

On optimal guillotine partitions approximating optimal d -box partitions*

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Abstract

Given a set of n points, P , in E^d (the plane when $d = 2$) that lie inside a d -box (rectangle when $d = 2$) R , we study the problem of partitioning R into d -boxes by introducing a set of orthogonal hyperplane segments (line segments when $d = 2$) whose total $(d - 1)$ -volume (length when $d = 2$) is the least possible. In a valid d -box partition, each point in P must be included in at least one partitioning orthogonal hyperplane segment. Since this minimization problem is NP-hard and thus likely to be computationally intractable, we present an approximation algorithm to generate a suboptimal solution. This solution is obtained by finding an optimal guillotine partition, i.e., a special type of rectangular partition, which can be generated in $O(dn^{2d+1})$ time. We present a simple proof that the $(d - 1)$ -volume of an optimal guillotine partition is not greater than $2d - 4 + 4/d$ times the $(d - 1)$ -volume of an optimal d -box partition.

Key words: Approximation algorithms; d -Box partitions; Polynomial time complexity; Guillotine partitions, Computational geometry

1. Introduction

Given a set of points, P , in E^2 that lie inside a rectangle R , we study the problem of partitioning R into rectangles by introducing a set of orthogonal line segments whose total length is the least possible, and each point in P is included in at least one of the partitioning line segments. We shall refer to this problem as the RG- P_2 problem,

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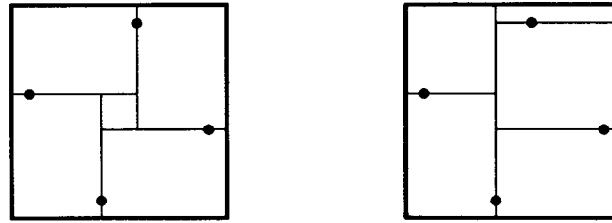
Table 1
Approximation algorithms for partitioning a rectangle

Approximation Bound	Time Complexity Bound	Reference	Method
4	$O(n \log n)$	[3], [10]	divide the conquer
3	$O(n^4)$	[5]	transformation
1.75	$O(n^5)$	[6]	dynamic programming

where RG stands for rectangle, P for a set of points, and the two for 2-D. We use $E(I)$ to represent a set of partitioning line segments in a feasible solution for $I = (R, P)$. The RG- P_2 problem was shown to be NP-hard in [8]. Approximation algorithms for the RG- P_2 problem appear in [3–6]. The currently best approximation algorithms for the RG- P_2 problem are summarized in Table 1.

In VLSI design, the problem of dividing routing regions into channels can be reduced to the RG- P_2 problem with interior holes instead of points ([11]). Several approximation algorithms for this more general problem exist (see [1, 7, 9, 10]). The algorithms with the smallest approximation bound are the ones reported in [9] and [10]. The algorithm given in [10] invokes the algorithm reported in [4] for the RG- P_2 problem as a sub-procedure. The algorithms in [4, 6, 3] generate guillotine partitions, but the ones in [6], and [3] are in the worst case ‘closer’ to optimal. Therefore, a smaller approximation bound for the general problem can be obtained by using these algorithms in the procedure reported in [10].

Let R be a rectangle and let P be a set of points inside R . A rectangular partition $E(I)$, for $I = (R, P)$, has a *guillotine cut* if there is a line segment in $E(I)$ that partitions the rectangular boundary R into two rectangles. We say that a rectangular partition $E(I)$ is a *guillotine partition* if either $E(I)$ is empty or $E(I)$ has a guillotine cut that partitions R into R_1 and R_2 , and both $E(I_1)$ (edges from $E(I)$ in R_1) and $E(I_2)$ (edges from $E(I)$ in R_2) are guillotine partitions for $I_1 = (R_1, P_1)$ and $I_2 = (R_2, P_2)$, respectively. The minimum edge length *guillotine partitioning problem* consists of finding a guillotine partition of least total length, i.e., the sum of the lengths of the edges in the partition. It is simple to see that any guillotine partition is a rectangular partition, but the converse is not necessarily true (see Fig. 1). Du et al. [2] studied the problem of finding an optimal guillotine partition for any instance $I = (R, P)$ of the RG- P_2 problem. They showed that such a partition can be found by dynamic programming in $O(n^5)$ time, where n is the total number of points. Du et al. [2] also showed that the length of an optimal guillotine partition is at most twice the length of an optimal rectangular partition for the RG- P_2 problem. Their proof of this bound is lengthy and complex. Our proof, which can be easily generalized to the problem defined over an arbitrary number of dimensions, is simpler. A complex proof showing that for 2-D the length of an optimal guillotine partition is within 1.75 times the length of an optimal rectangular partition is given in [6].



