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k-Capture in multiagent pursuit evasion, or the lion and the hyenas ☆,☆☆

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ABSTRACT

We consider the following generalization of the classical pursuit–evasion problem, which we call *k-capture*. A group of n pursuers (hyenas) wish to capture an evader (lion) who is free to move in an m-dimensional Euclidean space, the pursuers and the evader can move with the same maximum speed, and at least k pursuers must *simultaneously* reach the evader's location to capture it. If fewer than k pursuers reach the evader, then those pursuers get destroyed by the evader. Under what conditions can the evader be k-captured? We study this problem in the discrete time, continuous space model and prove that k-capture is possible if and only if there exists a time when the evader lies in the interior of the pursuers' k-Hull. When the pursuit occurs inside a compact, convex subset of the Euclidean space, we show through an easy constructive strategy that k-capture is always possible.

1. Introduction

We consider a variant of the pursuit–evasion game in which multiple pursuers must simultaneously reach the evader's location to capture it. Specifically, an evader $\mathbf{e}$, who is free to move in an m-dimensional Euclidean space, is being pursued by n agents $\mathbf{p}_1, \ldots, \mathbf{p}_n$. The evader and the pursuers have identical motion capabilities and, in particular, have equal maximum speed. Unlike the classical pursuit evasion, our game requires at least k pursuers to *simultaneously* reach the evader's location to capture it, for some given value of $k \leqslant n$. If fewer than k pursuers attack (reach) the evader, then those pursuers are destroyed by the evader. We assume that no two players ever occupy the same position in the environment *except* at the moment of capture; that is, co-location either ends the game or only one player survives among the co-located ones. By disallowing co-location, we are assuming a weaker model of pursuers, which may also be more realistic because in many physical systems only one agent can occupy a point in the space. We call this version the *k-capture pursuit evasion*, and investigate necessary and sufficient conditions, as well as worst-case time bounds, for the k-capture.

Pursuit–evasion games provide an elegant setting to study algorithmic and strategic questions of exploration or monitoring by autonomous agents. Their rich mathematical history can be traced back to at least 1930s when Rado posed the now-classical Lion-and-Man" problem [1]: *a lion and a man in a closed arena have equal maximum speeds; what tactics should the lion employ to be sure of his meal?* The problem was settled by Besicovitch who showed that the man can escape regardless of the lion's strategy [1]. An important aspect of this pursuit–evasion problem, and its solution, is the assumption of *continuous time*: each player's motion is a continuous function of time, which allows the lion to get arbitrarily close to the man

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but never capture him. If, however, the players move in discrete time steps, taking alternating turns but still in continuous space, the outcome is different, as first conjectured by Gale [2] and proved by Sgall [3].

The distinction between continuous and discrete time models is significant albeit subtle. Formulations based on the continuous time lead to differential games, whose solution requires solving the Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation. This is a partial differential equation, whose solution becomes intractable in complex scenarios. (See the seminal work of Isaacs [4] on several continuous time classical games including the Homicidal Chauffeur game and the Game of Two Cars.) Besides this *theoretical* difficulty, one also faces the *practical* problem that continuous time solutions usually are expressed as a feedback law requiring an *instantaneous* measurement of each player's position and its communication to the opponent. This is impractical from an implementation point of view, and especially problematic for non-smooth motions.

Consequently, discrete time alternate moves versions of pursuit evasion have been favored in recent past, especially due to their algorithmic tractability. In these formulations, the evader and the pursuers move in alternating time instants, with the evader moving first. We note that a capture in this formulation is equivalent to the evader being inside a specified small neighborhood of the pursuer in the continuous time formulation. In the discrete time model, Sgall [3] is able to circumvent the problem of lion approaching but not reaching the man in the continuous formulation, and shows that the lion can always capture the man in finite time inside a semi-open bounded environment.

When the evader is free to move inside an *unbounded* environment, multiple pursuers are clearly required to keep the evader from escaping. The capture condition is the same as before: if at some time t, *any* pursuer can reach the position of the evader then the latter is captured. In this setting, it is known that the evader can be captured if and only if it lies in the convex hull of the pursuers [5]. Many other pursuit evasion problems have also been studied, with focus on different types of environments [6,7], characterization of environments in which a certain capture strategy works [8], visibility-based pursuit–evasion [9], sensing limitations [10,11] etc. Finally, if both time *and* space are assumed to be discrete, then the underlying space is represented as a graph with nodes and edges, and on each move a player can move from one node to another by traversing the edge(s) connecting them. The techniques in this formulation tend to be different, and we refer the reader to a representative set of papers [12–15].

Our objective in this paper is to study the k-capture problem in the unbounded continuous space and discrete time framework. In particular, we assume that a group of n pursuers (hyenas) wish to capture an evader (lion) who is free to move in the m-dimensional Euclidean space. The players take turns: the evader moves first, the pursuers move next and all of the pursuers can move simultaneously. On its turn, each player can move anywhere inside a unit disk centered at its current position. (In other words, the maximum speed of the players is normalized to one.) We assume that no two players may occupy the same position in the environment *except* at the moment of capture. Technically, this assumption is used only to rule out the possibility of a trivial pursuer strategy where they partition themselves into size k subgroups, with each subgroup moving as a "meta pursuer". Co-location may also be unrealistic in many physical systems, and by disallowing it we only strengthen our results because pursuers without co-location are weaker in power than those with co-location.

We say that the evader is k-captured, for some specified value of k, if after a finite time, at least k pursuers reach the evader's location. However, if fewer than k pursuers reach the evader's location, then the evader is able to capture (or destroy) those pursuers. In other words, if at the end of a pursuer move, the evader occupies the same position as some of the pursuers, then the game either ends (k-capture occurs), or all the j, where $j < k$, pursuers at that location are captured leaving only the evader. We study the necessary and sufficient conditions under which such a k-capture is possible, and derive bounds on the worst-case time needed to achieve this. Additionally, we address a version of this problem played in a compact and convex subset of a Euclidean space.

In particular, our paper makes four main contributions. First, we show that a necessary condition for k-capture is that the evader must be located inside the k-Hull of the pursuers at the beginning of every evader move. The k-Hull is the set of all points p such that any line through p divides the given points into two sets of at least k points each. Second, we show that this simple k-Hull condition is also sufficient. In other words, if there is ever a time when this condition is satisfied, and in particular if it holds at time $t = 0$, then the pursuers can k-capture the evader in finite time. Our proof of sufficiency is constructive, and based on a new multi-pursuer strategy. Third, we derive an upper bound for the time needed to capture the evader, as a function of the initial positions of the pursuers and of the evader. Finally, for a version of this problem played in a compact and convex environment in a Euclidean space, we design a novel strategy and show that the evader is k-captured using k pursuers.

