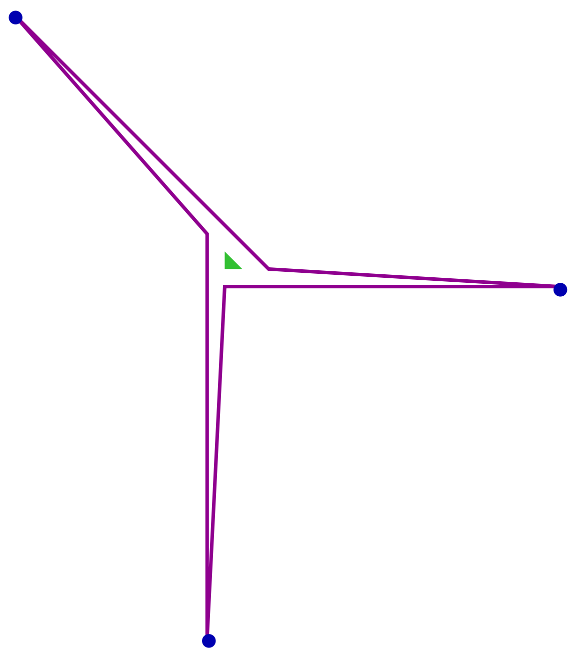
Trying it Out

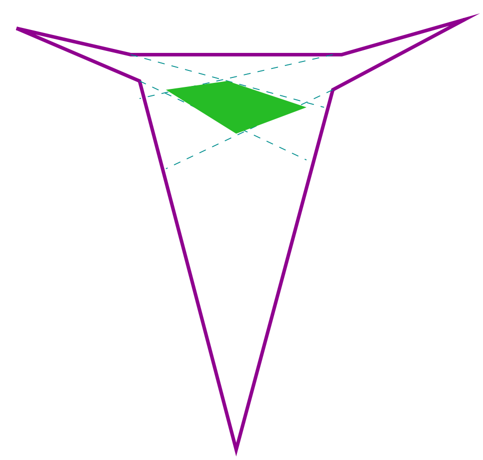
1. **For $n = 3, 4, 5$, we can check that $g(n) = 1$.**



2. **Is there a general formula in terms of n?**

Pathological Cases



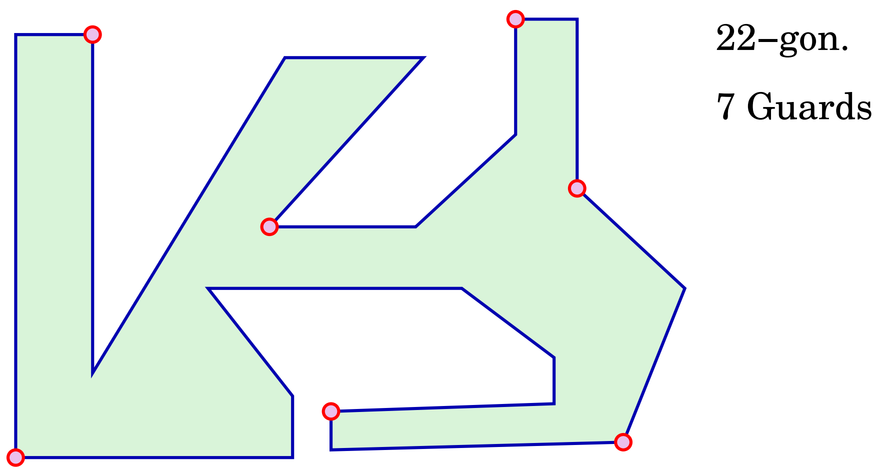


1. Fig. on left shows that seeing the boundary $\neq$ seeing the whole interior!

2. Even putting guards at every other vertex is not sufficient.

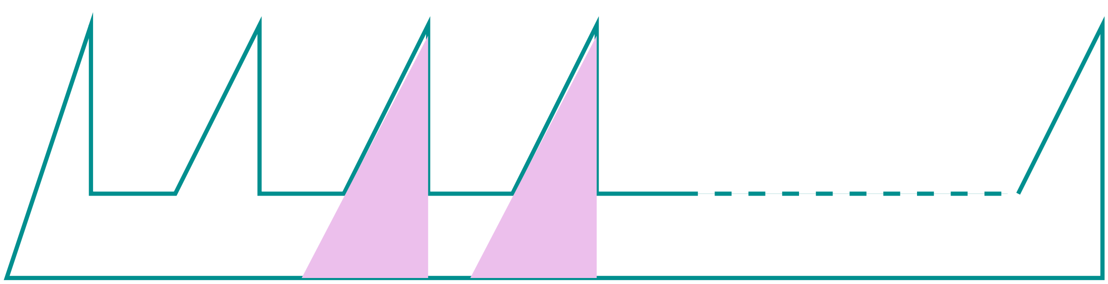
3. Fig. on right shows that putting guards on vertices alone might not give the best solution.

Art Gallery Theorem



Theorem: $g(n) = \lfloor n/3 \rfloor$

1. **Every n-gon can be guarded with $\lfloor n/3 \rfloor$ vertex guards.**
2. **Some n-gons require at least $\lfloor n/3 \rfloor$ (arbitrary) guards.**

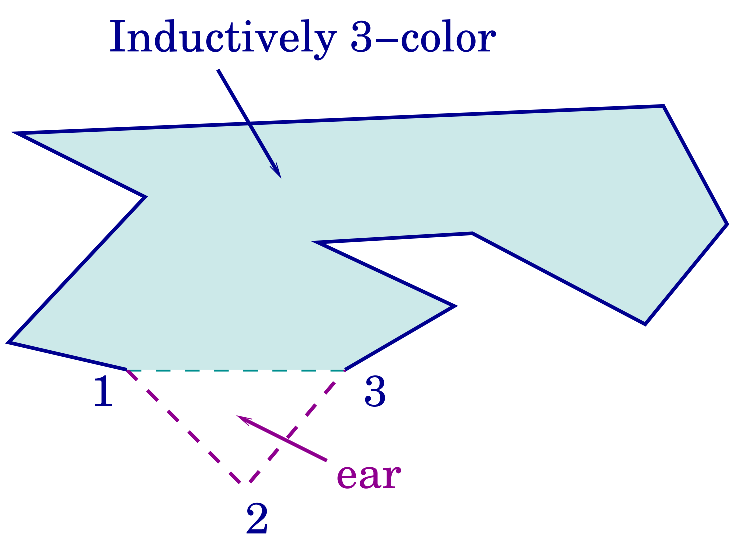


Necessity Construction

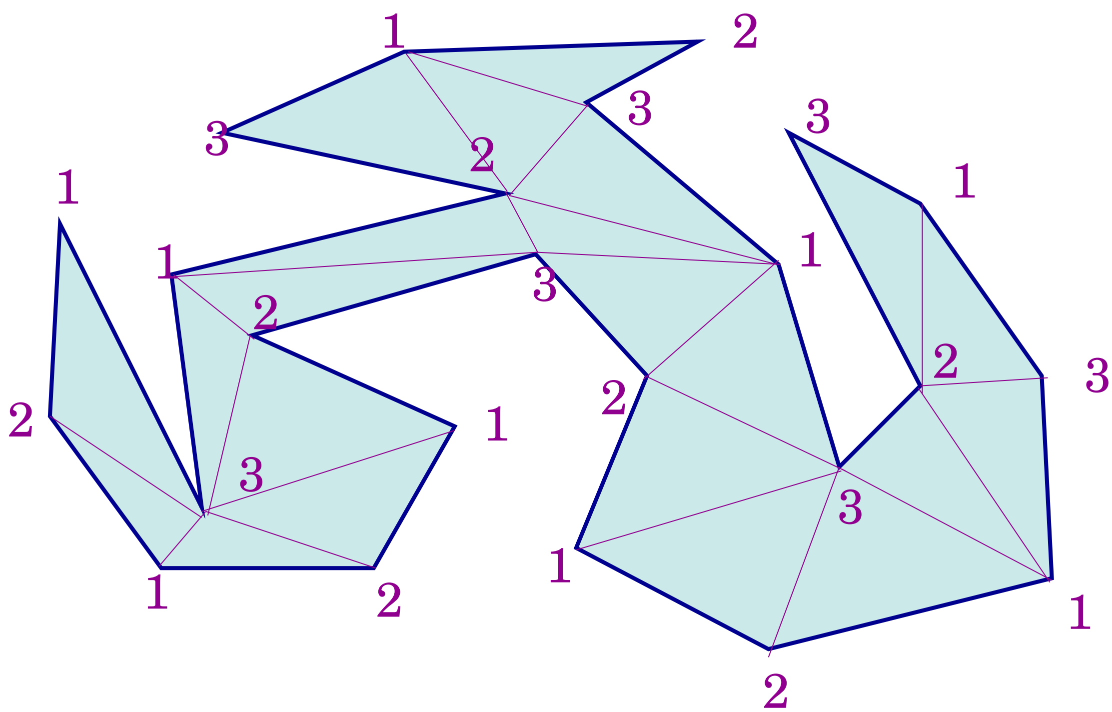
Fisk's Proof

Lemma: **Triangulation graph can be 3-colored.**

- **P plus triangulation is a planar graph.**
- **3-coloring means vertices can be labeled 1,2, or 3 so that no edge or diagonal has both endpoints with same label.**
- **Proof by Induction:**
  1. **Remove an ear.**
  2. **Inductively 3-color the rest.**
  3. **Put ear back, coloring new vertex with the label not used by the boundary diagonal.**