(a) Optimal Rectangular Partition. (b) Optimal Guillotine Partition.

Fig. 1. Rectangular and guillotine partitions.

We also consider a more general version $\text{RG-}P_2$, denoted by $\text{RG-}P_d$, which is defined over E^d . Specifically, given a set P of points in E^d that lie inside a d -box R , we study the problem of partitioning R into d -boxes by introducing a set of orthogonal hyperplane segments¹ whose total $(d - 1)$ -volume is the least possible. In a valid partition, each point in P must be included in at least one partitioning orthogonal hyperplane segment. In what follows, when we refer to a *guillotine cut (partition)* we mean the obvious d -dimensional generalization of a guillotine cut (partition).

Gonzalez, Razzazi and Zheng [3] present an $O(dn \log n)$ algorithm that generates solutions that are within $2d$ of optimal for the $\text{RG-}P_d$ problem. In Section 2, we present a simple proof that the $(d - 1)$ -volume of an optimal guillotine partition is not greater than $2d - 4 + 4/d$ times the $(d - 1)$ -volume of an optimal d -box partition. In theorem 2, we show that an optimal guillotine partition can be obtained via dynamic programming in $O(dn^{2d+1})$ time, where n is the number of points in P . Therefore, an optimal guillotine partition provides a polynomial time approximation to the d -box partition problem when d is bounded by some fixed constant. It is worthwhile mentioning that the algorithm in this paper always generates solutions that are not farther from optimal than the ones generated by the algorithm in [3], since both algorithms generate guillotine partitions, and our algorithm generates an optimal guillotine partition. However, the algorithm in [3] is always faster.

For more than two dimensions, our problem is far removed from its original application; however, it has other applications when $d = 3$. For example, suppose we have a solid block of material and we know exactly the location of all its ‘impure’ points. Our problem consists of partitioning the block into sub-blocks without impure material inside them in such a way that the total area of the cuts is the least possible. This is in general a good estimate for the cost of partitioning the block into sub-blocks. The objective function can be modified to include the more traditional setup cost to align the machine in the various directions. Our dynamic

¹By a d -box (in E^d space) we mean a d -dimensional rectangle, whereas by hyperplane we mean a $(d - 1)$ -dimensional plane. Whenever we refer to a hyperplane segment, we mean a $(d - 1)$ -dimensional d -box.

programming algorithm can be easily modified so that it finds optimal guillotine partitions with respect to this criteria. However, it is not clear whether or not this type of partition would also be a good approximation for the corresponding d -box partition problem.

2. Bounds for d -dimensional guillotine partitions

An instance $I = (R = (o, X), P)$ of the $\text{RG-}P_d$ problem consists of a d -box R (where $o = (o_1, o_2, \dots, o_d)$ is the ‘lower-left’ corner of the boundary R (*origin* of R), and $X = (X_1, X_2, \dots, X_d)$ are the dimensions or sizes of R) and a set of points $P = \{p_1, p_2, \dots, p_n\}$ inside R . Let $E(I)$ denote the set of orthogonal hyperplane segments in a valid d -box partition of I , and let $G(I)$ denote the set of orthogonal hyperplane segments in a minimum $(d - 1)$ -volume guillotine partition. We show that $V(G(I)) \leq (2d - 4 + 4/d) \cdot V(E(I))$, where $V(S)$ represents the sum of the $(d - 1)$ -volume of the segments in set S . To prove our bound we define, via procedure TRANS_E_TO_G , a set of orthogonal hyperplane segments $E'(I)$ such that $E'(I) \cup E(I)$ forms a guillotine partition for I and $V(E'(I) \cup E(I)) \leq (2d - 4 + 4/d) \cdot V(E(I))$.