This paper is organized as follows. The problem formulation and the necessity of the k-Hull condition for capture are presented in Section 2. Our multi-pursuer capture strategy and the sufficiency of the k-Hull condition is presented in Section 3. A version of this problem played in a compact and convex environment is analyzed in Section 4. The conclusions and future directions for this work are summarized in Section 5.

2. Problem formulation and necessary condition for *k*-capture

Our pursuit–evasion game is played in an m-dimensional Euclidean space, with n pursuers $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ and a single evader $\mathbf{e}$. The positions of these agents at any time t are denoted as $\mathbf{p}_j(t)$, for $j = 1, 2, \ldots, n$, and $\mathbf{e}(t)$, where $t \in \mathbb{Z}_{\geqslant 0}$. In Section 4, we also consider the capture problem in a compact convex environment.

We assume that the game is played in discrete time using alternate moves: on a turn, the evader moves first, all the pursuers simultaneously move next. We assume a normalized maximum speed of one, meaning that each player can move to any position inside a closed ball of radius one centered at the player's current position. More precisely, the players' motions are described by the following equations:

$$\mathbf{e}(t+1) = \mathbf{e}(t) + \mathbf{u}_e\big(t, \mathbf{p}_1(t), \dots, \mathbf{p}_n(t)\big),$$
$$\mathbf{p}_j(t+1) = \mathbf{p}_j(t) + \mathbf{u}_{\mathbf{p}_j}\big(t, \mathbf{e}(t), \mathbf{e}(t+1), \mathbf{p}_1(t), \dots, \mathbf{p}_n(t)\big),$$

where $\mathbf{u}_e$ and $\mathbf{u}_{\mathbf{p}_j}$ are unit vectors, termed as *strategies* of the evader $\mathbf{e}$ and the pursuer $\mathbf{p}_j$, respectively. These motion equations say that each agent's strategy depends on the current positions of all other players, and that each agent can move to any position within distance one of its current position. (The apparent asymmetry in the equations of the evader and the pursuers is due to the fact that the evader moves first, so the pursuers' moves can depend on the evader's positions at times t *and* $t+1$.)

The capture occurs when evader is at the same location as some of the pursuers. The k-capture of the evader requires at least k pursuers, while fewer than k pursuers are themselves captured by the evader.[1] Formally, we say that the evader is *k-captured* if there exists a finite time T and a subset $C \subset \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ of k pursuers such that $\|\mathbf{p}_j(T) - \mathbf{e}(T)\| = 0$ but $\|\mathbf{p}_j(t) - \mathbf{e}(t)\| > 0$, for all $t < T$ and all $\mathbf{p}_j \in C$. In other words, the k-capture occurs at a time T if at least k pursuers simultaneously reach the evader's location at time T, and none of these pursuers have ever been captured in the past.[2] We say that the evader *escapes* if there exists no finite time at which the pursuers k-capture the evader. Finally, we require that no two players occupy the same point in the environment *except* at the time of capture.

We now formulate a necessary condition for k-capture, which is then complemented by Section 3 that shows that this condition is also sufficient. Our necessary condition prescribes the location of the evader relative to the locations of the pursuers for the k-capture to occur. This condition is independent of the pursuers' strategy: that is, if the condition is violated, then there always exists an evader strategy for escape regardless of the pursuers' strategy.

Naturally, the convex hull of the pursuers' locations plays a key role in the game. This is not surprising because the convex hull is precisely the set of all evader locations that are capturable in the classical single pursuer game, as is well-known [5].

Lemma 2.1. *If the evader's initial location is not inside the interior of the convex hull of the pursuers, then it cannot be k-captured, even for $k = 1$.*

Proof. If the evader is not in the interior of the convex hull, then there exists a hyperplane through the evader's location such that all the pursuers lie in one (closed) half-space defined by the hyperplane. The evader simply escapes by moving perpendicular to this hyperplane, away from the pursuers, at maximum speed. □

2.1. The k-Hull

When $k > 1$, we need a generalized notion of the convex hull. The standard convex hull of a set of points can be defined as the set of points with the property that every hyperplane tangent to the hull contains at least one point of the hull in each of the two closed half-spaces. If we require that at least k points lie in each half-space, then we get a structure called k-Hull, introduced by Cole, Sharir and Yap [16], which also has intimate connections to other fundamental structures in computational geometry such as k-levels and k-sets [17].

Definition 1 *(k-Hull).* Let S be a set of n points in the plane, and let k be an integer. The k-Hull of S denoted by $\text{Hull}_k(S)$ is the set of points p such that, for any hyperplane $\ell(p)$ through p, there are at least k points of S in each closed half-space of $\ell(p)$.

Clearly, the standard convex hull is the same as the 1-Hull, and it is also easy to see that the $(k+1)$-Hull is contained in the k-Hull. One can also show, using Helly's Theorem [18], that the k-Hull is always non-empty for $k \leqslant \lceil n/(m+1)\rceil$, where m is the dimension of the underlying Euclidean space. We, therefore, assume throughout this paper that $1 \leqslant k \leqslant \lceil n/(m+1)\rceil$. In particular, the standard convex hull in two dimensions, is well-defined for 3 or more non-collinear points, but 2-Hull requires at least $n = 5$ points in the plane. As an illustration, Fig. 1 shows two of the possible configurations for $n = 5, k = 2$ for a planar environment.

Remark 2.2 *(k-Hull computation).* While computational complexity is the not focus of our paper, we do point out that k-Hulls are also efficiently computable. Under the point-hyperplane duality, they correspond to the level k in an arrangement of hyperplanes, and therefore computed easily in $O(n^m)$ time in m dimensions. The bound can be improved somewhat using

[1] We remark, however, that in the discrete time alternate moves model, the evader cannot force a pursuer's capture because the pursuers move *after* the evader. Indeed, if the evader moves to the current location of a pursuer $\mathbf{p}$, then $\mathbf{p}$ can always move away from the evader at its turn. However, one cannot rule out a pursuers' strategy that involves *sacrificing* some of them to ultimately achieve k-capture.