Proof



- **Triangulate P. 3-color it.**
- **Least frequent color appears at most $\lfloor n/3 \rfloor$ times.**
- **Place guards at this color positions—a triangle has all 3 colors, so seen by a gaurd.**
- **In example: Colors 1, 2, 3 appear 9, 8 and 7 times, resp. So, color 3 works.**

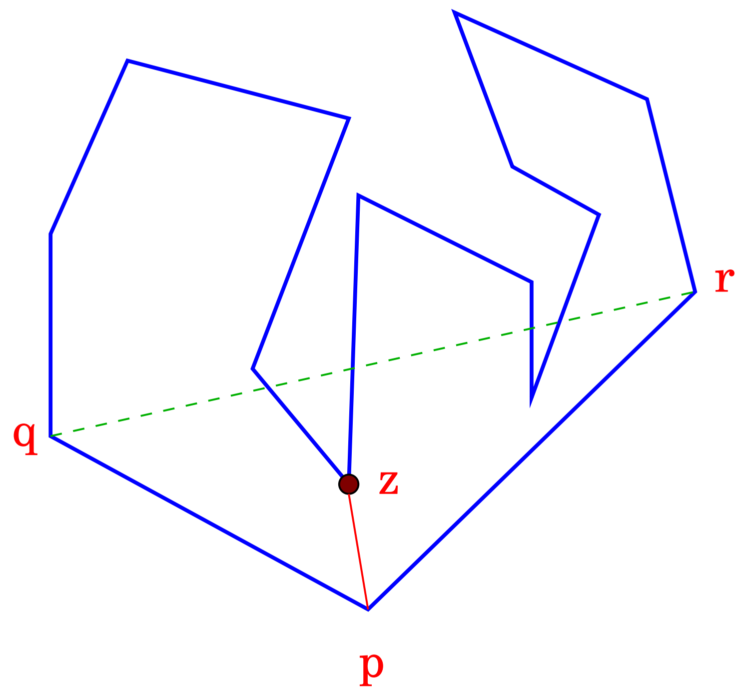
3D Curiosity

- **In 3D, even n vertex guards do not suffice!!!**
- **Put our BSP picture here....**

Triangulation: Theory

Theorem: Every polygon has a triangulation.

- **Proof by Induction. Base case $n = 3$.**

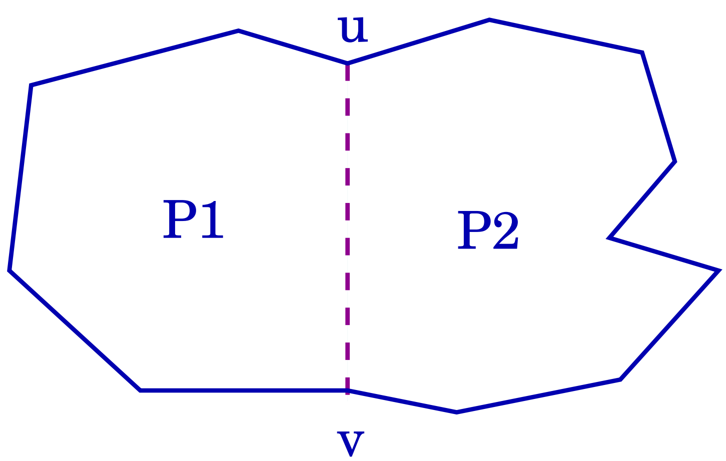


- **Pick a convex corner p. Let q and r be pred and succ vertices.**
- **If qr a diagonal, add it. By induction, the smaller polygon has a triangulation.**
- **If qr not a diagonal, let z be the reflex vertex farthest to qr inside $\triangle pqr$.**
- **Add diagonal pz; subpolygons on both sides have triangulations.**

Triangulation: Theory

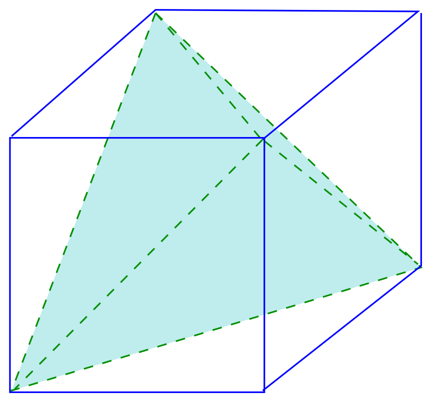
Theorem: Every triangulation of an n-gon has $n - 2$ triangles.

- **Proof by Induction. Base case $n = 3$.**

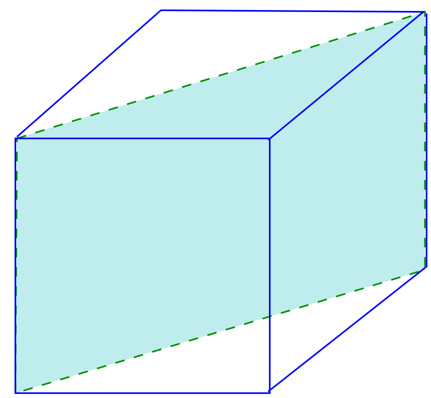


- **Let $t(P)$ denote the number of triangles in any triangulation of P.**
- **Pick a diagonal uv in the given triangulation, which divides P into P_1, P_2.**
- $t(P) = t(P_1) + t(P_2) = n_1 - 2 + n_2 - 2$.
- **Since $n_1 + n_2 = n + 2$, we get $t(P) = n - 2$.**

Triangulation in 3D



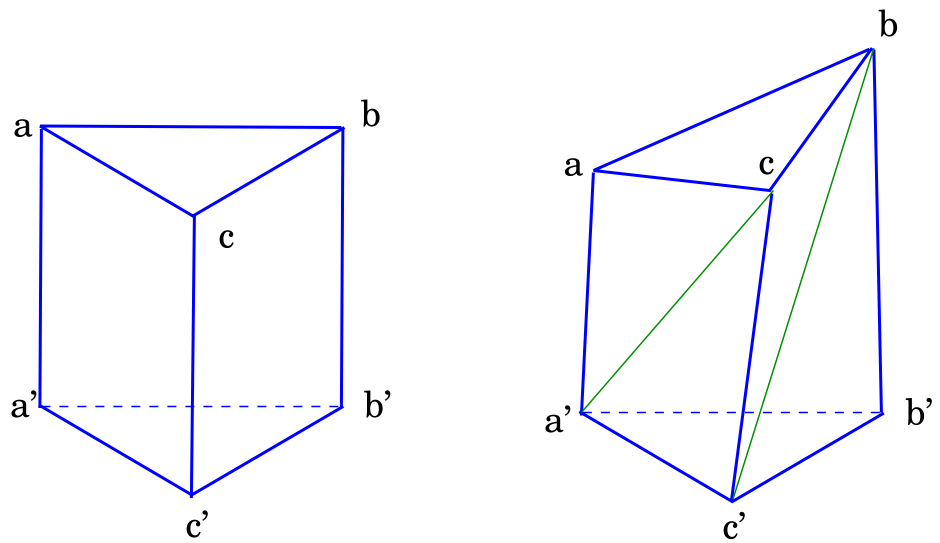
5 Tetrahedra



6 Tetrahedra

- **Different triangulations can have different number of tetrahedra (3D triangles).**

Untriangulable Polyhedron



- Smallest example of a polyhedron that cannot be triangulated without adding new vertices. (Schoenhardt [1928]).
- It is NP-Complete to determine if a polyhedron requires Steiner vertices for triangulation.
- Every 3D polyhedron with N vertices can be triangulated with $O(N^2)$ tetrahedra.