In what follows we refer to the dimensions of E^d by the integers $1, 2, \dots, d$ (in 2-space we have the first dimension or 1st-axis, and the second dimension or 2nd-axis). Let $E_j(I)$ be the set of hyperplane segments orthogonal to axis j in $E(I)$. An orthogonal hyperplane segment to the j th-axis that partitions the d -box R into two d -boxes is called a j th-axis *orthogonal cut*. A d -box partition $E(I)$ has a *half j th-axis overlapping cut* l if l is a j th-axis orthogonal cut such that $V(I \cap E(I)) \geq 0.5 \prod_{i \neq j} X_i$. We say that $E(I)$ has a *j th-axis guillotine cut*, l , if l is a j th-axis orthogonal cut such that $V(l \cap E(I)) = \prod_{i \neq j} X_i$. Suppose R is partitioned by a j th-axis orthogonal cut into two d -boxes, R_1 and R_2 . With respect to such a partition we define $E(I_1)$ and $E(I_2)$ as the set of hyperplane segments $E(I)$ inside R_1 and R_2 , respectively.

Assume that $E(I)$ is non-empty, and assume without loss of generality that the d -box has been rotated so that $V(E_j(I)) \leq V(E_d(I))$ for all $1 \leq j < d$. As we shall see later on, this is a very useful property in the proof of theorem 1. It is very important to remember that dimension d will remain dimension d throughout the transformation process. We use $E'(I)$ to denote all of the portions of the segments introduced by procedure TRANS_E_TO_G that do not overlap with the segments in $E(I)$. Initially $E'(I)$ is empty and none of the facets² of R are colored. When a facet of R is colored it indicates that procedure TRANS_E_TO_G has introduced a hyperplane segment (or part of it) whose $(d - 1)$ -volume is equal to the $(d - 1)$ -volume of this facet, and which has not yet been accounted for by hyperplane segments in $E(I)$. Procedure

²By facets of R we mean only the $2d$ outer facets of R , and not the facets of the d -boxes within R .

TRANS_E_TO_G, which we formally define below, first checks if there is a half d -overlapping cut and if so, it introduces a d th-axis orthogonal cut along the half d th-axis overlapping cut. As a result of introducing this cut two subinstances are generated and a hyperplane segment is added to $E'(I)$. It is simple to verify that the $(d - 1)$ -volume of the newly introduced segment in $E'(I)$ is at most equal to the $(d - 1)$ -volume of the segments in $E(I)$ that overlap with the d th-axis orthogonal cut. Such a segment is not part of $E(I_1)$ or $E(I_2)$. The facets of R_1 and R_2 that were parts of colored facets of R remain colored. At this point we invoke procedure TRANS_E_TO_G recursively on the two resulting subinstances. If there is no half d th-axis overlapping cut, TRANS_E_TO_G checks if there is a 1st-axis guillotine cut, and if so, it partitions the rectangle through one such cut and at most two subinstances are generated. All of the facets of R_1 and R_2 that were parts of colored facets of R lose their coloring at this point. The reason for this is that the $(d - 1)$ -volume of the 1st-axis guillotine cut will account for the $(d - 1)$ -volume of the colored facets of R . We apply procedure TRANS_E_TO_G recursively to each non-empty subinstance. If there is no 1st-axis guillotine cut, TRANS_E_TO_G introduces a 1st-axis orthogonal cut, called a *mid-cut*, that intersects the center of the d -box (i.e., it includes the point $(o_1 + X_1/2, o_2 + X_2/2, \dots, o_d + X_d/2)$). As we shall prove later on, when a mid-cut is introduced each of the two resulting subinstances has at least one segment inside it which is not orthogonal to the d -axis. Suppose that R_1 contains point o of R . The facets in R_1 and R_2 that were colored in R remain colored. The facet in R_1 orthogonal to the 1st-axis which does not include point o gets colored at this point. The $(d - 1)$ -volume of such facet is equal to the $(d - 1)$ -volume of the mid-cut just introduced. The procedure is applied recursively to the resulting subinstances. The procedure is formally defined below. The first invocation involves an uncolored d -box partition $E(I)$ of R .

procedure TRANS_E_TO_G($I = (o, X)$, $E(I)$, d)

begin

Relabel the dimensions so that $X_1 \geq X_2 \geq \dots X_{d-1}$;

/*Note that dimension d is never relabeled. Extreme care must be taken after this step since the axes have been relabeled. For clarity we do not include all the details needed because of the relabeling.*/

/*In what follows we partition R into R_1 and R_2 by introducing a hyperplane segment along a cut. We assume that R_1 contains point o of R .*/

case

: $E(I)$ has a half d th-axis overlapping cut:

Introduce a hyperplane segment in R along one such cut³;

³Remember that we are interested in $E'(I)$, the portions of the segments introduced by the procedure that do not overlap with the segments in $E(I)$.

