INTEGRATION, the VLSI journal 13 (1992) 153–177
Elsevier

Grid stretching algorithms for routing multiterminal nets through a rectangle

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Abstract. The rectangle routing (RRP) consists of routing a set of multiterminal nets through a rectangle R under the knock-knee model. We present, for the three-terminal-net RRP problem, an $O(n \log n)$ time algorithm that constructs a layout inside a rectangle R^f such that asymptotically $A(R^f)/A(R) < 24/13 < 1.85$, where $A(W)$ is the area of rectangle W. For the unrestricted RRP problem, we present an $O(n \log n)$ time algorithm that generates a layout inside a rectangle R^f such that asymptotically $A(R^f)/A(R) < 3.5$. Our approach consists of stretching the grid, and introducing a set of wires such that the given RRP problem instance is transformed into a two-terminal-net routable RRP instance, which can be routed by an existing $O(n \log n)$-time algorithm.

Keywords. VLSI design automation; layouts; switch-box routing; routing through a rectangle; minimizing area; approximation algorithms.

1. Introduction

Let R be a rectangle uniformly partitioned by $w-1$ vertical line segments and $h-1$ horizontal line segments. The set of lines (which include the rectangle boundary sides) is called the *grid* and the lines are called *grid lines*. The intersection of two grid lines is referred to as a *grid point*. A subset of grid points on the boundary of R without including the corners of R are referred to as *terminal points*. A terminal t is denoted by a pair $(x(t), y(t))$ of the x and y coordinate values of t. It is important to note that in order to simplify our notation we eliminated the possibility of terminal points being located on the corners of R; however, our algorithms can be easily modified to handle this situation by introducing a constant number of additional grid lines (the number is very small). The approximation bounds for the modified algorithms are asymptotically identical to the bounds reported in this paper. The vertical (horizontal) grid lines are called *columns (rows)*. The columns (rows) are labeled from left to right (bottom to top) with the integers 0 to w (0 to h). The set of terminal points is partitioned into m sets, $N_1, N_1, \ldots, N_m$. Each set N_i is called a *net*, and the set of nets is denoted by N. The problem of routing though a rectangle, which we call the RRP *problem* (which is also referred to as the *switch-box routing problem*), is denoted by $I = (R, N)$, and consists of finding a *layout* under the knock-knee wiring model for the set N of nets inside R. A layout under the knock-knee model for the set N of m nets consists of m edge-disjoint connected subgraphs $W_1, W_2, \ldots, W_m$ of R such that each W_i connects all terminals in N_i. It is well known that any knock-knee layout is wirable in four layers by using the algorithm in [1]. An RRP problem in which every net has at most k terminals is called an RRP *of degree k* or a *k-terminal-net* RRP *problem*.

We define a *vertical cut* of R as the region between a pair of adjacent columns $(c, c+1)$. Note that the two columns are not included in the cut. The *capacity of a vertical cut* is $H+1$, the number of rows between column c and column $c+1$ (i.e. the total number of rows in R). The *density* of a vertical cut $(c, c+1)$, denoted by $d_c^{\mathrm{v}}(N)$, is the number of nets with at least one terminal to the left of the cut and least one terminal to the right of the cut. A vertical cut $(c, c+1)$ is *not saturated* if its capacity exceeds its density. These notions are similarly defined for horizontal cuts. We define $d^{\mathrm{v}}(N) = \max\{d_c^{\mathrm{v}}(N) \mid 0 \leqslant c < w\}$ and $d^{\mathrm{h}}(N) = \max\{d_c^{\mathrm{h}}(N) \mid 0 \leqslant c < h\}$ as the *vertical density* and *horizontal density* of I, respectively.

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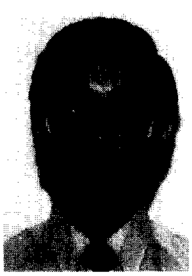

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The two-terminal-net RRP has been extensively studied. The fundamental theorem for routability of a two-terminal-net RRP was established by Frank [2] and Mehlhorn and Preparata [5].

Theorem 1.1. *A two-terminal-net* RRP *is routable iff the revised row and column criteria hold* [2]. *Furthermore, if these conditions are satisfied a layout can be constructed in* $O(n \log n)$ *time, where n is the number of terminals* [5].

The concept of revised row and column criteria is required in Theorem 1.1. However, since this concept is not relevant to our discussion we do not elaborate on it. Interested readers can find additional details in [2] and [5]. The following corollary of Theorem 1.1 allows us to simplify the presentation of our results.

Corollary 1.1. For a two-terminal-net RRP, $I=(R, N)$, if every vertical cut and horizontal cut of R is not saturated, then I has a layout in R [2]. Furthermore, such a layout can be generated in $O(n \log n)$ time, when n is the number of nets [5].

It is important to point out the difference between Theorem 1.1 and Corollary 1.1. Corollary 1.1 guarantees a layout solution for two-terminal-net RRP instances $I=(R, N)$ such that $d^{\mathrm{v}}(N)<h+1$ and $d^{\mathrm{h}}(N)<w+1$, whereas Theorem 1.1 guarantees a layout for some two-terminal-net RRP instances even when $d^{\mathrm{v}}(N)=h+1$ and/or $d^{\mathrm{h}}(N)=w+1$.

The problem of determining whether or not an RRP instance of arbitrary degree is routable is an NP-complete problem [9], and thus it is unlikely that an efficient algorithm for its solution exists. However, any RRP problem instance is routable if enough rows and columns are introduced. In [5], an algorithm for routing any instance of the RRP problem by introducing additional rows and columns is presented. This algorithm is based on Theorem 1.1. For any rectangle R we use $A(R)$ to represent the area of R. We say that rectangle R' is a *stretched version of rectangle R* if R' is obtained from R by adding zero or more rows and columns. We say that OPT is an *optimal area layout* for $I=(R, N)$ if R^*, the smallest rectangle that includes OPT, is a stretched version of rectangle R, and $A(R^*) \leqslant A(R')$ for any rectangle R' that is a stretched version of R and (R', N) is a routable RRP problem instance. Note that our definition of optimality is with respect to all layouts with a number of rows and columns that is at least as large as the number of rows and columns of R, respectively. Let R'' be the rectangle obtained from R by adding $d^{\mathrm{v}}(N)-(h+1)$ $(d^{\mathrm{h}}(N)-(w+1))$ columns (rows) between two adjacent columns (rows) if $d^{\mathrm{v}}(N)>h+1(d^{\mathrm{h}}(N)>w+1)$. Clearly, $A(R'')$ (greater than or equal to $A(R)$) is a lower bound for the area of an optimal layout for R. Hereafter, we assume that for any given RRP, $I=(R, N)$, $d^{\mathrm{v}}(N)<h+1$ and $d^{\mathrm{h}}(N)<w+1$; and we use $A(R)$ as a lower bound for $A(R^*)$, where R^* is the smallest rectangle enclosing OPT. This assumption allows us to apply Corollary 1.1 to simplify our algorithms and their

analyses. Our algorithms (and their analyses) can be easily modified when we choose to use Theorem 1.1. The difference would be at most one additional row and one additional column. For a rectangle R with height h and width w, the *aspect ratio of* R, denoted as $r(R)$, is defined as $\max\{h, w\}/\min\{h, w\}$. We assume without loss of generality that $w \leqslant h$. In practice w and h are large. Because of these properties, we only derive asymptotic bounds for the approximation, i.e., our bounds hold when w is larger than some fixed constant. For all h and w, the exact approximation bound (by this we mean the approximation bound for all h and $w \geqslant 1$) for our algorithm is equal to the asymptotic one plus $O((h+w)/(hw))$. It is important to note that the exact approximation bounds for our algorithms differ from the asymptotic ones, because there are several (at most five) additional rows and columns that are not accounted for in the analyses.