[2] While it is sufficient to ensure the safety of only the k pursuers who perform the k-capture, in our strategy, *all* the pursuers will remain safe.

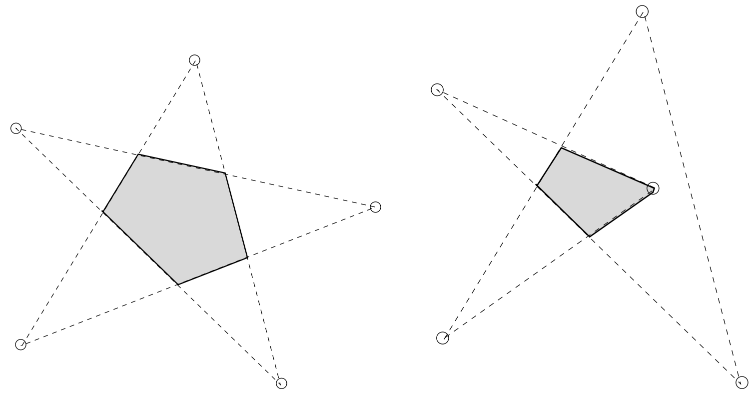

Fig. 1. Illustrating the k-Hull. Two of the possible configurations for 2-Hull of $n = 5$ points in the plane.

more sophisticated algorithms and analysis. For instance, in the two-dimensional plane, k-Hull contains at most $O(nk^{1/3})$ vertices, and using dynamic convex hull data structures, it can be computed in worst-case time $O(nk^{1/3}\log^2 n)$ [16].

2.2. The necessary condition

In our generalization of 1-capture, it turns out that the k-Hull of the pursuers' locations is precisely the set of evader locations that are k-capturable. Throughout, we will use the notation Hull_k^o to denote the *interior* of the k-Hull. Here, the ^{o} notation is to highlight that it is the interior of the set. We have the following theorem stating our necessary condition.

Theorem 2.3. *The evader* $\mathbf{e}$ *can be* k*-captured by pursuers* $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ *only if*

$$\mathbf{e}(t) \in \mathrm{Hull}_k^o\big(\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t)\big),$$

at all times $t \in \mathbb{Z}_{\geqslant 0}$*, where* Hull_k^o *is the interior of the* k*-Hull of the pursuers.*

Proof. The proof is by contradiction. Suppose that the evader's position $\mathbf{e}(t)$ violates the condition of the theorem at some time t. Then, there exist a hyperplane $\bar{H}$ through $\mathbf{e}(t)$ that separates a subset C of the pursuers from the rest, which we call $\bar{C}$, such that $|C| < k$, and therefore $|\bar{C}| \geqslant n - k + 1$. In this case, the evader can escape by moving according to the following strategy:

Move with maximum speed in the direction normal to $\bar{H}$ *towards the side containing* C.

We observe that $\mathbf{e}(t) \notin \mathrm{Hull}_1^o(\bar{C})$—this is because $\mathbf{e}(t)$ lies on $\bar{H}$, the hyperplane defining the half-space that contains $\bar{C}$. By Lemma 2.1, therefore, none of the $n - k + 1$ pursuers in $\bar{C}$ can catch the evader when the evader uses the above-mentioned strategy. Therefore, only the (at most) $k - 1$ pursuers in C can reach the evader, and the k-capture of evader is not possible. This completes the proof. □

The necessary condition asserts that if there is ever a time when the evader is outside the k-Hull of the pursuers, then it has an escape strategy. The main result of our paper, presented in the following section, shows that this necessary condition is also sufficient. In particular, if the evader lies in the pursuers' k-Hull at the initial time instant $t = 0$, then the pursuers are able to k-capture it. (Clearly, if evader is not inside the k-Hull initially, then it can escape unless it plays sub-optimally and move inside the pursuers' k-Hull at a later time, allowing them to capture it.)

3. Proof of sufficiency

In this section, we prove our main result, which is to show that the necessary condition of Theorem 2.3 is also sufficient. The proof, which is constructive, outlines a strategy for the pursuers and derives an upper bound on the time needed for the capture. Our analysis exploits properties of the pursuers' k-Hull, and so we begin with some geometric preliminaries.

3.1. Geometric preliminaries and an orientation-preserving strategy

In general, the orientations of the pursuers with respect to the evader will change once the pursuit begins. We will show, however, that pursuers can coordinate their moves to preserve their individual directions relative to the evader's location.

Such a strategy will allow us to conclude that if the evader is in the k-Hull of the pursuers at the initial instant, then it will remain in the k-Hull at all subsequent instants.

Let us call a pursuers' strategy *orientation-preserving* if the orientations of the vectors $\mathbf{p}_i - \mathbf{e}$ are preserved throughout the pursuit. We will prove that there is an orientation-preserving k-capture strategy for the pursuers. But first, we establish a key geometric lemma about such a strategy.

Let $\mathbf{u}_e(t)$ denote the evader's move at time t, where the vector $\mathbf{u}_e$ is a point on the m-dimensional sphere $\mathbb{S}$. Let $\theta_i(\mathbf{u}_e)$ denote the (smaller) angle between vectors $\mathbf{p}_i(t) - \mathbf{e}(t)$ and $\mathbf{u}_e(t)$. Define,

$$g(\mathbf{u}_e) := \min_k \{\theta_1(\mathbf{u}_e), \ldots, \theta_n(\mathbf{u}_e)\},$$

where $\min_k$ refers to the k-th smallest of the n quantities. Additionally, at time t, let $\beta_{\max}(t) \in [0, \pi]$, be the maximum possible value of g over all allowable $\mathbf{u}_e$, i.e.,

$$\beta_{\max}(t) := \max_{\mathbf{u}_e(t) \in \mathbb{S}} g(\mathbf{u}_e(t)). \tag{1}$$

The following result states that as long as the pursuers follow an orientation-preserving strategy, one can always find k favorable pursuers at each instant of time, for whom the k respective θ's are all less than a number which remains invariant at all times and which is strictly less than $\pi/2$.