Triangulation History

1. A really naive algorithm is $O(n^4)$: check all n^2 choices for a diagonal, each in $O(n)$ time. Repeat this $n-1$ times.

2. A better naive algorithm is $O(n^2)$; find an ear in $O(n)$ time; then recurse.

3. First non-trivial algorithm: $O(n \log n)$ [GJPT-78]

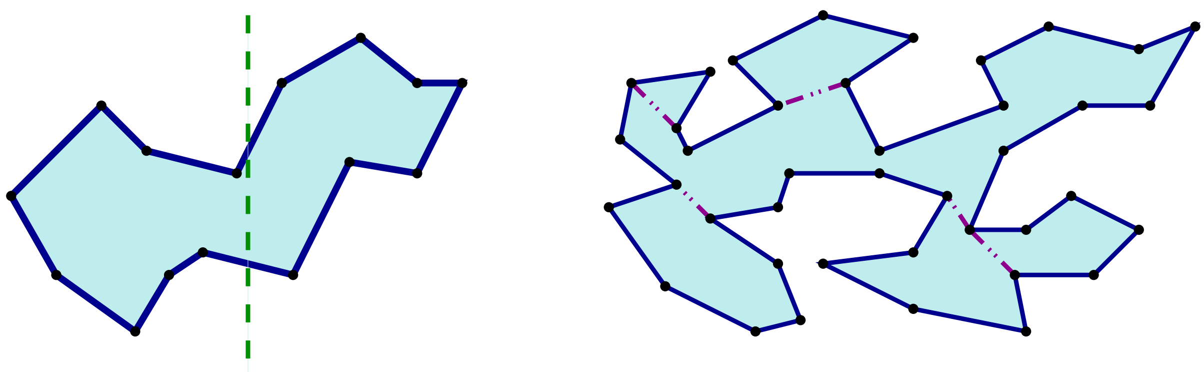
4. A long series of papers and algorithms in 80s until Chazelle produced an optimal $O(n)$ algorithm in 1991.

5. Linear time algorithm insanely complicated; there are randomized, expected linear time that are more accessible.

6. We content ourselves with $O(n \log n)$ algorithm.

Algorithm Outline

1. Partition polygon into trapezoids.
2. Convert trapezoids into monotone subdivision.
3. Triangulate each monotone piece.

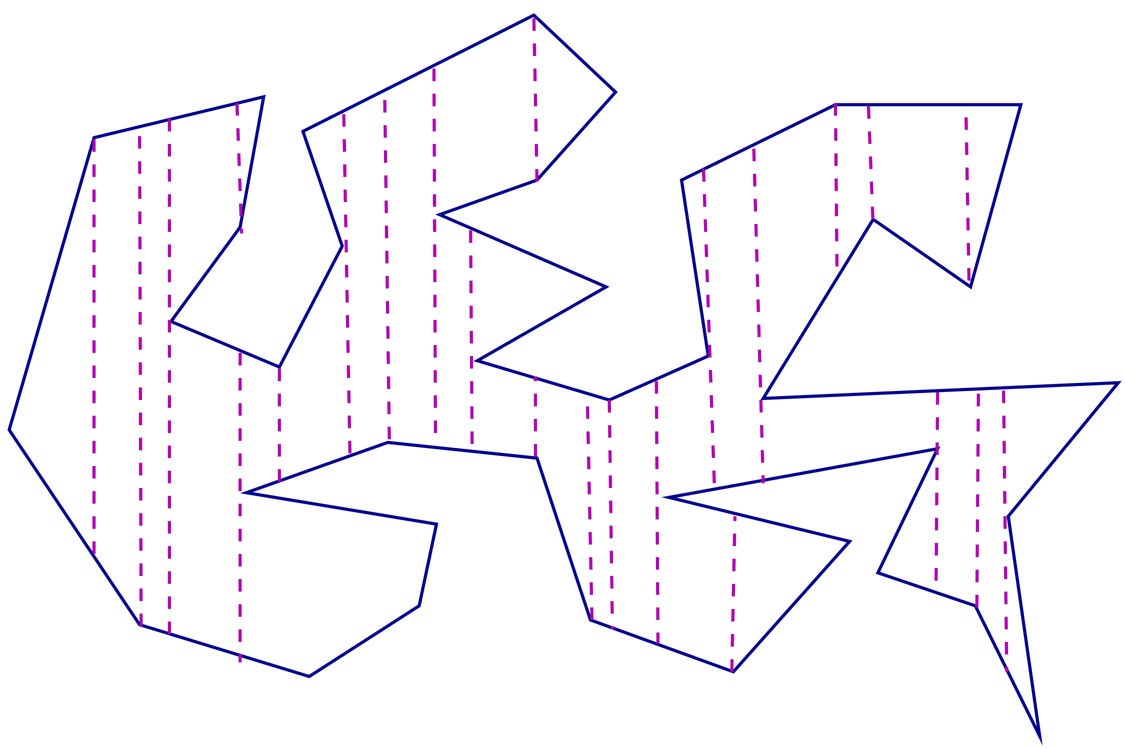


x−monotone polygon Monotone decomposition

4. A polygonal chain C is monotone w.r.t. line L if any line orthogonal to L intersects C in at most one point.
5. A polygon is monotone w.r.t. L if it can be decomposed into two chains, each monotone w.r.t. L.
6. In the Figure, L is x-axis.

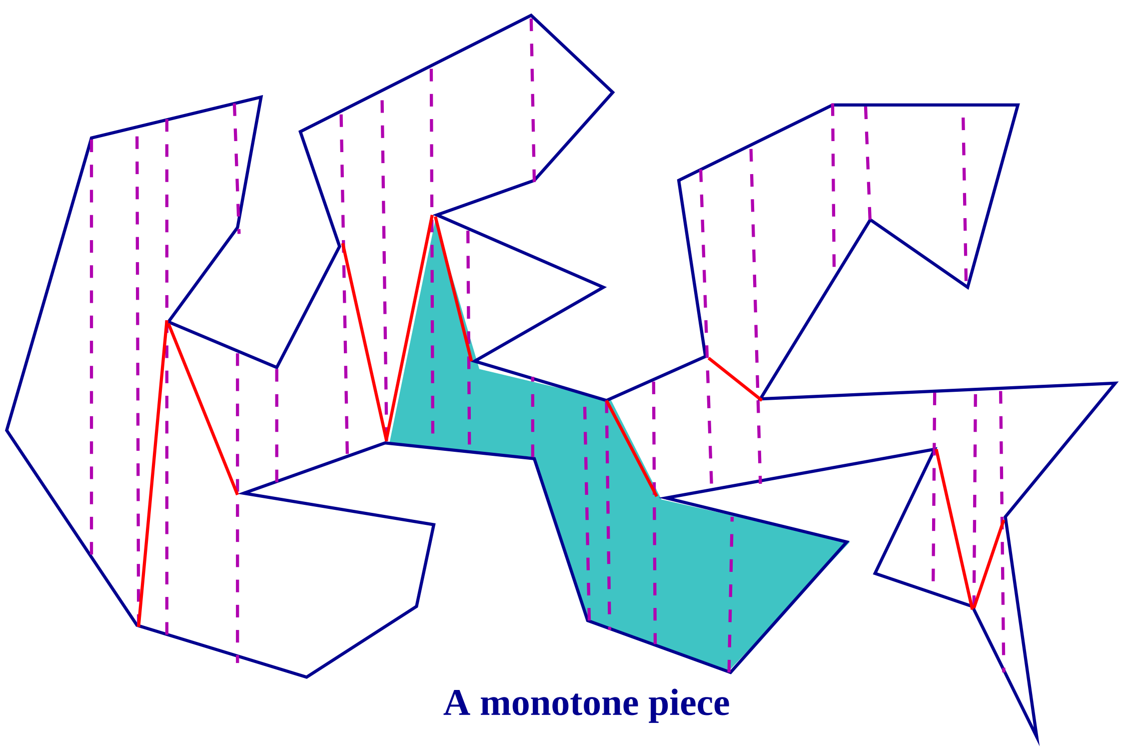
Trapezoidal Decomposition

- **Use plane sweep algorithm.**
- **At each vertex, extend vertical line until it hits a polygon edge.**
- **Each face of this decomposition is a trapezoid; which may degenerate into a triangle.**
- **Time complexity is $O(n \log n)$.**