In [5] it is shown that for any RRP problem $I=(R, N)$ a layout can be constructed inside R^{f}, a stretched version of rectangle R, such that asymptotically $A(R^{\mathrm{f}})/A(R) \leqslant 4$. The idea behind this algorithm is to stretch R into R^{f} and introduce a set of wires so that $I=(R, N)$ is transformed into a routable two-terminal-net RRP problem instance $I'=(R', N')$. Since I' is routable, its layout can be constructed by the algorithm for the two-terminal-net RRP given in [5]. It is instructive to describe a modified version of their transformation in detail. Our version is based on the assumption that $d^{\mathrm{v}}(N) < h+1$ and $d^{\mathrm{h}}(N) < w+1$. As a result of this, several unnecessary rows and columns are introduced. Let R^{f} be the rectangle obtained from R by adding a grid line between every pair of adjacent grid lines, except between rows 0 and 1 (columns 0 and 1) in which case two rows (columns) are added. Make a copy of each terminal located on the left and right (top and bottom) sides of the rectangle on the unused boundary grid point located immediately below (to the left of) it. Each net N_i with p terminals is transformed into p nets with two terminals each. Let us refer to the new set of terminals by the integer $1, 2, \ldots, 2p$ and assume that for $1 \leqslant i \leqslant p$, terminals $2i-1$ and $2i$ have the property that one is an original terminal and the other is a new terminal which is a copy of the original one in the pair. The p nets constructed from net N_i are defined as follows: the jth net,

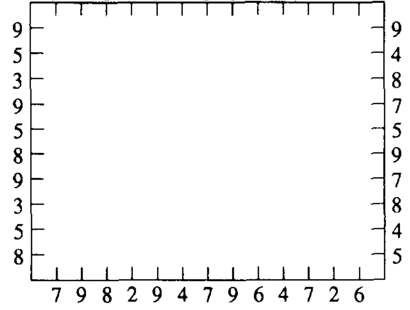

Fig. 1.1. Problem instance $I=(R, N)$.

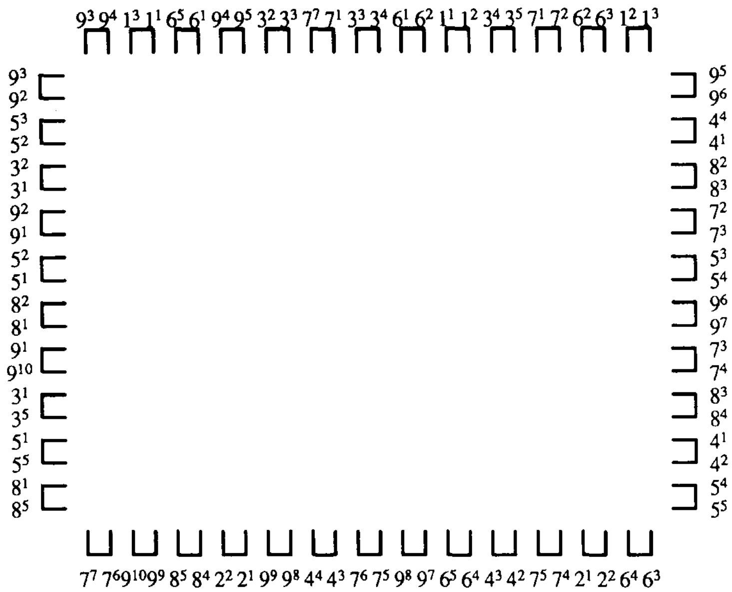

Fig. 1.2. Problem instance $I^f = -(R^f, n^f)$ after introducing the U wires.

$1 \leqslant j \leqslant p-1$, consists of terminals $2j$ and $2j+1$ and the pth net consists of terminals $2p$ and 1 (see Fig. 1.2). The set of nets resulting from this transformation is referred to as N^f. If the resulting problem after deleting two rows and two columns without terminal points from R^f is not routable, then add enough rows and columns to R^f so that it is routable. In Fig. 1.1 we show a problem instance $I = (R, N)$ and in Fig. 1.2 we show the problem instance $I^f = (R^f, N^f)$. Note that in this case only $h+1$ columns and $w+1$ rows are introduced. Let us now show that I^f is routable in O(n log n). Project each terminal point to the row or column closest and parallel to the boundary side where the terminal is located, and introduce U wires to connect each terminal point to its copy (see Fig. 1.2). Let $I' = (R', N')$ be the resulting problem, i.e., R' is R^f after deleting the boundary lines and N' is N^f projected from R^f to R'. Each row and column of the newly constructed two-terminal-net RRP satisfies the conditions of Corollary 1.1, therefore a layout can be constructed by the algorithm in [5] in O(n log n) time, where n is the total number of terminals of the nets in N. The final layout for I^f is the layout constructed by the algorithm in [5] plus all previously introduced U wires. Since $h^f \leqslant 2h+3$ and $w^f \leqslant 2w+3$, $A(R^f)$ is approximately $4 \cdot A(R)$. This is the currently best approximation algorithm for the RRP problem. It is worth noting that the above transformation preserves the aspect ratio of R, i.e., R^f and R have identical aspect ratios.

The area bound of four of the above transformation method results from the indiscriminating rule of introducing new grid lines. Are there other rules for extending R so that R^f has smaller area? How about when the degree of nets is small? The second question is very important because in practice nets have degrees bounded by a small constant [8]. In this paper, we present a set of transformations *different* from the ones given in [5] that provide smaller approximation bounds for the unrestricted RRP and the three-terminal-net RRP problems. In Section 2 we show that if every net in a routable RRP contains no more than three terminals, then a layout can be constructed in a rectangle R^f such that asymptotically $A(R^f)/A(R) < 24/13$. This compares favorably with the bound of 1.5 obtained for this problem when all terminals are located on two opposite sides of the rectangle [3,6]. For the unrestricted RRP problem, we present in Section 3 an algorithm that generates a layout in a rectangle R^f such that asymptotically $A(R^f)/A(R) < 3.5$.

2. Three-terminal-net RRP approximation algorithm

In this section we present an approximation algorithm for the three-terminal-net RRP problem. Given a three-terminal-net RRP problem $I = (R, N)$, our algorithm stretches R into R^f so that the three-terminal-net RRP problem $I^f = (R^f, N^f)$ is routable, and after introducing a set of wire segments the resulting problem is a two-terminal-net RRP problem $I' = (R', N')$, which we know can be routed by the algorithm given in [5]. Let us derive rules for splitting three-terminal nets into two-terminal nets such that $A(R^f)$ is as small as possible. Remember that we assumed that w and h are large, and $w \leqslant h$.