The facets of R_1 and R_2 which were colored in R remain colored (even if sliced);

: $E(I)$ has an 1st-axis guillotine cut:

Introduce a hyperplane segment in R along one such cut³;

Delete all colors from the facets of R_1 and R_2 ;

:else:

Introduce into R a 1st-axis orthogonal cut³ that intersects the center of R ;/*

mid-cut*/

All the facets of R_1 and R_2 which were parts of colored facets of R (even if sliced) remain colored;

Color the facet of R_1 orthogonal to the 1st-axis which does not include point o ;

end case

if $E(I_1) \neq \emptyset$ then invoke TRANS- E -TO- G recursively with $(I_1, E(I_1), d)$;

if $E(I_2) \neq \emptyset$ then invoke TRANS- E -TO- G recursively with $(I_2, E(I_2), d)$;

end

It is important to note that procedure TRANS- E -TO- G is only used to establish our approximation bound. Let $E'(I)$ be the set of hyperplane segments introduced by procedure TRANS- E -TO- G that do not overlap with the segments in $E(I)$. Clearly, $E(I) \cup E'(I)$ is a guillotine partition. Remember that $E_j(I)$ represents the set of hyperplanes in $E(I)$ orthogonal to axis j . We define $E'_j(I)$ similarly, but with respect to $E'(I)$. We define X_j as $\prod_{i \neq j} X_i$, for $1 \leq j \leq d-1$. In what follows, we assume that $X_1 \geq X_2 \geq \dots \geq X_{d-1}$. A d -box $R = (o, X)$ is said to be of type i ($1 \leq i \leq d-1$) if i is the largest integer such that $X_1 \leq 2X_i$. Since $X_1 \leq 2X_1$ for all I , the type of a d -box is uniquely defined. Before proving our main result (Theorem 1), we prove the following intermediate results (Lemmas 1 and 2). It is important to remember that once TRANS- E -TO- G begins, dimension d will not be relabeled.

Lemma 1. Every invocation made to procedure TRANS- E -TO- $G(I, E(I), d)$, satisfies the following conditions:

- (a) $E(I)$ is non-empty, and
- (b) if $E(I)$ has a d th-axis guillotine cut then none of the facets of R are colored; otherwise the only facets of R which could be colored are those facets orthogonal to the j th-axis which do not include point o , for $1 \leq j \leq i$, where i is the type of R .

Proof. Initially, $E(I) \neq \emptyset$ and none of the facets of R are colored. Therefore, the first invocation to procedure TRANS- E -TO- G satisfies (a) and (b). We now show that if upon entrance to the procedure the conditions are satisfied, then the invocations made directly from it will also satisfy (a) and (b). Assume $(I, E(I), d)$ satisfies conditions (a) and (b). There are three cases depending on the type of cut introduced by procedure TRANS- E -TO- G .

Case 1: Procedure TRANS- E -TO- G partitions R along a half d th-axis overlapping cut.

Suppose that $E(I)$ has a d th-axis guillotine cut. From (b) we know that none of the facets of R are colored, and no facet will get colored at this step. Since each invocation made directly from the current invocation is on non-empty problem instances, it follows that invocations made directly from this call satisfy both properties (a) and (b). On the other hand, if $E(I)$ does not have a d th-axis guillotine cut, then the half d th-axis overlapping cut cannot be a d th-axis guillotine cut. Therefore, there is at least one hyperplane segment orthogonal to the j th-axis, for some $1 \leq j < d$, on each side of the half d th-axis overlapping cut. Since a d th-axis orthogonal cut over this half d th-axis overlapping cut is introduced, the resulting two d -box partitions ($E(I_1)$ and $E(I_2)$) must be non-empty, otherwise $E(I)$ is not a d -box partition. Clearly, R_1 and R_2 are of type i (since the first $d - 1$ dimensions of R_1 , R_2 and R are identical), there are no d th-axis guillotine cuts (since R had no guillotine cuts and a cut orthogonal to the d -axis was introduced) and the only facets of R_1 and R_2 which could be colored are those facets orthogonal to the j th-axis which do not include point o , for $1 \leq j \leq i$. Therefore, each invocation made directly from the current invocation satisfy both properties (a) and (b). This completes the proof for this case.