Monotone Subdivision

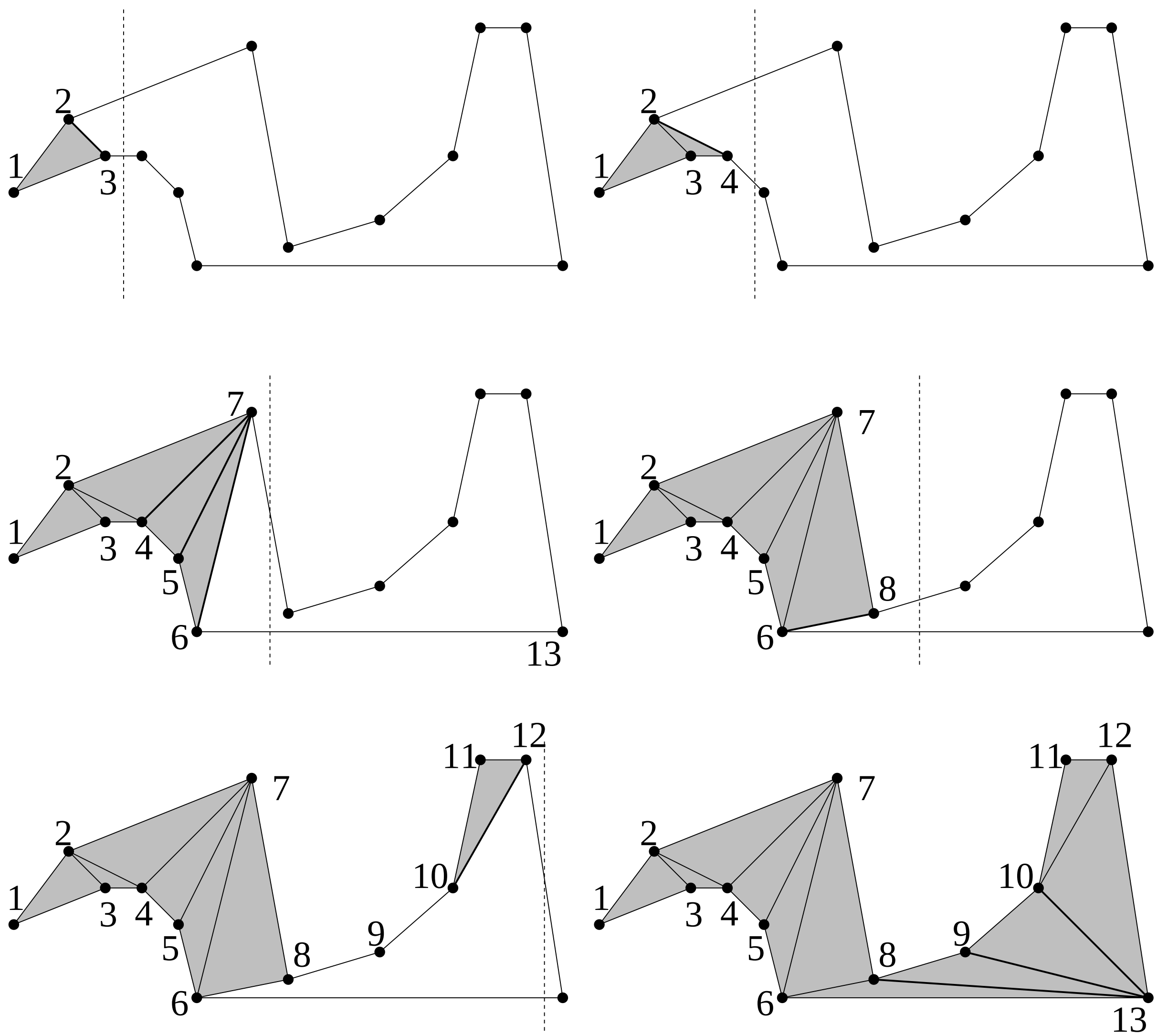
- Call a reflex vertex with both rightward (leftward) edges a split (merge) vertex.
- Non-monotonicity comes from split or merge vertices.
- Add a diagonal to each to remove the non-monotonicity.
- To each split (merge) vertex, add a diagonal joining it to the polygon vertex of its left (right) trapezoid.



Monotone Subdivision

- **Assume that trap decomposition represented by DCEL.**
- **Then, matching vertex for split and merge vertex can be found in $O(1)$ time.**
- **Remove all trapezoidal edges. The polygon boundary plus new split/merge edges form the monotone subdivision.**
- **The intermediate trap decomposition is only for presentation clarity—in practice, you can do monotone subdivision directly during the plane sweep.**

Triangulation



Triangulation

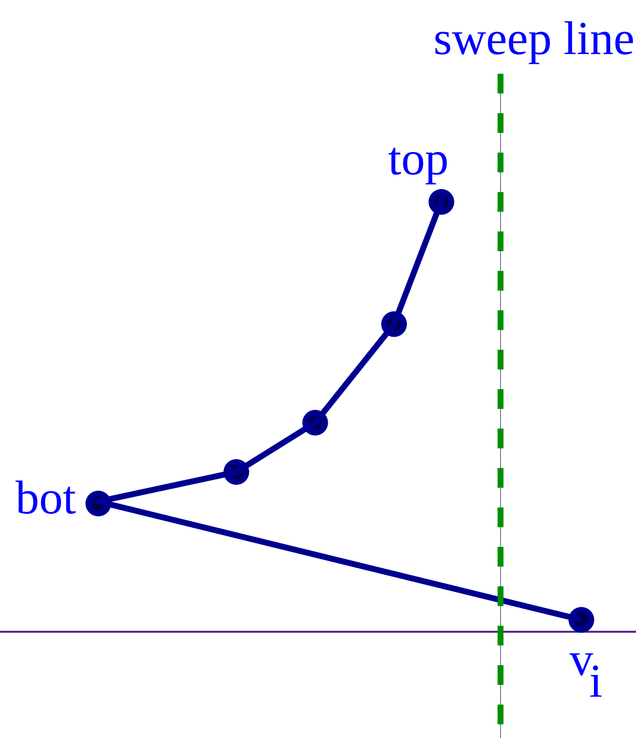
- $\langle v_1, v_2, \ldots, v_n \rangle$ **sorted left to right.**
- **Push** v_1, v_2 **onto stack.**
- **for** $i = 3$ **to** n **do**

  if v_i **and** $top(stack)$ **on same chain**

  Add diagonals $v_i v_j, \ldots, v_i v_k$**, where**
  v_k **is last to admit legal diagonal**
  Pop $v_j, \ldots, v_{k-1}$ **and Push** v_i

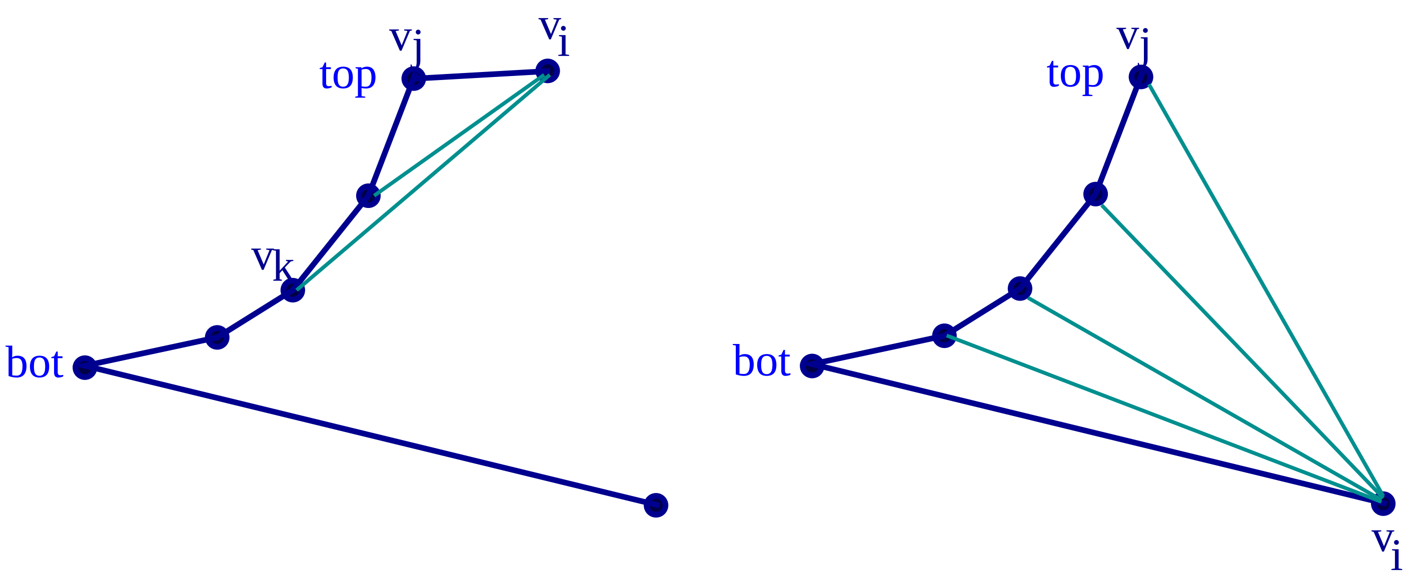
  else

  Add diagonals from v_i **to all vertices on the stack and pop them**
  Save v_{top}**; Push** v_{top} **and** v_i