We define a total ordering on terminal points as follows: we say that terminals $t' < t''$ iff $x(t') < x(t'')$, or $x(t') = x(t'')$ and $y(t') < y(t'')$. We define a net N_i with p terminals as a sequence $(t_{i,1}, t_{i,2}, \ldots, t_{i,p})$ such that $t_{i,j} < t_{i,j+1}$, $1 \leqslant j < p$. With respect to this ordering we say that $t_{i,2}$ is the *middle terminal* of the three-terminal net N_i. Note that with respect to the y ordering another terminal from net N_i might be the middle terminal. When we refer to the middle terminal we mean with respect to x. Let N^3 denote the set of three-terminal nets in N. The set N^3 is the same as set N in the problem instance given in Fig. 2.1. We partition N^3 into two subsets $\mathrm{TB} = \{N_i \mid N_i \in N^3$ and at least two of its terminals are located on the horizontal boundaries of the rectangle $R\}$ and $\mathrm{LR} = N^3 - \mathrm{TB}$. Let $\mathrm{tb} = |\mathrm{TB}|$ and $\mathrm{lr} = |\mathrm{LR}|$. Assume that tb and lr are even numbers. When tb (lr) is odd, an additional column (row) is required. For the problem instance given in Fig. 2.1, $\mathrm{TB} = \{N_1, N_2, N_5, N_7\}$,

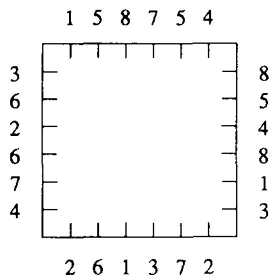

Fig. 2.1. Problem instance $I = (R, N)$.

$\mathrm{LR} = \{N_3, N_4, N_6, N_8\}$, tb = 4 and lr = 4. Our algorithm, ROUTE3, is given below.

Algorithm *ROUTE3*
(1) Let $l_1, l_2, \ldots, l_{\mathrm{tb}}$ be such that the middle terminals of the nets in TB appears in sorted order, i.e., $t_{l_1,2} < t_{l_2,2} < \cdots < t_{l_{\mathrm{tb}},2}$.
(2) Insert a row between rows $h - 1$ and h.
(3) Let net $N_{l_{2i-1}}$ and $N_{l_{2i}}$ form a pair p_i for $i = 1, 2, \ldots, \mathrm{tb}/2$.
(4) Transform each pair p_i of nets into two-terminal nets by following the rules given in the Appendix.
(5) Apply steps (1)–(4) to the nets in LR after rotating the rectangle 90 degrees.
(6) Add enough rows and columns so that the resulting two-terminal net problem $I^{\mathrm{f}} = (R^{\mathrm{f}}, N^{\mathrm{f}})$ is routable after deleting the row introduced by step (2) and the column introduced by step (5).
/* later on it will be evident why step (6) is required */
(7) Introduce the wire segment generated by the rules given in the Appendix (Case 3); project all terminal points one grid unit inside rectangle R^{f}; and let $I' = (R', N')$ be the resulting two-terminal-net routable RRP problem.
(8) Apply the routing algorithm given in [5] to the two-terminal-net routable RRP problem $I' = (R', N')$.
(9) Use the layout generated by the previous steps to construct a layout for N^{f} inside R^{f}.
end of *ROUTE3*

Algorithm ROUTE3 transforms the problem given in Fig. 2.1 into the one given in Fig. 2.2. The pairs formed by step (3) are $p_1 = (N_2, N_1)$ and $p_2 = (N_7, N_5)$. In step (5) the pairs formed by the algorithm are $p_1 = (N_8, N_4)$ and $p_2 = (N_6, N_3)$. Since the rules given in the Appendix do not introduce in this case fixed wire segments, our figures do not include the additional row and column introduced by steps (2) and (5). Note that no additional grid lines are introduced in step (6). Figure 2.3 shows a layout for I'. The layout was not constructed by the algorithm given in [5]. The reason is that for small problem instances a simple ad-hoc layout can be easily constructed. All of our figures were drawn this way. Figure 2.4 shows the final layout. Wire segments that emanate from a pseudo terminal and terminate at a solid dot (via) may be deleted.

Let us now consider the transformation applied to each pair of nets p_i in set TB. Let $X(p_i)$ be defined as the interval $(x(t_{l_{2i-1}}, 2), x(t_{l_{2i}}, 2))$. Note that by definition $X(p_i) \cap X(p_j) = \varnothing$ for $i \neq j$. Each time that we apply a transformation rule (Appendix) to a pair of nets p_i, d_c^{v} for each vertical cut $(c, c+1)$ in the interval $X(p_i)$ increases by at most one; d_c^{h} for each horizontal cut $(c, c+1)$ increases by at most one; and if Case 3 (in the Appendix) applies, additional fixed wires are introduced (on the top boundary). Also, the capacity of every horizontal cut is increased by one because we introduce a new column. Therefore, after step (3), d_c^{v} increases by at most one, d_c^{h} increases by at most tb/2, and tb/2 new columns are introduced. A similar situation arises for the nets in set LR. At most one row and one column are added in step (6). In summary I^{f} has the property that $d^{\mathrm{h}}(N^{\mathrm{f}}) \leqslant d^{\mathrm{h}}(N) + 1 + \mathrm{tb}/2$, $d^{\mathrm{v}}(N^{\mathrm{f}}) \leqslant d^{\mathrm{v}}(N) + 1 + \mathrm{lr}/2$, and R^{f} has at most $2 + \mathrm{tb}/2$ more columns and at most $2 + \mathrm{lr}/2$ more rows than R. As a result, I' is routable. Let us now analyze the performance of algorithm ROUTE3.

Theorem 2.1. *For any three-terminal-net* RRP *problem* $I = (R, N)$ *such that* $d^{\mathrm{v}}(N) < h + 1$ *and* $d^{\mathrm{h}}(N) < w + 1$, *algorithm* ROUTE3 *constructs a layout in* R^{f} *such that asymptotically* $A(R^{\mathrm{f}})/A(R) < 2$. *Furthermore, procedure* ROUTE3 *takes* $O(n \log n)$ *time, where n is the number of terminals.*

Proof. In our algorithm, each three-terminal net in N^3 is split into two two-terminal nets. For example, the RRP given in Fig. 2.1 is transformed into a new RRP shown in Fig. 2.2. It is easy to show that if $d^{\mathrm{v}}(N) < h + 1$ and $d^{\mathrm{h}}(N) < W + 1$, then $d^{\mathrm{v}}(N') < h' + 1$ and $d^{\mathrm{h}}(N') < w' + 1$, where h' and w' are the height and width of rectangle R', respectively. Therefore, a layout for I' can be constructed in R' by the two-terminal-net routing algorithm given in [5].