Lemma 3.1. *Suppose that the evader lies inside the k-Hull of the pursuers' initial locations, and the pursuers follow an orientation-preserving strategy throughout the pursuit. Then, the following facts hold at all times*:

- *At every time instant $t > 0$, $\beta_{\max}(t) = \beta_{\max}(0)$, and $\beta_{\max}(0) < \pi/2$.*
- *After any move by the evader at time $t+1$, there exist at least k pursuers $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}$ such that $\theta_{i_j} \leqslant \beta_{\max}(0)$, for all $j \in \{i_1, \ldots, i_k\}$.*

Proof. Since $\mathbf{e}(0) \in \mathrm{Hull}_k^o(\mathbf{p}_1(0), \ldots, \mathbf{p}_n(0))$, an orientation preserving strategy will ensure that $\mathbf{e}(t) \in \mathrm{Hull}_k^o(\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t))$, for all time instants t. Thus, for any t, the quantity $g(\mathbf{u}_e)$ is identically defined as for $t = 0$, and therefore for the first claim, it suffices to show that as defined in Eq. (1), $\beta_{\max}(0) < \pi/2$.

To see this, note that each $\theta_i(\cdot)$ is a continuous function of $\mathbf{u}_e$. Therefore, the k-th minimum, i.e., $g(\cdot)$ is a continuous function of $\mathbf{u}_e$. Since $\mathbf{u}_e$ lies on the m-dimensional unit sphere $\mathbb{S}$, which is a compact set, $g(\cdot)$ attains a maximum for some $\mathbf{u}_e^*$ in $\mathbb{S}$. It now remains to show that the maximum value $\beta_{\max}(0) = g(\mathbf{u}_e^*(0)) < \pi/2$. Since $\mathbf{e}(0) \in \mathrm{Hull}_k^o(\mathbf{p}_1(0), \ldots, \mathbf{p}_n(0))$, for every choice of $\mathbf{u}_e$, we must have at least k pursuers $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}$ such that $\theta_j < \pi/2$, for all $j \in \{i_1, \ldots, i_k\}$. If this were not the case for some $\bar{\mathbf{u}}_e$, then the hyperplane orthogonal to $\bar{\mathbf{u}}_e$ through $\mathbf{e}$ would separate $k-1$ pursuers from the remaining, implying that $\mathbf{e} \notin \mathrm{Hull}_k^o(\mathbf{p}_1, \ldots, \mathbf{p}_n)$. Thus, for every $\mathbf{u}_e \in \mathbb{S}$, $g(\mathbf{u}_e) < \pi/2$ and in particular, $g(\mathbf{u}_e^*) < \pi/2$. Thus, $\beta_{\max}(0) < \pi/2$, and the first claim is established.

The second claim follows trivially from the definitions of $g(\cdot)$ and $\beta_{\max}(0)$. □

Remark 3.2 *(General initial positions).* Throughout this section, we assume that no two pursuers are collinear with the evader, which implies that the vectors $\mathbf{p}_i(0) - \mathbf{e}(0)$ all have distinct orientations at $t = 0$, for all $1 \leqslant i \leqslant n$. We could easily ensure this condition by an initial move by the pursuers, as follows. Suppose $\angle \mathbf{p}_i(0)\mathbf{e}(0)\mathbf{p}_j(0) = 0$, for some i, j, where the notation $\angle \mathbf{pxq}$ denotes the (smaller) angle between vectors $\mathbf{p} - \mathbf{x}$ and $\mathbf{q} - \mathbf{x}$. Suppose that the evader's initial move is from position $\mathbf{e}(0)$ to $\mathbf{e}(1)$. Then, all the pursuers except $\mathbf{p}_i$ move parallel to $\mathbf{e}(1) - \mathbf{e}(0)$ with step size $\|\mathbf{e}(1) - \mathbf{e}(0)\|$. The pursuer $\mathbf{p}_i$ also moves with step size $\|\mathbf{e}(1) - \mathbf{e}(0)\|$ but in a direction making a sufficiently small but positive angle α with $\mathbf{e}(1) - \mathbf{e}(0)$. Since $\mathrm{Hull}_k^o(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is an open set and a continuous function of the pursuer locations, there exists a sufficiently small but positive angle α so that $\mathbf{e}(1)$ still lies inside $\mathrm{Hull}_k^o(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ at time $t = 1$. If there are multiple collinearities, then the same strategy can be used to break all of them while preserving the invariant that the evader lies inside the k-Hull.

We are now ready to describe our k-capture strategy and prove its correctness.

3.2. A strategy for k-capture

One simple-minded strategy for capture is to let each pursuer maximally advance towards the evader's new position at each move. Because the evader lies in Hull_k^o, this strategy reduces at least one pursuer's distance to $\mathbf{e}$. But it does not ensure that k pursuers reach the evader simultaneously and so cannot guarantee k-capture. Instead, we let only those pursuers that are *not the closest* to the evader execute this kind of move, while those closest to the evader carry out a *parallel* move that maintains their distance and angle to the evader. We call this the *advance move*. More specifically, the pursuers who are closest to the evader move to maintain their distance and angle to the evader, while the remaining pursuers advance towards the evader. Unfortunately, while this strategy keeps the pursuers safe, it also keeps them away from the evader, and in the worst-case all the pursuers may become equidistant to the evader and then stagnate. We, therefore, introduce

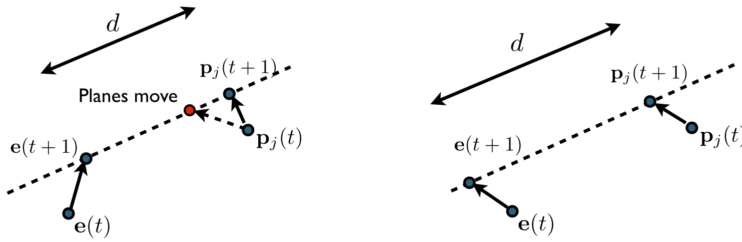

Fig. 2. Illustrating the Advance move. The left figure illustrates that, unlike the Planes move [5], our strategy moves the pursuer to a more conservative location with respect to the new position of the evader. The right subfigure shows the case when the evader's move is such that pursuer $\mathbf{p}_j$ is forced to move to preserve its orientation and distance from the evader.

a second move, called the *cone move*, which ensures that the distance of the closest pursuers itself decreases but in such a way that at least k pursuers remain closest to the evader.

The following algorithm describes at a pseudo-code level the overall strategy. The terms $\mathbf{P}_{\text{closest}}$ and Cone, respectively, denote the set of pursuers that are the closest to the evader in terms of distance and a Cone region, and are defined precisely following the algorithm.