Correctness

- **Invariant:** **Vertices on current stack form a single reflex chain. The leftmost unscanned vertex in the other chain is to the right of the current scan line.**



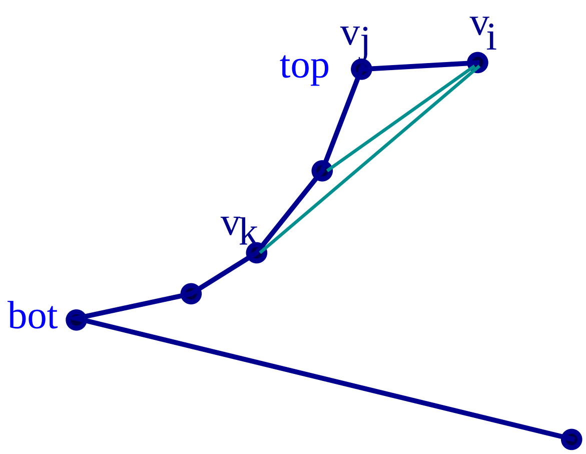
New stack: (bot, ..., vk, vi)

Case I

New stack: (vj, vi)

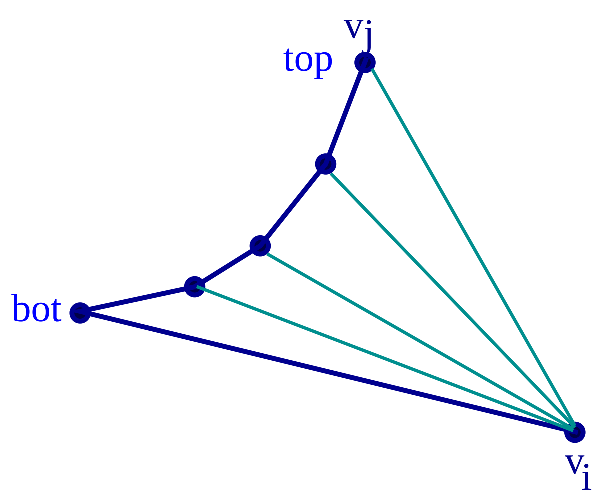
Case II

Time Complexity



New stack: (bot, ..., vk, vi)

Case I

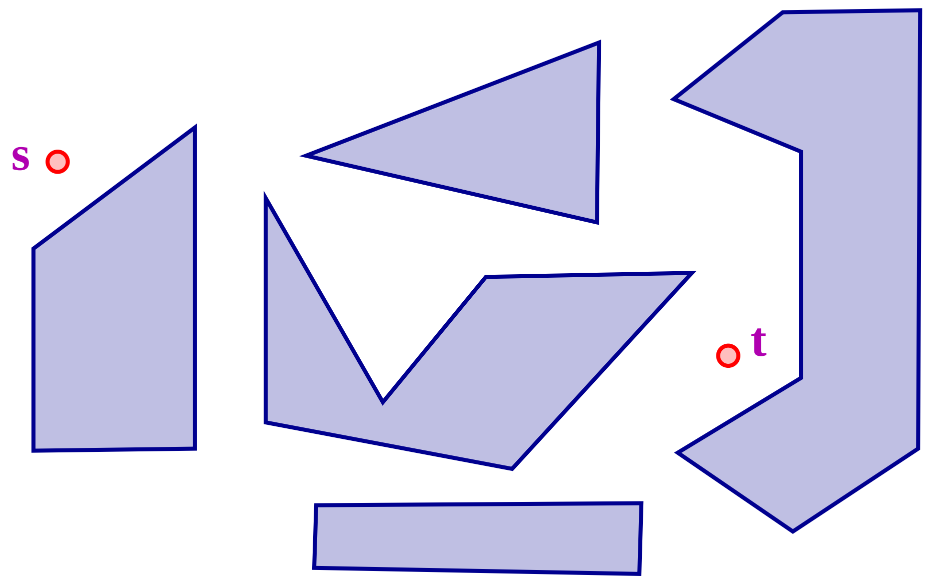


New stack: (vj, vi)

Case II

- **A vertex is added to stack once. Once it's visited during a scan, it's removed from the stack.**
- **In each step, at least one diagonal is added; or the reflex stack chain is extended by one vertex.**
- **Total time is $O(n)$.**
- **Total time for polygon triangulation is therefore $O(n \log n)$.**

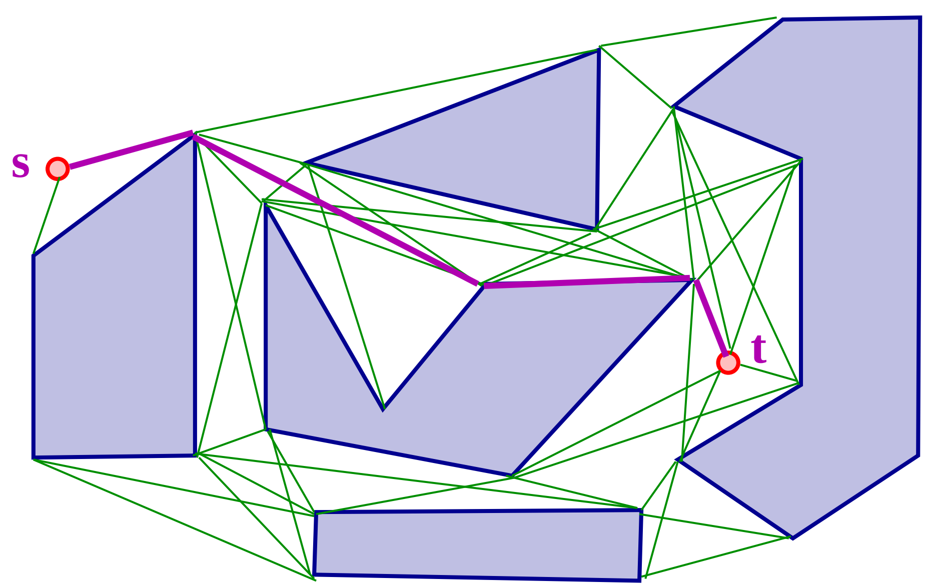
Shortest Paths



- A workspace with polygonal obstacles.
- Find shortest obstacle-avoiding path from s to t.
- Properties of Shortest Path:
  - Uses straight line segments.
  - No self-intersection.
  - Turns at convex vertices only.