Let m be the number of nets in N and let n be the total number of terminals from the nets in N. Clearly the number of terminals $n < 2h + 2w$ and the number of nets in N^3 is $\mathrm{tb} + \mathrm{lr} < (2h + 2w)/3$. From the discussion preceding this theorem it is simple to verify that

$$A(R^{\mathrm{f}}) = (w + \mathrm{tb}/2 + 2)(h + \mathrm{lr}/2 + 2).$$

Since w and h are large, we may eliminate the constants. Therefore, $A(R^{\mathrm{f}}) = (w + \mathrm{tb}/2)(h + \mathrm{lr}/2)$. Substituting $\mathrm{lr} < (2h + 2w)/3 - \mathrm{tb}$, we know that

$$\begin{aligned} A(R^{\mathrm{f}}) &< (w + \mathrm{t}/2)(4h/3 + w/3 - \mathrm{tb}/2) \\ &= 4hw/3 + w^2/3 + (2h/3 - w/3)\mathrm{tb} - \mathrm{tb}^2/4. \end{aligned} \tag{a}$$

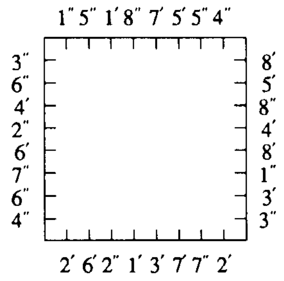

Fig. 2.2. Problem instance $I^{\mathrm{f}} = (R^{\mathrm{f}}, N^{\mathrm{f}})$ constructed from I (Fig. 2.1) by ROUTE3 by step (6). The wires introduced by step (7) are shown in the figure.

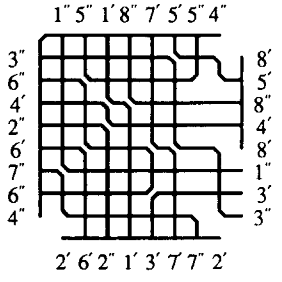

Fig. 2.3 Layout for I'.

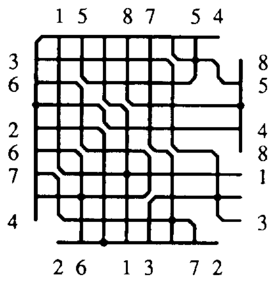

Fig. 2.4. Final layout for I^{f} (solid dots indicate wires belong to the same net).

The above equation achieves its maximum value when $\text{tb} = 4h/3 - 2w/3$ (this is obtained by deriving the function with respect to tb and equating it to zero, and observing that the sign of the second order term is negative). Since $\text{tb} \leqslant w - 1$, we know that $4h/3 - 2w/3 \leqslant w - 1$ iff $5w/4 - 3/4 \geqslant h$. Therefore, from the form of the objective function $A(R^{\text{f}})$ and the above bounds we know that if $5w/4 - 3/4 \geqslant h$ then the maximum occurs at $\text{tb} = 4h/3 - 2w/3$, otherwise (when $5w/4 - 3/4 \leqslant h$) the maximum occurs when $\text{tb} = w - 1$. Let us now establish our approximation bound by substituting in inequality (a) the value for tb that maximizes it. There are two cases.

Case 1: $5w/4 - 3/4 \geqslant h$.

Clearly, $4h/3 - 2w/3 \leqslant w - 1$. Therefore, the right hand side of inequality (a) achieves it maximum value when $\text{tb} = 4h/3 - 2w/3$. Substituting it in inequality (a), we know that

$$A(R^{\text{f}}) < (2w/3 + 2h/3)(2h/3 + 2w/3) = 8hw/9 + 4h^2/9 + 4w^2/9.$$

Since $A(R) = hw$, $A(R')/A(R) < 8/9 + 4h/(9w) + w/(9h)$. The maximum value of this function is when h/w is as large as possible. Therefore, replacing h by $5w/4 > 5w/4 - 3/4$ in the above inequality we know that

$$A(R^{\text{f}})/A(R) < 8/9 + (4/9)((5w/4)/w + w/(5w/4)) = 9/5.$$

Case 2: $5w/4 - 3/4 \leqslant h$.

Clearly, $4h/3 - 2w/3 \geqslant -1$. Therefore, the right-hand side of inequality (a) achieves its maximum value when $\text{tb} = w - 1$. Substituting it in inequality (a), we know that

$$\begin{aligned} A(R^{\text{f}}) &< (w + (w-1)/2)(4h/3 + w/3 - (w-1)/2) \\ &= (3w/2 - 1/2)(4h/3 - w/6 + 1/2) \\ &= 2hw - w^2/4 + 5w/6 - 2h/3 - 1/4. \end{aligned}$$

For $w \geqslant 4$, $A(R^{\text{f}}) < 2hw$. Since $A(R) = hw$, we know that asymptotically $A(R^{\text{f}})/A(R) < 2$.

Obviously, step (1) takes $O(n \log n)$ time. After the middle terminals are rearranged, steps (2), (3) and (4) takes $O(n)$ time. Steps (5) and (8) also take $O(n \log n)$ time. Steps (6), (7) and (9) can be easily implemented in $O(n)$ time. Hence, the total time required is $O(n \log n)$. It is worth noting that the output generated by our algorithm has at most two wire segments for each net. □

From the analysis of algorithm ROUTE3, we know that the worst case area may be achieved only when $h \geqslant 5w/4 - 3/4$. We also notice that the additional grid lines introduced by algorithm ROUTE3 can be either horizontal or vertical. Clearly, when $h \geqslant 5w/4 - 3/4$, transforming a three-terminal-net RRP into a two-terminal-net RRP by exclusively introducing additional rows may possibly result in smaller routing area. For this reason, we present the following algorithm for the three-terminal-net RRP problem.

Algorithm *ROUTE3_ALT*

(1) If $h \leqslant 13w/8$, then apply procedure ROUTE3 and stop;
/ * This bound for h was selected because it is a break even point for the asymptotic approximation bounds of ROUTE3 and steps (2)–(6). We are not claiming that for each problem instance if $h \leqslant 13w/8$ then steps (2)–(6) generates a worse solution; or conversely, if $h > 13w/8$ then ROUTE3 generates a worse solution. */.

(2) Let R^1 be a copy of R and let N^1 be N. Let α_t (α_b) be the number of nets with least two terminals located on the top (bottom) side of R. We introduce α_t (α_b) rows between the topmost (bottommost) row and the (bottom) side of R^1. The topmost (bottommost) rows are used to route the nets with two or more terminals located on the top (bottom) side of R^1. The layout for these nets is constructed by the algorithm given in [4]. For each net with exactly one terminal on the top (bottom) side of R^1, we project this terminal to the topmost (bottommost) empty row and for each net with exactly two terminals on the top (bottom) side of R^1, we project one of these two terminals to the topmost (bottommost) empty row.

(3) Let R^2 be the empty portion (without wires) of R^1. At this point there are two- and three-terminal nets. All the middle terminals (with respect to y) of the three-terminal nets are located on the left or right side of R^2. This routing problem is referred to as $I^2 = (R^2, N^2)$. The remaining three-terminal nets are split into two-terminal nets and rows are introduced using the transformation rules given in step (5) of algorithm ROUTE3. If the rules in the Appendix introduce fixed wire segments, add a column between columns 0 and 1. When this additional column is introduced, project each terminal point located on the left side of the rectangle one unit towards the inside of the rectangle. Add enough columns so that the resulting problem, which we call $I^3 = (R^3, N^3)$, is routable.