Algorithm 1: k-Capture

Assumes: $\mathbf{e}(0)$ satisfies the k-Hull necessary condition.
For each $t = 1, 2, \ldots$ and for each $j \in \{1, \ldots, n\}$,
Determine

$$d_{\min}(t) = \min_{\mathbf{p} \in \mathbf{P}_{\text{closest}}(t)} \|\mathbf{p} - \mathbf{e}(t)\|$$

if $\mathbf{p}_j$ is among k pursuers $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}$ that are in $\mathbf{P}_{\text{closest}}(t)$ and in $\text{Cone}(k, t)$ **then**
  $\mathbf{p}_j$ uses Cone move corresponding to $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}$
else
  $\mathbf{p}_j$ uses Advance move with parameter $d_{\min}(t)$
end for

In the following, we give a precise definition of the Advance move and the Cone move. Informally, the *Advance* move is used by a pursuer to reduce its distance from the evader if it is sufficiently far from the evader. The *Cone* move is used by a pursuer *together* with at least $k-1$ other pursuers, if all of them are among the closest to the evader, and if the evader has made a move which is favorable for those pursuers. When both the moves are possible for a pursuer, the Cone move has the priority.

Definition 2 *(Advance move).* Suppose the evader moves from $\mathbf{e}(t)$ to $\mathbf{e}(t+1)$. Then, given a parameter $d \geqslant 0$, the *Advance move* of a pursuer $\mathbf{p}_j$ is the following:

- Draw a line through $\mathbf{e}(t+1)$ parallel to the vector $\mathbf{p}_j(t) - \mathbf{e}(t)$.
- Move to the position $\mathbf{p}_j(t+1)$ on this line for which $|d - \|\mathbf{e}(t+1) - \mathbf{p}_j(t+1)\||$ is minimized.

Fig. 2 illustrates two possibilities of applying the Advance move. Our modification to the original Planes algorithm [5] is the inclusion of the parameter d. This parameter keeps a pursuer $\mathbf{p}_j$ from moving straight towards $\mathbf{e}$ *if* that pursuer is the closest one to the evader. For example, with the parameter setting $d = \|\mathbf{e}(t) - \mathbf{p}_j(t)\|$, the Advance move will force the pursuer $\mathbf{p}_j$ to move *parallel* to the evader, and with exactly the same step size as the evader, as is shown in the right subfigure. The left subfigure shows a generic application of the Advance move.

We now describe the Cone move, which is used by k or more pursuers when they are among the closest pursuers to the evader, and when they are located inside a Cone region, which we define next. We show later (cf. Lemma 3.4) that after a finite time, there will be at least k closest pursuers, so the following discussion focuses on such pursuers.

Let $\mathbf{P}_{\text{closest}}(t)$ denote the set of pursuers that are closest to the evader $\mathbf{e}(t)$ at time t. That is,

$$\mathbf{P}_{\text{closest}}(t) := \{\mathbf{p}_i(t)\colon\ i \in \operatorname{argmin}_{1,\ldots,n} \|\mathbf{p}_i(t) - \mathbf{e}(t)\|\}.$$

Definition 3 *(Cone).* The closed positive cone formed with vertex at $\mathbf{e}(t)$, the axis along $\mathbf{e}(t+1) - \mathbf{e}(t)$ (i.e., along $\mathbf{u}_e(t)$), and with half angle equal to $\beta_{\max}$ is called the $\text{Cone}(k, t)$.

Next, we formalize the Cone move, which is defined only when there are at least k pursuers inside $\text{Cone}(k, t)$.

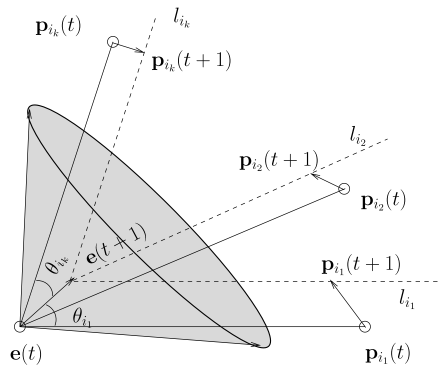

Fig. 3. Illustrating the Cone move for k pursuers. The shaded region is Cone(k,t). If the evader moves into Cone(k,t), then the pursuers $\mathbf{p}_{i_1},\dots,\mathbf{p}_{i_k}$ move so that their distances to the evader decrease by the same finite amount.

Definition 4 *(Cone move).* If some k pursuers $\mathbf{p}_{i_1}(t),\dots,\mathbf{p}_{i_k}(t)$ are in $\mathbf{P}_{\text{closest}}(t)$ and also in Cone(k,t), then the Cone move for $\mathbf{p}_{i_1},\dots,\mathbf{p}_{i_k}$ is defined as follows:

- draw a line l_j through $\mathbf{e}(t+1)$, parallel to $\mathbf{p}_j(t)-\mathbf{e}(t)$, for all $j\in\{i_1,\dots,i_k\}$.
- $\mathbf{p}_j(t+1)$ is the point on line l_j that minimizes $\|\mathbf{p}_j(t+1)-\mathbf{e}(t+1)\|$ subject to the constraint that $\|\mathbf{p}_{i_1}(t+1)-\mathbf{e}(t+1)\|=\dots=\|\mathbf{p}_{i_k}(t+1)-\mathbf{e}(t+1)\|$.

Fig. 3 offers a geometric interpretation of the Cone move. The key intuition behind the Cone move is that the k closest pursuers in Cone(k,t) can reduce their distance to the evader, and remain in $\mathbf{P}_{\text{closest}}(t+1)$, using the Cone move. The constraint in the second part of the definition of the Cone move gets satisfied if the k pursuers follow a prescribed set of moves. We give a more precise expression for the new locations in a Cone move, as follows.

Assume without loss of generality that $\theta_{i_1}\geqslant\theta_j, \forall j\in\{i_1,\dots,i_k\}$. Then, choose $\mathbf{p}_{i_1}(t+1)$ satisfying the following conditions:

- $\|\mathbf{p}_{i_1}(t+1)-\mathbf{p}_{i_1}(t)\|=1$, and
- $\angle\mathbf{p}_{i_1}(t+1)\mathbf{p}_{i_1}(t)\mathbf{e}(t)=\arcsin(u_e(t)\sin\theta_{i_1})$, where $u_e(t)=\|\mathbf{e}(t+1)-\mathbf{e}(t)\|$.