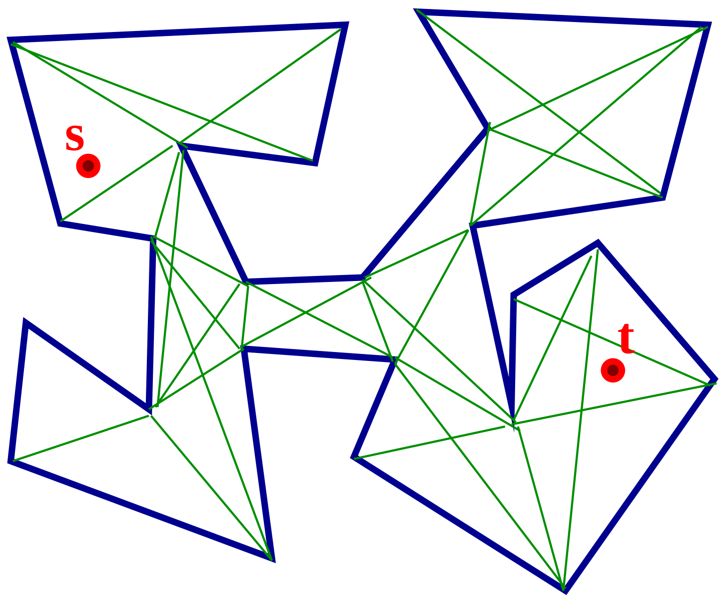
Visibility Graph

- **Construct a visibility graph $G = (V, E)$, where V is set of polygon vertices (and s, t), E is pairs of nodes that are mutually "visible".**
- **Give each edge (u, v) the weight equal to the Euclidean distance between u and v.**
- **The shortest path from s to t in this graph is the obstacle avoiding shortest path.**
- **G can have between $c_1 n$ and $c_2 n^2$ edges. Run Dijkstra's algorithm.**



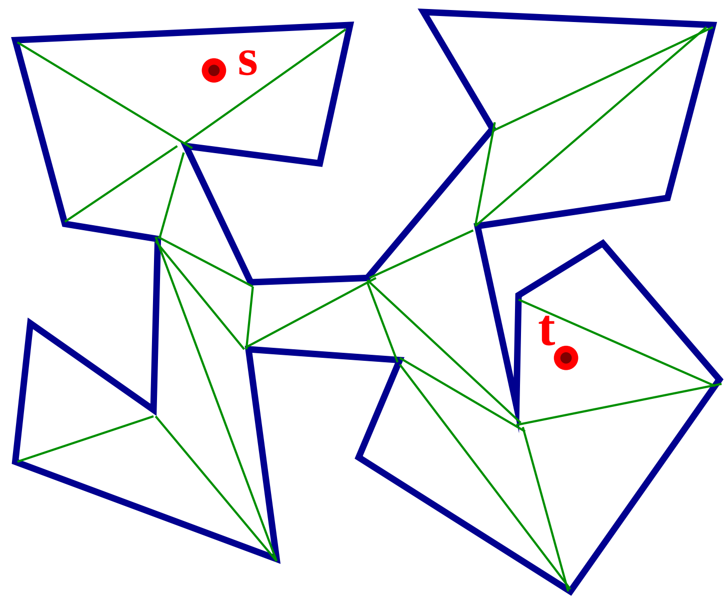
Paths in a Polygon

- Workspace interior of a simple polygon.
- Can we compute a shortest path faster?
- The visibility graph can still have $\Theta(n^2)$ edges.



- Using polygon triangulation, we show an $O(n \log n)$ time algorithm.

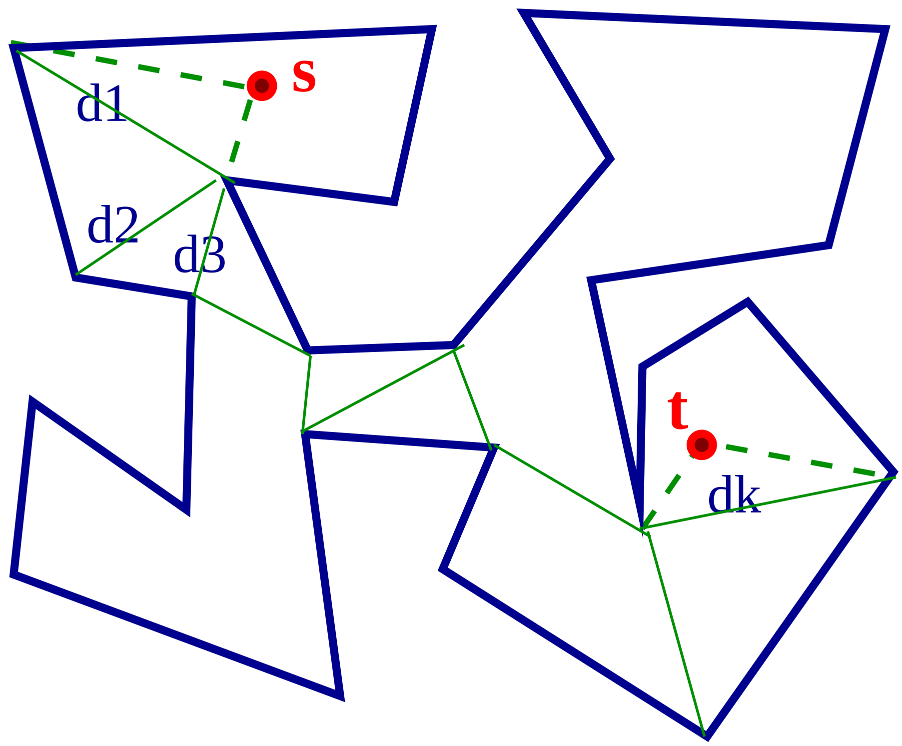
Fast Algorithm



- **Let P be a simple polygon and s, t be source and target points.**
- **Let T be a triangulation of P.**
- **Call a diagonal d of T essential if s, t lie on opposite sides of d.**
- **Let $d_1, d_2, \ldots, d_k$ be ordered list of essential diagonal.**