(4) Let $I' = (R', N')$ be the resulting problem;

(5) Construct a layout for the two-terminal-net RRP problem I' using the algorithm given in [5];

(6) Construct from the layout for N' in R' and the partial layouts constructed in previous steps the final layout. Let R^f be the smallest rectangle enclosing the final layout.

end of *ROUTE3_ALT*

Let us now apply steps (2)–(6) to the problem instance given in Fig. 2.1. Note that such instance does not satisfy the condition $h > 13w/8$. Step (2) introduces the wire segments shown in Fig. 2.5 and the new problem instance is given in Fig. 2.6. The resulting problem after step (3) of procedure ROUTE3_ALT is given in Fig. 2.7. Note that the rules in the Appendix do not introduce fixed wire segments for this example.

Figure 2.8 shows a layout for problem I' constructed by an ad-hoc method rather than by the algorithm given in [5] for step (4) of procedure ROUTE3_ALT. Figure 2.9 shows the final layout.

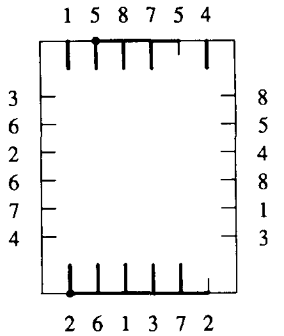

Fig. 2.5. Partial layout constructed by step (2).

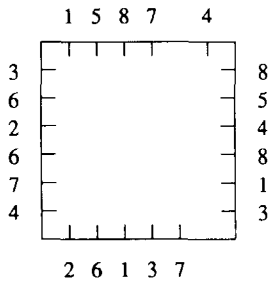

Fig. 2.6. Problem $I^2 = (R^2, N^2)$.

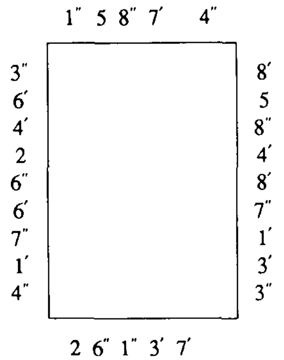

Fig. 2.7. Problem I^3 after introducing wires.

If $h \leqslant 5w/4 - 3/4$, then since $5w/4 - 3/4 < 13w/8$ algorithm ROUTE3_ALT uses algorithm ROUTE3 to construct a layout in R^{f} extended from R. Therefore, by the approximation bound given in case 1 of the proof of Theorem 2.1, $A(R^{\mathrm{f}})/A(R) < 8/9 + (4/9)(h/w + w/h)$. Since $h \leqslant 5w/4 - 3/4$, we know that $A(R^{\mathrm{f}})/A(R) < 9/5 = 1.8$.

When $5w/4 - 3/4 \leqslant h \leqslant 13w/8$ algorithm ROUTE3_ALT uses algorithm ROUTE3 to construct a layout in rectangle R^{f} extended from R. Therefore, by the approximation bound given in case 2 of the proof of Theorem 2.1, $A(R^{\mathrm{f}})/A(R) < 2 - w/(4h) + 5/(6h) - 2/(3w) - 1/(4hw)$. Since $h \geqslant 5w/4 - 3/4$, then $5/(6h) - 2/(3w) - 1/(4hw) \leqslant 0$. Therefore, $A(R^{\mathrm{f}})/A(R) < 2 - w/(4h)$. Substituting $h \leqslant 13w/8$, we know that $A(R^{\mathrm{f}})/A(R) < 24/13 < 1.85$.

When $h > 13w/8$, algorithm ROUTE3_ALT constructs a layout by performing steps (2)–(6). In this case at most two columns are introduced and the number of rows introduced is $\alpha + \beta$, where $\alpha = (\alpha_{\mathrm{t}} + \alpha_{\mathrm{b}})$ and β is the number of middle terminals (with respect to y) located on the left and right sides of R

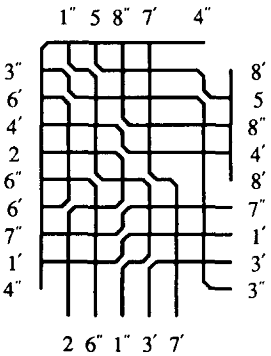

Fig. 2.8. Layout constructed for I' by step (4).

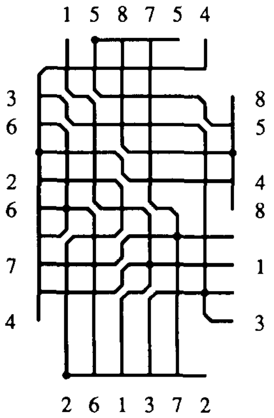

Fig. 2.9. Final layout.

(after constructing the partial layout) divided by two. It is simple to verify that for each of the α rows introduced in step (2) at least one three-terminal net is transformed into two-terminal nets and for each of the $\beta = \mathrm{lr}/2$ rows introduced in step (3) two three-terminal nets are transformed into two-terminal nets (assuming that lr is even). Therefore, $\alpha = 2\beta \leqslant 2h/3 + 2w/3$. Let us now establish a bound for $A(R^{\mathrm{f}})/A(R)$ when h and w are large. After eliminating constants, $A(R^{\mathrm{f}}) \leqslant (h + \alpha + \beta)(w)$. Substituting $\beta \leqslant h/3 + w/3 - \alpha/2$ in the above equation we know that $A(R^{\mathrm{f}}) \leqslant 4hw/3 + \alpha w/2 + w^2/3$. Since $A(R) = hw$, $A(R^{\mathrm{f}})/A(R) \leqslant 4/3 + \alpha/(2h) + w/(3h)$. Since $\alpha < w$, $A(R^{\mathrm{f}})/A(R) < 4/3 + 5w/(6h)$. By the condition $h > 13w/8$, we know that $A(R^{\mathrm{f}})/A(R) < 24/13 < 1.85$ also holds. This analysis is summarized in the following theorem.

Theorem 2.2. *For any three-terminal-net* RRP *problem $I = (R, N)$ such that $d^{\mathrm{v}}(N) < h + 1$ and $d^{\mathrm{h}}(N) < w + 1$, algorithm* ROUTE3_ALT *constructs a layout in R^{f} such that asymptotically $A(R^{\mathrm{f}})/A(R) < 24/13 < 1.85$ in* O(n log n *time, where n is the number of terminals.*

Proof. The proof follows from the above discussion. □

3. Approximations for the RRP problem of arbitrary degree

In this section we present an approximation algorithm for the unrestricted RRP problem. As in the previous section, the idea behind the algorithm is to transform the given RRP problem $I = (R, N)$ into a routable two-terminal-net RRP problem $I'' = (R'', N'')$, such that after introducing some additional wire segments in any layout for I'', we obtain a layout for N^{f} inside R^{f}, where R^{f} is a stretched version of R. Let us derive rules for splitting nets into two-terminal nets such that $A(R^{\mathrm{f}})$ is as small as possible. Remember that we assumed that h and w are large, and $w \leqslant h$.