The positions $\mathbf{p}_j(t+1)$, for all $j\in\{i_2,\dots,i_k\}$, are chosen to satisfy the following conditions:

- $\|\mathbf{p}_j(t+1)-\mathbf{p}_j(t)\|^2=1+u_e^2(\cos\theta_j-\cos\theta_i)^2+2u_e(\cos\theta_j-\cos\theta_i)\sqrt{1-u_e^2\sin^2\theta_i}$.
- $\angle\mathbf{p}_j(t+1)\mathbf{p}_j(t)\mathbf{e}(t)=\arcsin(u_e(t)\sin\theta_j)$.

3.3. Proof of k-capture sufficiency

In the rest of this section, we prove that Algorithm 1 succeeds. We begin with the observation that k-capture is orientation-preserving, since throughout the algorithm, the direction of the vectors $\mathbf{p}_j-\mathbf{e}$ remains invariant for each j.

Proposition 3.3 *(Orientation preserving). The Algorithm k-capture is orientation-preserving.*

Our proof of k-capture depends on three technical lemmas showing, respectively, that some k pursuers become closest to the evader, that every cone move reduces the minimum distance by a finite amount, and that irrespective of the evader's strategy, the minimum distance decreases by a finite amount. Throughout the following discussion, it is assumed that the pursuers all follow the Algorithm k-capture.

The bound on the capture time depends on $d_{\min}(0):=\min_{i=1}^n\|\mathbf{p}_i(0)-\mathbf{e}(0)\|$ and $d_{\max}(0):=\max_{i=1}^n\|\mathbf{p}_i(0)-\mathbf{e}(0)\|$, which are the minimum and the maximum distance between a pursuer and the evader at the initial time $t=0$. Further, since $\beta_{\max}(0)$ remains constant under an orientation preserving strategy (cf. Lemma 3.1), we will drop the time dependence while referring to $\beta_{\max}$ for notational brevity. The following lemma proves the closest pursuers property.

Lemma 3.4 *($\mathbf{P}_{\text{closest}}$ cardinality). After a finite time upper bounded by $n(1+d_{\max}/\cos(\beta_{\max}))$, some k pursuers are in the set $\mathbf{P}_{\text{closest}}$.*

Proof. From statement 2 of Lemma 3.1, at every instant of time and for any move of the evader, there exist some k pursuers $\mathbf{p}_{i_1},\dots,\mathbf{p}_{i_k}$ such that for all $j\in\{i_1,\dots,i_k\}$, $\theta_j\leqslant\beta_{\max}$. If all of these k pursuers are in $\mathbf{P}_{\text{closest}}(t)$, then this result stands proved. Otherwise, for every t, there exists some pursuer (say $\mathbf{p}_j(t)$) out of the k pursuers, which is not in $\mathbf{P}_{\text{closest}}(t)$, and is such that $\theta_j \leqslant \beta_{\max}$. So at time $t+1$, the Advance move by $\mathbf{p}_j$ will ensure that either $\|\mathbf{p}_j(t+1) - \mathbf{e}(t+1)\| \leqslant \|\mathbf{p}_j(t) - \mathbf{e}(t)\| - \cos\beta_{\max}$ or $\mathbf{p}_j(t+1) \in \mathbf{P}_{\text{closest}}(t+1)$.

Thus, in the worst case, some k pursuers can be at a distance of at most $d_{\max}$ from the evader, which implies that it would take at most $1+d_{\max}/\cos\beta_{\max}$ time instants for a pursuer to be among the closest to the evader. Further, there can be at most n pursuers that are furthest from the evader initially. Therefore, in the worst case, after at most $n(1+d_{\max}/\cos\beta_{\max})$ time instants, some k pursuers must be $\mathbf{P}_{\text{closest}}$. □

Recall that $d_{\min}(t)$ is the distance of the closest pursuer from the evader at time t. Once k pursuers are in $\mathbf{P}_{\text{closest}}$, the following lemma establishes a lower bound on the decrease of $d_{\min}$ assuming that a Cone move occurs, which is favorable for the pursuers.

Lemma 3.5. *Let $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k} \in \mathbf{P}_{\text{closest}}$ be k pursuers closest to the evader at time t. If these pursuers' next move is a Cone move, then after the pursuers' move, we have*

$$d_{\min}(t+1) \leqslant d_{\min}(t) - \cos\beta_{\max}.$$

Proof. Let θ_j be the largest among the angles $\theta_{i_1}, \ldots, \theta_{i_k}$. Using the new locations of the pursuers in the Cone move, we obtain,

$$\begin{aligned} d_{\min}(t) - d_{\min}(t+1) &= u_e \cos\theta_j + 1 \cdot \cos\angle \mathbf{p}_j(t+1)\mathbf{p}_j(t)\mathbf{e}(t) \\ &= u_e \cos\theta_j + \sqrt{1 - u_e^2 \sin^2\theta_j} \\ &\geqslant \cos\theta_j \geqslant \cos\beta_{\max}, \end{aligned}$$

since $\theta_j \leqslant \beta_{\max}$ from the definition of the Cone region. The lemma follows. □

Finally, the next lemma derives a lower bound on the decrease of $d_{\min}$ for the worst-case evader move, while the pursuers follow the strategy of Algorithm k-capture.

Lemma 3.6. *If some k pursuers become closest to the evader at some time t, then the following holds*:

- *after every subsequent pursuer move, some k pursuers are in $\mathbf{P}_{\text{closest}}$, and*
- *after at most $n(1+d_{\max}/\cos\beta_{\max})$ pursuer moves, $d_{\min}$ decreases by at least $\cos\beta_{\max}$.*

Proof. Let A and B be two groups of pursuers in $\mathbf{P}_{\text{closest}}$ at time t, of which group A comprises of some k pursuers. If all pursuers of group A are in the Cone region at time t, then group A will make a Cone move which ensures that all pursuers in A are in $\mathbf{P}_{\text{closest}}$ at time $t+1$. Thus, the first claim trivially holds. Otherwise, all pursuers in A move parallel to the evader at time $t+1$. Now, if group B does not contain k pursuers, then at time $t+1$, all pursuers in group B are forced to move parallel to the evader, since they do not satisfy the criterion to make a Cone move. Thus, the pursuers in group A satisfy the first claim at time $t+1$. Finally, if group B contains some k pursuers and are in the Cone region at time t, then these k pursuers make a Cone move and satisfy the first claim at time $t+1$. Thus, the first claim holds at all times.