Case 2: Procedure TRANS-E-TO-G partitions R along a 1st-axis guillotine cut.

Since just after partitioning R all of the facets in R_1 and R_2 lose their coloring, and we invoke procedure TRANS-E-TO-G only if their corresponding d -box partition is non-empty, then the invocations made in this case satisfy (a) and (b). This completes the proof of this case.

Case 3: Procedure TRANS-E-TO-G introduces a mid-cut.

Since $E(I)$ does not have a 1st-axis guillotine cut, because Case 2 does not apply, then each of the resulting d -box partitions has at least one hyperplane segment orthogonal to the j th-axis for some $1 \leq j \leq d - 1$. Since R is of type i , and a mid-cut is introduced orthogonal to the 1st-axis, then the type of R_1 and R_2 is k , for some integer $k \geq i$. Clearly, the second subinstance satisfies (a) and (b). Since o is in R_1 , the facet orthogonal to the 1st-axis which does not include o is not colored in R_1 when the mid-cut is introduced. Immediately after that step it gets colored. Therefore, the first subinstance also satisfies (a) and (b). This completes the proof for this case and the lemma. $\square$

Lemma 2. *For any non-empty d -box partition $E(I)$ of any instance I of the RG- P_d problem, procedure TRANS-E-TO-G generates a set $E'(I)$ of $(d - 1)$ -dimensional hyperplane segments such that*

$$V(E'_d(I)) \leq V(E_d(I)),$$

and

$$\sum_{j=1}^{d-1} V(E'_j(I)) \leq (2d - 3) \sum_{j=1}^{d-1} V(E_j(I)).$$

Proof. First we show that $V(E'_d(I)) \leq V(E_d(I))$. This is simple to prove because d th-axis orthogonal cuts are only introduced over half d th-axis overlapping cuts. Each

time a d th-axis orthogonal cut is introduced the segment l' added to $E'_d(I)$ has $(d - 1)$ -volume that is at most equal to the $(d - 1)$ -volume of the segments in the half d th-axis overlapping cut l in $E_d(I)$. Since neither l nor l' are in the interior of the resulting subinstances after this operation, these hyperplane segments of $E_d(I)$ are never charged more than once, and thus $V(E'_d(I)) \leq V(E_d(I))$.

Let us now show that

$$\sum_{j=1}^{d-1} V(E'_j(I)) \leq (2d - 3) \sum_{j=1}^{d-1} V(E_j(I)).$$

It is simple to verify that procedure TRANS- E -TO- G does not color a facet more than once, and at the end none of the facets are colored. The only place where facets lose their color is when procedure TRANS- E -TO- G introduces a 1st-axis guillotine cut. Let us examine this case in more detail. Since R is of type i , and the only facets which could be colored are those that do not include o and which are orthogonal to axis j for $1 \leq j \leq i < d$ (by the Lemma 1), we know that the $(d - 1)$ -volume which could be colored is at most $\sum_{j=1}^i X_{\bar{j}}$. The $(d - 1)$ -volume of the 1st-axis guillotine cut is $X_{\bar{1}}$. Since $X_1 \leq 2X_j$, for $1 \leq j \leq i < d$, we know that $2X_{\bar{1}} \geq X_{\bar{j}}$. Therefore,

$$\sum_{j=1}^i X_{\bar{j}} \leq X_{\bar{1}} + 2(d - 2)X_{\bar{1}} = (2d - 3)X_{\bar{1}}.$$

Since the $(d - 1)$ -volume of each segment introduced in $E'_j(I)$, $1 \leq j \leq d - 1$, corresponds to the $(d - 1)$ -volume of every facet colored by procedure TRANS- E -TO- G , and the $(d - 1)$ -volume of a set of j th-axis guillotine cuts identified by procedure TRANS- E -TO- G is at most $(2d - 3)$ times the $(d - 1)$ -volume of the facets colored by procedure TRANS- E -TO- G , it then follows that

$$\sum_{j=1}^{d-1} V(E'_j(I)) \leq (2d - 3) \sum_{j=1}^{d-1} V(E_j(I)).$$

This concludes the proof of the lemma. $\square$

Theorem 1. Let $E_{\text{opt}}(I)$ be an optimal d -box partition for any instance I of the RG- P_d problem, and let $G(I)$ be an optimal guillotine partition for I . Then $V(G(I)) \leq (2d - 4 + 4/d)V(E_{\text{opt}}(I))$.