We call net N_i a *k-side net* if its terminals are located on exactly k sides of R. Let $s = s_1 s_2 \cdots s_k$ be a string such that $1 \leqslant k \leqslant 4$. String s consists of no more than two h's followed by no more than two v's. If s contains two h's (v's) then the first one is labeled top (left) and the second one is labeled bottom (right); on the other hand, if s contains one h (v), it is labeled top and bottom (left and right). We partition net set N depending on the sides where the terminals are located into the sets O^{h}, O^{v}, O^{hv}, O^{hh}, O^{vv}, O^{hhv}, O^{hvv}, and O^{hhvv}. These sets are defined in Table 1. As an example, for the problem instance given in Fig. 1.1, $O^{\mathrm{h}} = \{N_1, N_2\}$, $O^{\mathrm{v}} = \varnothing$, $O^{\mathrm{hv}} = \{N_3, N_4\}$, $O^{\mathrm{vv}} = \{N_5\}$, $O^{\mathrm{hh}} = \{N_6\}$, $O^{\mathrm{hhv}} = \{N_7\}$, $O^{\mathrm{hvv}} = \{N_8\}$, and $O^{\mathrm{hhvv}} = \{N_9\}$.

Let $s = s_1 s_2 \cdots s_k$ and $t = t_1 t_2 \cdots t_k$ be two strings such that $1 \leqslant k \leqslant 4$, and $t_i \in \{1, 2, +\}$ for $1 \leqslant i \leqslant k$. Net $N_j \in O^s$ is said to belong to set O_t^s if for all $1 \leqslant i \leqslant k$ there is a label $x \in$ {top, bottom, left, right} associated with s and net N_j has on side x exactly one terminal if $t_i = 1$, exactly two terminals if $t_i = 2$, and at least one terminal if $t_i = +$. For example the set O_{12+}^{hhv} consists of all

Table 1
Definition of O sets

Set	Nets in the set
O^{h}	Set of 1-side nets with all terminals located at the top or bottom side of R
O^{v}	Set of 1-side nets with all terminals located at the left or right side of R
O^{hv}	Set of 2-side nets with all terminals located on adjacent sides of R
O^{vv}	Set of 2-side nets with all terminals located at the left and right side of R
O^{hh}	Set of 2-side nets with all terminals located at the top and bottom side of R
O^{hhv}	Set of 3-side nets with terminals located at the top and bottom side of R
O^{hvv}	Set of 3-side nets with terminals located at the left and right side of R
O^{hhvv}	Set of 4-side nets

3-side nets with exactly one terminal located on the top side of R, exactly two terminals located on the bottom side of R and at least one terminal located on either the left or right side of R; and the set O_{21+}^{hhv} consists of all 3-side nets with exactly two terminals located on the top side of R, exactly one terminal located on the bottom side of R and at least one terminal located on the either left or right side of R. Note that any two distinct O_t^s s are mutually disjoint. Similarly, we define sets F_t^s and F^s (S_t^s and S^s) for the net set N' (N''), which will be defined shortly.

We refer to the leftmost (rightmost) terminal of net N_i located on the top side of R as the *left representative (right representative)* of N_i on the top side of R. Similarly, we define left representative and right representative for the terminals on the bottom side of R. A net with at least two terminals located on the top (bottom) side of R will be referred to as t (b) net. Let d_t (d_b) be the vertical density of all t (b) nets when considering only their terminals located on the top (bottom) side of R.

We apply two transformations that introduce new rows and a set of wires on these new rows to transform the given multiterminal-net RRP problem into a routable two-terminal-net RRP problem. Let us consider the first transformation. Let R^0 be a copy of R and let each terminal point in R^0 be in exactly the same position as in R. Between rows h (0) and $h-1$ (1) in R^0 add d_t (d_b) rows. The bottommost (topmost) of these rows will be called the *top (bottom) free row*. In case, d_t (d_b) is zero the top (bottom) boundary is called the top (bottom) free row. The first transformation takes all the t (b) nets and connects all their terminal points located on the top (bottom) side of R^0 by a set of wires which are routed on the topmost (bottommost) d_t (d_b) rows in R^0 by the algorithm given in [4]. We select either the left or right representative from each net with at least one terminal point located on the top (bottom) side of R^0 and project it to the top (bottom) free row of R^0 (see Fig. 3.1). The rectangle, which we call R', is defined by the left and right boundary together with the top and bottom free row of R^0. We shall refer to the top (bottom) free row as row h (0) of R' (see Fig. 3.2). By these operations, the original is transformed to one in which each net has at most one terminal located on the top side of R' and at most one

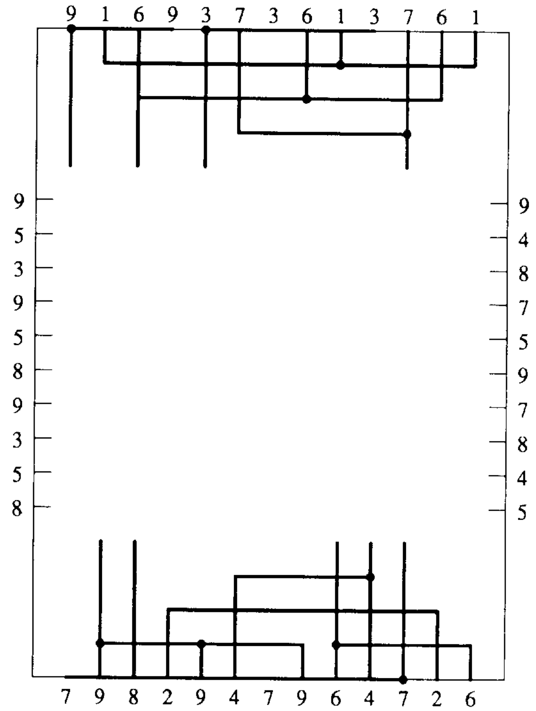

Fig. 3.1. Partial layout constructed by the algorithm.

terminal located on the bottom side of R'. The new set of nets is referred to as the set N'. Let us now explain this process in detail.

Algorithm *FIRST_TRANS*

Let R^0 be a copy of R and let each terminal point in R^0 be in exactly the same position in R. Introduce d_t (d_b) rows between row h (0) and row $h-1$ (1) in R^0;

The terminals located on the top (bottom) side of R^0 from all the t (b) nets are connected by wires which are routed in the topmost (bottommost) d_t (d_b) rows in R^0. This partial layout is constructed by the algorithm given in [4]. The rectangle, which we call R', is defined by the left and right boundary together with the top and bottom free rows of R^0. We shall refer to these rows as row h and row 0 of R'.

For each net $N_i \in N$ with at least one terminal located on the top (bottom) side of R^0 perform the following projection operation:

(i) If the net does not have terminals located on the left and right side of R^0 and N_i is a 2-side net, project the left representative of N_i to the top (bottom) side of R'.

(ii) If the topmost (bottommost) terminal located on the left or right side of R^0 is located on the left side of R^0, project the left representative of N_i to the top (bottom) side of R' and skip (iii).