Now, let us consider the second claim. From Proposition 3.3 and statement 2 of Lemma 3.1, at every instant of time and for any move of the evader, there exists some k pursuers $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}$ such that $\theta_j(t) \leqslant \beta_{\max}$, for all $j \in \{i_1, \ldots, i_k\}$. We need to consider two cases:

- [All of $\mathbf{p}_{i_1}(t), \ldots, \mathbf{p}_{i_k}(t)$ are in $\mathbf{P}_{\text{closest}}(t)$:]
  In this case, the claim follows from Lemma 3.5 because all of these pursuers lie in $\text{Cone}(k,t)$.
- [At least one of out of the k pursuers, say $\mathbf{p}_j(t)$ is not in $\mathbf{P}_{\text{closest}}(t)$:]
  Without loss of generality, assume that $\mathbf{p}_j(t) \notin \mathbf{P}_{\text{closest}}(t)$. Then, at time $t+1$, the Advance move by $\mathbf{p}_j$ will ensure that either $\|\mathbf{p}_j(t+1) - \mathbf{e}(t+1)\| \leqslant \|\mathbf{p}_j(t) - \mathbf{e}(t)\| - \cos\beta_{\max}$ or $\mathbf{p}_j(t+1) \in \mathbf{P}_{\text{closest}}(t+1)$. Thus, in the worst-case, it requires at most $n(1+d_{\max}/\cos\beta_{\max})$ moves before all n pursuers are in $\mathbf{P}_{\text{closest}}$. Then, the next pursuer move is necessarily a Cone move, because for any choice of the evader move, there exists some k pursuers which are now equidistant from the evader, which lie in the Cone region. By Lemma 3.5, the distance of the k closest pursuers from the evader strictly decreases by at least $\cos\beta_{\max}$.

This completes the proof of the lemma. □

We can now state our main theorem on k-capture.

Theorem 3.7. *If the evader lies in the interior of the pursuers' k-Hull at $t=0$, i.e., $\mathbf{e}(0) \in \text{Hull}_k^o(\mathbf{p}_1(0), \ldots, \mathbf{p}_n(0))$, then it can be k-captured in at most $n(1+d_{\max}/\cos\beta_{\max})^2$ moves.*

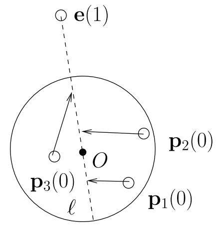

Fig. 4. Illustrating the initializing move. It is always possible to ensure that the pursuers are collinear with the evader and within a unit distance of each other. In this figure, the circle centered at O, has radius equal to half unit.

Proof. By Lemma 3.4, after at most

$$n\big(1 + d_{\max}/\cos(\beta_{\max})\big)$$

moves, some k pursuers are in $\mathbf{P}_{\text{closest}}$. Thereafter, Lemma 3.6 ensures that the distance of some k closest pursuers to the evader decreases by at least $\cos\beta_{\max}$ after every $n(1 + d_{\max}/\cos\beta_{\max})$ moves. Since capture is defined after the pursuers' move, after at most $n(1 + d_{\max}/\cos\beta_{\max})d_{\max}/\cos\beta_{\max}$ pursuer moves, we obtain $d_{\min} = 0$, that is, the evader and some k pursuers are coincident, which satisfies the conditions of k-capture. An upper bound on the time taken for the k-capture of the evader follows by summing the bounds of Lemma 3.4 and Lemma 3.6. This completes the proof of the theorem. □

Remark 3.8 *(Lower bound on capture time).* A lower bound on the time taken to capture is $d_{\max}/\cos\beta_{\max}$. To see this, consider the following initial condition and evader strategy. The evader's strategy is to move along a fixed vector $\mathbf{u}_e$ with unit step. Let $\mathbf{p}_1, \ldots, \mathbf{p}_k$ be furthest from the evader initially, and be located on the boundary of the resulting Cone$(k, 0)$. The rest of the pursuers are located outside Cone$(k, 0)$. This evader strategy and the initial pursuer locations ensure that the evader is captured after a time of at least $d_{\max}/\cos\beta_{\max}$, independent of the pursuers' strategy.

4. Bounded environments

In this section, we show a simple strategy for k-capture that always succeeds in a compact and convex subset of a Euclidean space. If every pursuer were to use an established strategy by Sgall [3] independently of the other pursuers, at each instant of time, then the distance between each pursuer and the evader would decrease to zero, but at different instants in time. Although this approach does not guarantee k-capture in general, it suggests that intuitively, it should be possible to coordinate the moves of each pursuer to achieve k-capture from any set of initial locations in the environment. Therefore, in contrast with the previous sections wherein there existed a necessary condition for k-capture, we will now directly present a strategy which requires k pursuers, and which achieves k-capture of the evader in at most $O(D^2)$ time steps, where D is the diameter of the environment.

Our strategy comprises of two phases. The first phase is an *initializing* move, which gets the pursuers in a favorable formation so that they can apply the steps in the second phase. In particular, the initializing move will show that it is possible to achieve a configuration of the pursuers and the evader such that $k - 1$ pursuers are located between a *lead* pursuer and the evader.

The second phase will mimic Sgall's strategy [3] for the lead pursuer, while the other $k - 1$ pursuers will maintain an invariant of being located between the lead pursuer and the evader at all times. The initial locations of the pursuers being sufficiently close to each other ensures that the evader gets captured if it moves to the location of any pursuer. We show that this phase terminates into the evader being k-captured.

Let us begin with the Initializing move.

4.1. Initializing move

In this phase, the pursuers first group themselves such that they are located inside a sphere of radius equal to half. This essentially means that every pursuer can reach the location of any other pursuer, in one time step.

Now, consider a closed sphere O of radius half which contains the pursuers at time $t = 0$. Let ℓ denote the intersection of the sphere O with line joining the evader's location at time $t = 1$ to the center of O. Now, independent of the location $\mathbf{e}(1)$, it is always possible to find k distinct locations $\mathbf{p}_1(1), \ldots, \mathbf{p}_k(1)$ each contained in ℓ, such that $\mathbf{p}_1(1), \ldots, \mathbf{p}_k(1)$ are collinear with $\mathbf{e}(1)$ and $\mathbf{p}_2(1), \ldots, \mathbf{p}_k(1)$ lie between $\mathbf{p}_1(1)$ and $\mathbf{e}(1)$. Fig. 4 shows an illustration of this move.