Algorithm

- **Essential diagonals $d_1, d_2, \ldots, d_k$.**

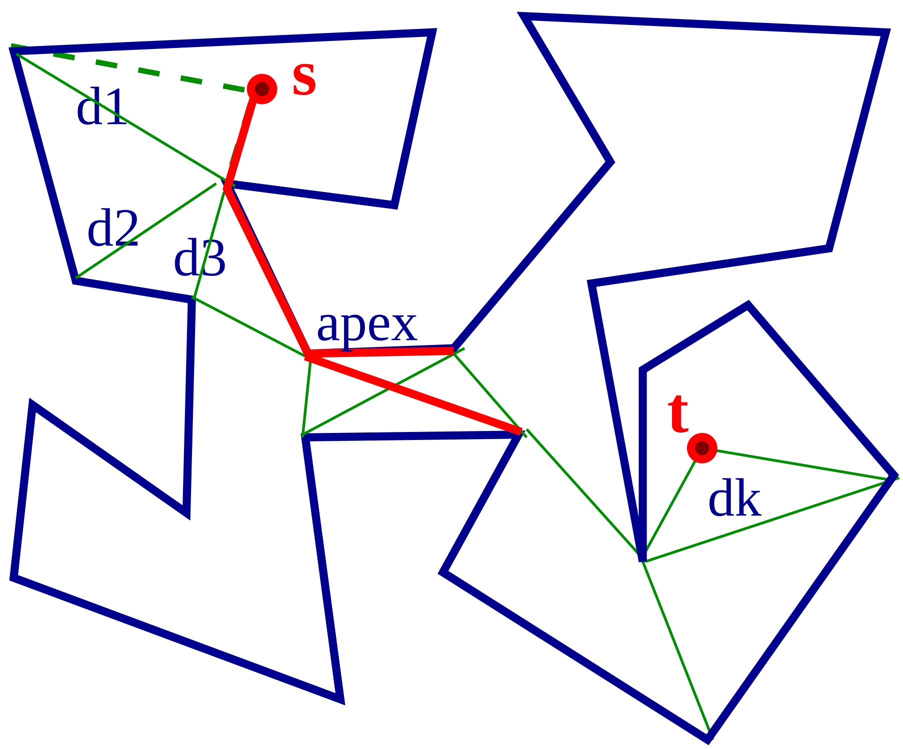


- **The algorithm works as follows:**

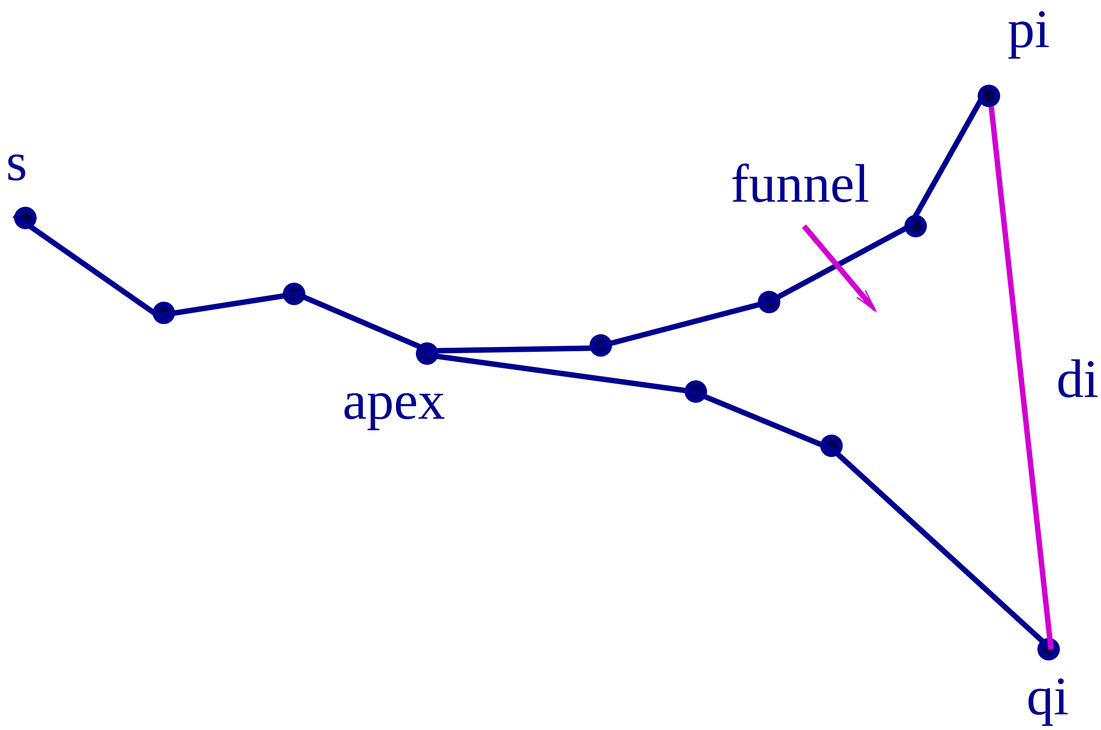
  1. **Start with $d_0 = s$.**
  2. **for $i = 1$ to $k + 1$ do**
  3. **Extend path from s to both endpoints of d_i**

Path Extending: Funnel

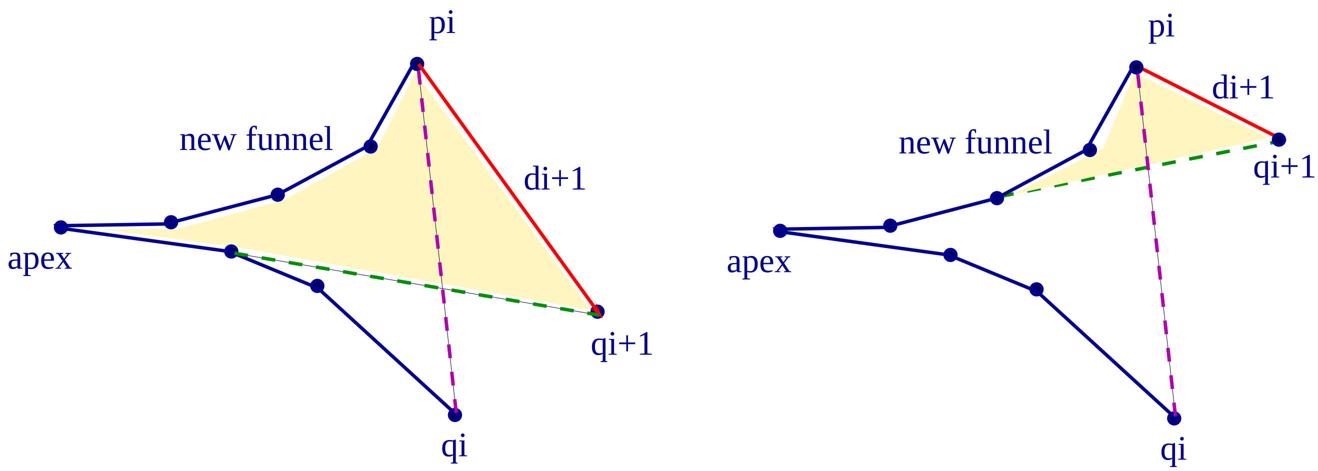
- **Union of** $path(s, p_i)$ **and** $path(s, q_i)$ **forms a funnel.**
- **The vertex where paths diverge is called apex.**



Funnel

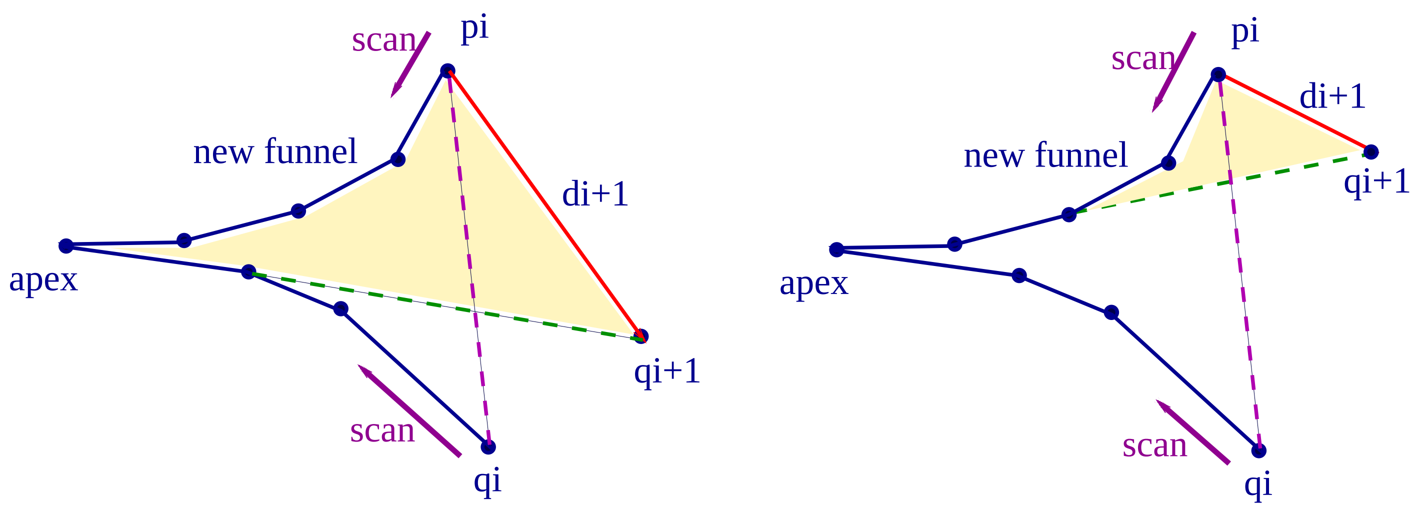


Path Extending



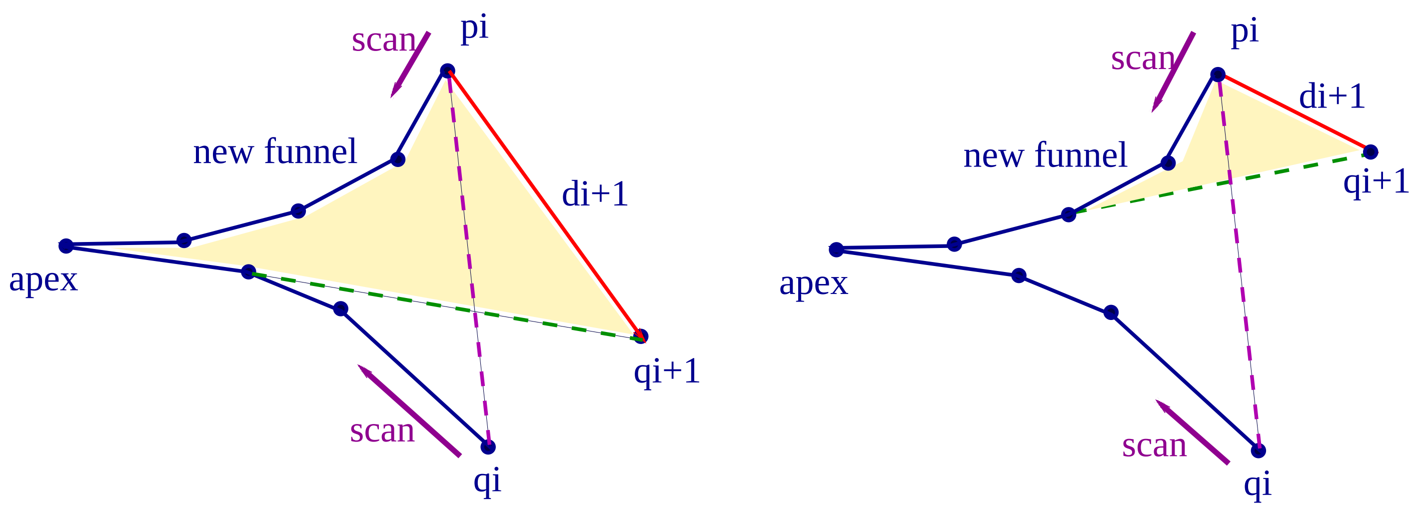
- **Two cases of how to extend the path.**
- **In case I, funnel contracts.**
- **In case II, apex shifts, tail extends, funnel contracts.**
- **In each case, funnel property maintained.**