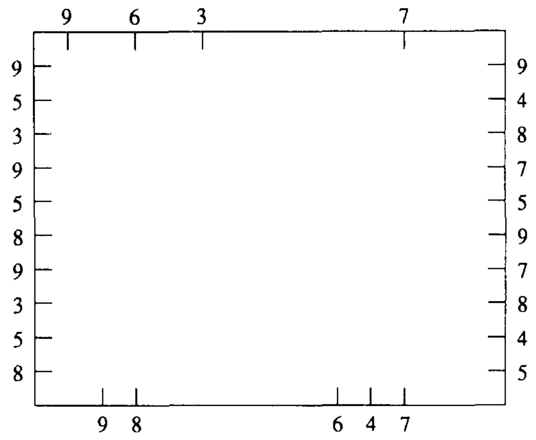

Fig. 3.2. Subproblem $I' = (R', N')$.

(iii) If the topmost (bottommost) terminal located on the left or right side of R^0 is located on the right side of R^0, project the right representative of N_i to the top (bottom) side of R'.

After these projection operations, we transform each net N_i into another net N_i' which is identical to the original one if N_i does not have two or more terminals on the top or bottom sides of R^0. On the other hand, if N_i has at least two terminals located on the top or bottom sides of R^0, then N_i' is N_i without all the terminal points located on the top and bottom side of R^0 except the projected one.

/* see Fig. 3.2 */

Let N' be the set of all nets N_i'. The set N' is partitioned into subsets F_t^s.

end *FIRST_TRANS*

Figure 3.1 shows the partial layout constructed by the procedure, and Fig. 3.2 shows the resulting subproblem $I' = (R', N')$.

For the problem given in Fig. 1.1, procedure FIRST_TRANS generates the following sets of nets: $F_{1+}^{\mathrm{hv}} = \{N_3', N_4'\}$, $F_{++}^{\mathrm{vv}} = \{N_5'\}$, $F_{11}^{\mathrm{hh}} = \{N_6'\}$, $F_{11+}^{\mathrm{hhv}} = \{N_7'\}$, $F_{1++}^{\mathrm{hvv}} = \{N_8'\}$, and $F_{11++}^{\mathrm{hhvv}} = \{N_9'\}$. Note that for this example, procedure FIRST_TRANS generates empty N' nets for N_1 and N_2 simply because all their terminals are located on either the top or bottom side of rectangle R. The effects of FIRST_TRANS are given in the following table, which explicitly shows that each net in set O_t^s has been transformed into a net which belongs to set $F_{t'}^{s'}$. The transformation applied by FIRST_TRANS has the following properties that we state without a proof. The vertical (horizontal) density at each vertical (horizontal) cut for N' is never larger than its corresponding density in N, i.e., $d^{\mathrm{v}}(N) \geqslant d^{\mathrm{v}}(N')$ and $d^{\mathrm{h}}(N) \geqslant d^{\mathrm{h}}(N')$.

procedure *SECOND_TRANS*

Let R^1 be R' with all the terminal points from the nets N_i';

Add a column between columns 0 ($w-1$) and 1 (w);
for $j = 1$ **to** $h-1$ **do**
  if there is a terminal at any of the boundary grid points in row j of R^1 **then**
    begin
    Insert a new row between row j and row $j-1$ of R^1;
    if the left boundary point in row j of R^1 is a terminal t from net N_i'
      then make a copy of t at the left boundary point of the newly introduced row;
    if the right boundary point in row j of R^1 is a terminal t from net N_i'
      then make a copy of t at the left boundary point of the newly introduced row;
    end
endfor
Add U wires to connect adjacent terminal from the same net and project each terminal one unit to the inside of the rectangle;
Let R'' be the rectangle after deleting the left and right boundary of R^1 and let R'' contain all the terminal points in R^1;
/* Label all terminals and copies of terminals on R^1 as follows */
for each N_i' with terminals on the left or right side of R^1 **do**
  /* Each net N_i' is split into several two terminal nets. The kth of such nets is identified by the label i^k */
  Let u be a terminal of N_i' on the left or right side of R^1 with the smallest y-coordinate value;
  if N_i' does not have a terminal on the bottom side of R^1
    then assign label i^1 to u; $j \leftarrow 1$;
    else
      begin
      Assign label i^1 to the copy of u and the terminal in N_i' located on the bottom side of R^1;
      if there is another terminal v of N_i' such that $y(v) \geqslant y(u)$
        then assign label i^2 to u **else** $j \leftarrow 2$;
      end
  while there is an unlabeled terminal of N_i' located on the left or right side of R'' **do**
    $j \leftarrow j+1$;
    Let u be the unlabeled terminal of N_i' with the smallest y-coordinate value;
    Assign label i^j to u and the copy of u, respectively;
  endwhile
  if N_i' has a terminal located on the top side of R^1
    then assign label i^j to the terminal of N_i' located on the top side of R^1;
    else delete the last label i^j;
endfor
end of procedure *SECOND_TRANS*

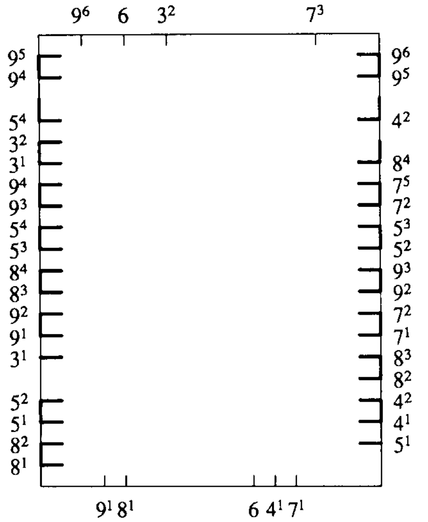

Fig. 3.3. Effects of procedure SECOND_TRANS on I'.

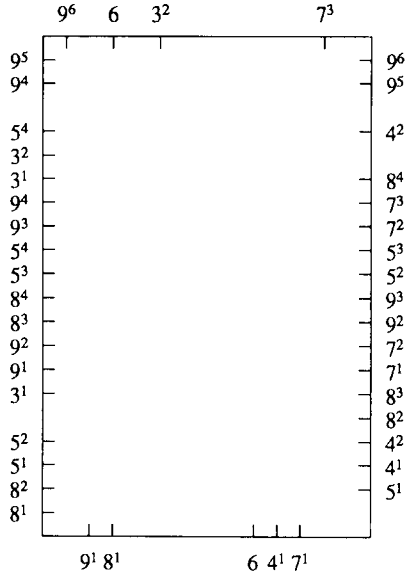

Fig. 3.4. Effects of procedure SECOND_TRANS on I'.

Figure 3.3 and 3.4 show the resulting subproblem after applying procedure SECOND_TRANS to the problem instance I' shown in Fig. 3.2.. The terminal points labeled 3^2, 4^1, 7^1, 7^3, 8^1, 9^1 and 9^6 belong to S_{11}^{hv}; the one labeled 5^1, 5^2, 5^3, 8^2, 8^3, 8^4, 9^2, 9^3, and 9^5 belong to S_{11}^{vv}; the ones labeled 3^1, 4^2, 5^4, 7^2, and 9^4 belong to S_2^{v}; and terminals labeled 6 belong to a net in S_{11}^{hh}.