This terminates the initializing move, and we are now ready to present the k-capture strategy.

4.2. An algorithm for k-capture

At each time instant t, $\mathbf{p}_1$ makes a Sgall-like move, described as below.

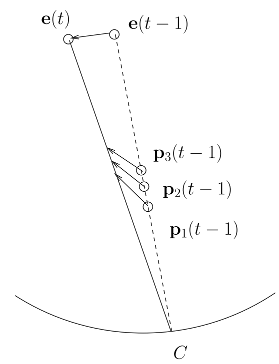

Fig. 5. Illustrating a move of Algorithm 2. Pursuer $\mathbf{p}_1$ follows the Sgall move with respect to the special location C, while all the others pick distinct points between $\mathbf{p}_1$ and $\mathbf{e}$ to move to.

1. Join $\mathbf{e}(t-1)$ and $\mathbf{p}_1(t-1)$ and extend it beyond $\mathbf{p}_1(t-1)$ to intersect the environment at C.
2. Move to the point closest to $\mathbf{e}(t)$ and on the line joining $\mathbf{e}(t)$ and C.

All other pursuers pick distinct points between $\mathbf{p}_1(t)$ and $\mathbf{e}(t)$. This strategy is illustrated in Fig. 5, and is summarized in Algorithm 2.

Remark 4.1 *(Convexity of the environment).* Recently, it was pointed out in [19] that Sgall's algorithm [3] may lead to situations where the pursuer's position falls outside the environment. This situation cannot arise in our version of the Sgall move because of our choice of the center C, which remains fixed throughout the second phase of the game, due to the way our move is defined. At every time instant t, since both C and $\mathbf{e}(t)$ lie inside the environment, due to the convexity assumption on the environment, every point on the line segment between $\mathbf{e}(t)$ and C also lies inside the environment, and therefore, $\mathbf{p}_1(t)$ lies inside the environment.

Algorithm 2: Sgall-like strategy

Assumes: The players are in a configuration resulting from the Initializing move. The location C is obtained by extending the line joining $\mathbf{e}(0)$ and $\mathbf{p}(0)$ beyond $\mathbf{p}(0)$ and intersecting it with the environment.

For each $t = 1, 2, \ldots,$
if *the evader is k-capturable,* **then**
  Every pursuer moves to $\mathbf{e}(t)$
else
  $\mathbf{p}_1$ makes the *Sgall-like Move* with respect to the point C.
  For each $j \in \{2, \ldots, k\}$,
    $\mathbf{p}_j$ moves to a distinct point between $\mathbf{p}_1(t)$ and $\mathbf{e}(t)$, and on the line joining $\mathbf{p}_1(t)$ and $\mathbf{e}(t)$.
  end for
end for

Thus, we obtain the following result.

Proposition 4.2. *With the initializing move and subsequently Algorithm 2, the pursuers k-capture the evader in $O(D^2)$ number of time steps, where D is the diameter of the compact environment.*

Proof. We first show that at each time step during the game, all the remaining $k-1$ pursuers maintain the following invariant: they lie on the line segment connecting the pursuer $\mathbf{p}_1$ with the evader. Since the pursuers and evader have the same step size, if we join C and $\mathbf{e}(t)$, from geometry we can conclude that there is always a set of candidate points for every pursuer $\mathbf{p}_j$ (between C and $\mathbf{e}(t)$) which can be reached from $\mathbf{p}_j(t-1)$. By picking a point from this set for the location $\mathbf{p}_j(t)$, every pursuer ensures that it is located between $\mathbf{p}_1$ and $\mathbf{e}$ at time t. Further, a distinct choice for each pursuer ensures that the additional $k-1$ pursuers do not already achieve collocation with the evader while the game has not yet terminated.

Second, we observe that if $\mathbf{e}(t)$ is reachable from $\mathbf{p}_1(t-1)$, then $\mathbf{e}(t)$ is simultaneously reachable from every other pursuer, and therefore, is k-captured at time t. To see this, consider the unit spheres centered at $\mathbf{e}(t-1)$ and $\mathbf{p}_1(t-1)$. Since $\mathbf{e}(t)$ is reachable from $\mathbf{p}_1(t-1)$, $\mathbf{e}(t)$ is contained in the intersection of the two unit spheres. But this intersection is contained within the unit sphere centered at every point between $\mathbf{e}(t-1)$ and $\mathbf{p}_1(t-1)$, and thus, $\mathbf{e}(t)$ is reachable from every other pursuer.

Third and finally, we observe that if, at any instant t in time, the evader's move $\mathbf{e}(t)$ is inside the circle centered at C and with radius $\|\mathbf{p}_1(t-1) - C\|$, then $\mathbf{e}(t)$ is reachable from $\mathbf{p}_1(t-1)$. This is because any point within the described circle is closer to $\mathbf{p}_1(t-1)$ than to $\mathbf{e}(t-1)$. From the Cosine rule applied to the triangle $\mathbf{p}_1(t-1)\mathbf{p}_1(t)C$, where $\mathbf{p}_1(t)$ is obtained from the Sgall-like move at time t, we conclude that the quantity $\|\mathbf{p}_1(t) - C\|^2$ increases by at least unity at every time step t. Therefore, Algorithm 2 terminates in finite time, since the environment has a finite diameter D. This yields the $O(D^2)$ number of steps for capture, and the k-capture claim is complete after unifying with the first two arguments. □

5. Closing remarks

In this paper, we introduced a new variant of the classical pursuit–evasion problem in an m-dimensional Euclidean space, which requires multiple pursuers to simultaneously reach the evader for capture. We showed that, for k-capture to occur, the evader must lie inside the k-Hull, in a pleasing generalization of the convex hull rule for the single pursuer capture. The main result of the paper was to show that this simple necessary condition is also sufficient. The proof of this sufficiency required a new pursuit strategy, requiring both an Advance move, which is a modified version of a known Planes algorithm and a new type of Cone move, which requires a careful coordination among the pursuers. For a version of this problem played in a compact and convex subset of Euclidean space, we showed that k-capture is always possible.

Our work suggests a number of intriguing problems for future research. Interesting directions include improving the upper bound on the time taken to capture the evader and addressing versions of this problem in general environments, with obstacles.

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