Proof. Let $E(I)$ be any d -box partition for I . Assume without loss of generality that $V(E_d(I)) \geq V(E_j(I))$ for $1 \leq j \leq d - 1$ (i.e., the sum of the $(d - 1)$ -volume of the hyperplane segments orthogonal to the j th-axis is larger than the ones orthogonal to the i th-axis), as otherwise a simple initial relabeling is performed before we start the transformation. It is important to keep in mind that this relabeling is performed only once. Apply procedure TRANS- E -TO- G to $E(I)$. Combining the fact that

$V(E_d(I)) \geq V(E_j(I))$ for $1 \leq j \leq d-1$ together with Lemma 2 and the assumption that $d \geq 2$, we know that

$$V(E(I) \cup E'(I)) \leq (2d-2) \sum_{j=1}^{d-1} V(E_j(I)) + 2V(E_d(I)).$$

This function achieves its maximum value when $V(E_j(I)) = V(E(I))/d$. Therefore,

$$V(E(I) \cup E'(I)) \leq ((2d-2)(d-1) + 2) V(E(I))/d = (2d-4 + 4/d) V(E(I)).$$

Since $E(I) \cup E'(I)$ is a guillotine partition for I ,

$$V(G(I)) \leq V(E(I) \cup E'(I)).$$

Hence,

$$V(G(I)) \leq (2d-4 + 4/d) V(E(I)). \quad \square$$

Theorem 2. *An optimal guillotine partition for a RG- P_d problem may be constructed in $O(dn^{2d+1})$ time.*

Proof. Let $g(I_1, I_2, \dots, I_d)$ be the $(d-1)$ -volume of an optimal guillotine partition for the d -box defined by the intervals $I_1, I_2, \dots, I_d$, where $I_j = [l_j, r_j]$, $l_j \leq r_j$, and $l_j(r_j)$ is the j th-coordinate value of a point in P . For any $I_1, I_2, \dots, I_d$, one can easily compute $g(I_1, I_2, \dots, I_d)$ recursively by trying all dn guillotine cuts (where n is the number of points in the d -box formed by $I_1, I_2, \dots, I_d$) and then selecting the minimum one. Since there are $O(n^{2d})$ of such g 's that need to be computed in order to find the $(d-1)$ -volume of an optimal partition, then by using dynamic programming the overall time complexity is $O(dn^{2d+1})$. By recording all valid guillotine partitions while computing volumes, this procedure can be easily modified to construct an optimal guillotine partition, rather than only computing the optimal $(d-1)$ -volume. $\square$

3. Discussion

For any $d \geq 2$, it is simple to find a problem instance I such that $V(G(I))$ is about $1.5 V(E_{\text{opt}}(I))$ [2]. One of such problem instances is an obvious generalization of the d -box partition in Fig. 2. Although we have obtained the upper bound $2d-4 + 4/d$, it is not clear whether or not there are problem instances that achieve (asymptotically) this bound for $d > 2$.

In our transformation procedure, TRANS- E -TO- G , one may replace a 1st-axis guillotine cut by a j th-axis guillotine cut ($1 \leq j < d$). This has no effect in the worst case approximation bound obtained in this paper; however, most of the time it will introduce a set of hyperplane segments with smaller $(d-1)$ -volume.

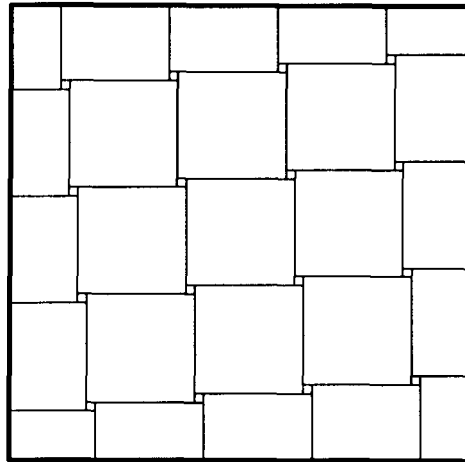


Fig. 2.

By using standard speed-up techniques [12] it seems impossible to reduce the $O(dn^{2d+1})$ time complexity bound. It would be of theoretical interest to try to improve the time required to find an optimal guillotine partition.

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