Data Structure & Update



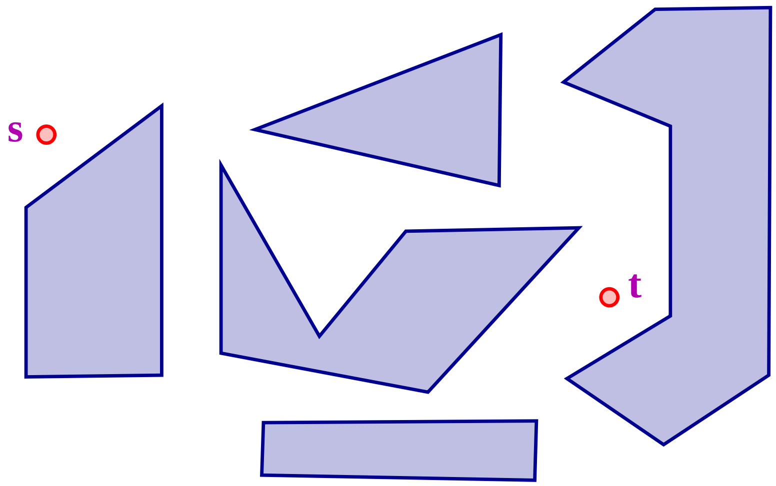
- **How to determine tangent to funnel?**
- **Can't afford to spend $O(n)$ time for each tangent.**
- **Idea: If x edges of funnel are removed by the new tangent, spend $O(x)$ time for finding the tangent.**
- **How to tell a tangent?**

Data Structure & Update



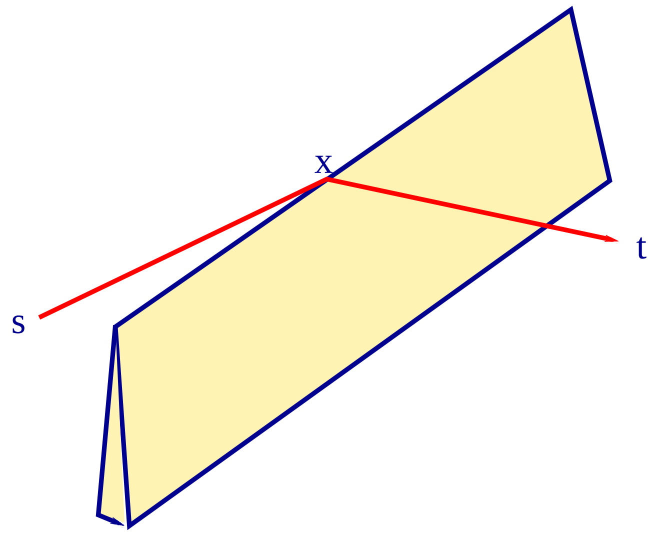
- **Start scanning the funnel from both ends, until tangent determined.**
- **At most $2x + 2$ vertices scanned.**
- **Since each vertex inserted once, and deleted once, total cost for all the tangents is $O(n)$.**
- **Data structure for the funnel: Double-ended queue. Insert/delete in $O(1)$ time.**

Paths Among Obstacles



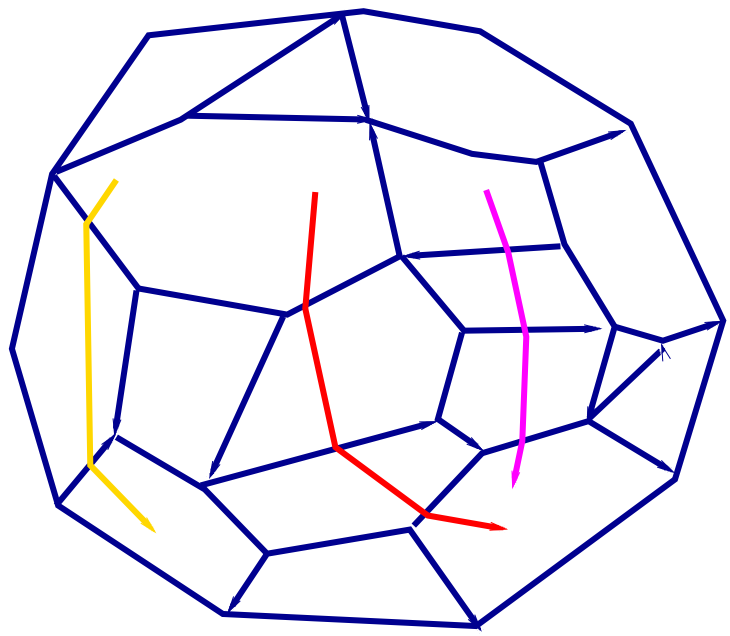
Approach	Complexity	Reference
Vis. Graph	$O(n^3)$	**L. Perez, Wesley ’79**
	$O(n^2 \log n)$	**Sharir-Schorr ’84**
	$O(n^2)$	**Welzl, AAGHI ’86**
	$O(E + n \log n)$	**Ghosh-Mount ’91**
SP Map	$O(k^2 + n \log n)$	**Kapoor-Maheshwari ’**
	$O(nk \log n)$	**Reif-Storer ’91**
	$O(n^{5/3+\varepsilon})$	**Mitchell ’93**
	$O(n \log n)$	**Hershberger-S ’93**

Paths in 3D



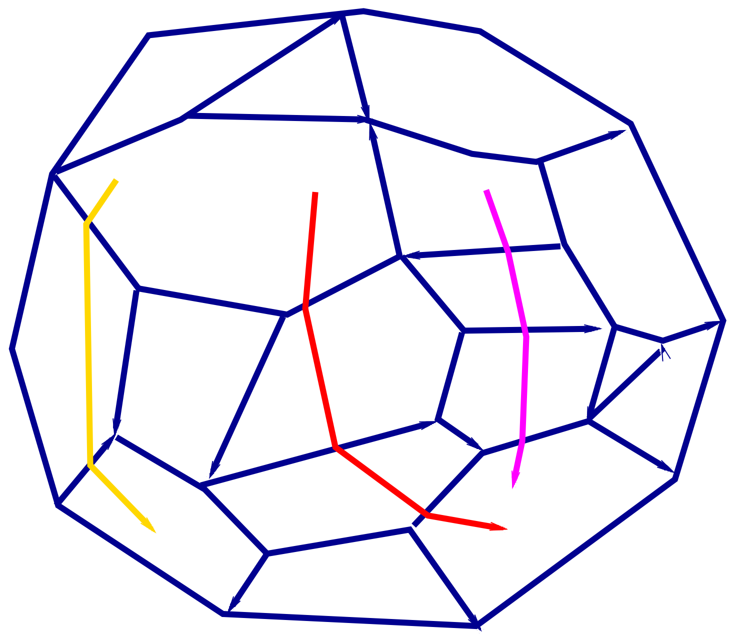
1. Shortest Euclidean length path among convex polyhedral obstacles.
2. Visibility Graph approach breaks down: No finite size graph. Contact points like x can be anywhere on the edge.
3. NP-Hard! Canny-Reif [1986-87].
4. Approximation algorithms (FPTAS) exist, which compute paths of length at most at most $(1 + \epsilon)$ times the optimal.
5. Running time polynomial in n and $1/\epsilon$.
6. Papadimitriou 1985, Aleksandrov et al. 2000, etc.]

Paths on a Convex Surface



- Shortest paths on a surface can be determined fast.
- Paths still turn at interior points of edges, but satisfy a crucial unfolding property.
- Given polytope P, points s, t, a shortest path from s to t on P's surface can be determined in $O(n^2)$ time. [Sharir-Schorr '84, Mount '86, Chen-Han '90].

Paths on a Convex Surface



- **Approximately shortest paths on a convex surface even more efficiently.**
- **Hershberger-Suri '95 give an $O(n)$ time 2-approximation algorithm.**
- **[AHSV '96] generalize it to $(1+\epsilon)$-approximation algorithm in $O(\frac{n \log n}{\epsilon} + \frac{1}{\epsilon^3})$.**