Each net N_i' may be split into several two-terminal nets by procedure SECOND_TRANS. The kth of such nets is defined by the label i^k. We use R'' to denote the rectangle extended from R^1 by SECOND_TRANS, and use N'' to denote all two-terminal nets defined on the boundary of R''. At this point, N'' is partitioned into three subsets: S_{11}^{hv}, S_{11}^{vv} and S_{11}^{hh}. Referring to Table 2, each net N_i' in set F_i^s is transformed into one or more two-terminal nets belonging to

Table 2
Sets of O, F and S before and after each transformation

original net	after first transformation	after second transformation
O^{h}	$\varnothing$	$\varnothing$
O^{v}	$\varnothing$	$\varnothing$
O^{hv}	F_{1+}^{hv}	$S_{11}^{\mathrm{hv}}, S_2^{\mathrm{v}}$
O^{vv}	F_{++}^{vv}	$S_2^{\mathrm{v}}, S_{11}^{\mathrm{vv}}$
O^{hh}	F_{11}^{hh}	S_{11}^{hh}
O^{hhv}	F_{11+}^{hhv}	$S_{11}^{\mathrm{hv}}, S_2^{\mathrm{v}}$
O^{hhv}	F_{1++}^{hhv}	$S_{11}^{\mathrm{hv}}, S_2^{\mathrm{v}}, S_{11}^{\mathrm{vv}}$
O^{hhvv}	F_{11++}^{hhvv}	$S_{11}^{\mathrm{hv}}, S_2^{\mathrm{v}}, S_{11}^{\mathrm{vv}}$

the S net sets shown in the next column of the table. For example, a net in set F^{hhv}_{11+} is transformed into one net in S^{hv}_{11} and several nets in S^{v}_{2}. A net in F^{hvv}_{1++} is transformed into one net in S^{hv}_{11} and one or more nets in $S^{v}_{2} \cup S^{vv}_{11}$. The transformation applied by SECOND_TRANS has the following property that we state without a proof. With respect to the horizontal density, $d^{h}(N'') \leqslant d^{h}(N')$. Note that this does not explicitly take into account the U wires. Since all of these wires connect adjacent terminals located on the left and right side of R'', all these connections can be carried out on two additional columns.

Lemma 3.1. After procedure SECOND_TRANS is executed, $|S^{vv}_{11}| \leqslant 2(h-1) - |N-(O^{h} \cap O^{hh})|$.

Proof. Let v_i be the number of terminals of net N_i located on the left and right sides of R. For each $N_i \in N-(O^{h} \cup O^{hh})$ at most $v_i - 1$ two-terminal nets in S^{vv}_{11} can be generated by SECOND_TRANS. Therefore, $|S^{vv}_{11}| \leqslant \Sigma_{N_i \in N-(O^{h} \cup O^{hh})}(v_i - 1)$. Since $\Sigma_{N_i \in N} v_i \leqslant 2(h-1)$, we know that $|S^{vv}_{11}| \leqslant 2(h-1) - |N-(O^{h} \cup O^{hh})|$. This completes the proof of the lemma. □

Our algorithm for the unrestricted RRP problem is given below.

Algorithm *ROUTE_MULTINET*
  Apply procedure FIRST_TRANS to obtain a routing instance defined in R' for net set N';
  Apply procedure SECOND_TRANS to obtain a routing instance defined in R'' for net set N'';
  Add enough rows and columns so that I'' is a routable two-terminal-net RRP problem instance;
  Use the algorithm in [5] to route I'';
  Construct a layout for N^{f} in R^{f} from the layout in N'' in R'' and the partial layouts constructed in previous steps.
end *ROUTE_MULTINET*

Figure 3.5 shows the layout constructed for (R'', N'' by an ad-hoc method rather than by the algorithm given in [5]. Figure 3.6 shows the final layout.

Theorem 3.1. *For any multiterminal* RRP *defined in a rectangle R such that* $h \geqslant w$ *and* $d^{h} < w+1$, *algorithm* ROUT_MULTINET *constructs a layout in a rectangle* R^{f} *extended from R such that asymptotically* $A(R^{f})/A(R) < 2 + (3/2)/r(R) \leqslant 3.5$, *where r(R) is the aspect ratio of R. Furthermore, such a layout can be constructed in* $O(n \log n)$ *time.*

Proof. To estimate the maximum total number of additional rows introduced by FIRST_TRANS and SECOND_TRANS, we partition S^{hv}_{11} into two subsets S^{hv}_{11}[top] and S^{hv}_{11}[bottom], where S^{hv}_{11}[top] (S^{hv}_{11}[bottom]) contains all two-terminal nets in S^{hv}_{11} with a terminal located on the top (bottom) side of R'', and insert to zero or more rows into R'' until the total number of rows in R'' is exactly

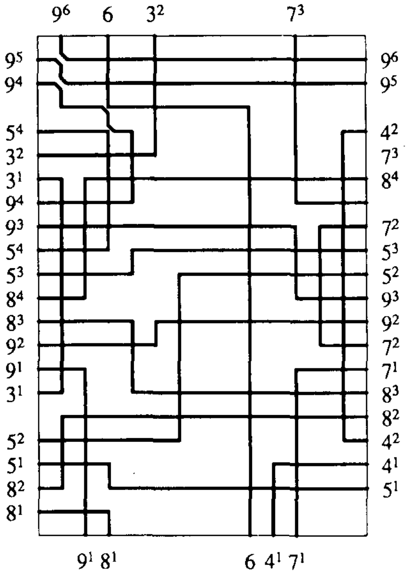

Fig. 3.5. Layout for (R'', N'').

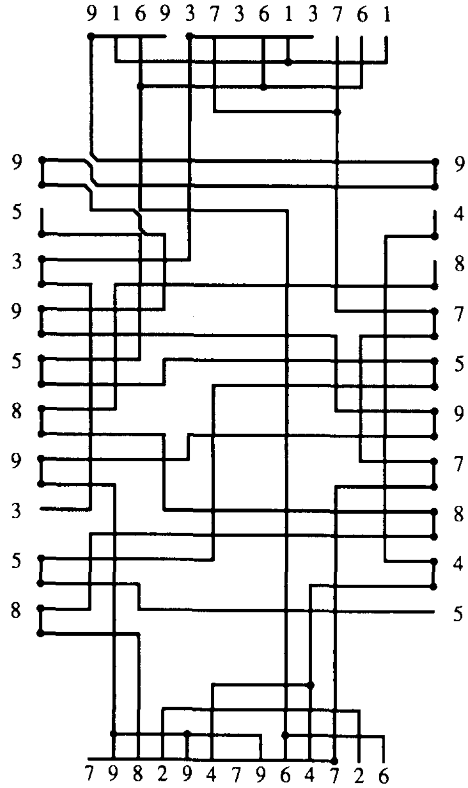

Fig. 3.6. Final layout.

$2(h-1)$. Since for each $N_i \in N-(O^{\mathrm{h}} \cup O^{\mathrm{hh}})$ procedure SECOND_TRANS introduces at most one net to $S_{11}^{\mathrm{hv}}[\mathrm{top}]$ and at most one net to $S_{11}^{\mathrm{hv}}[\mathrm{bottom}]$, it then follows that $|S_{11}^{\mathrm{hv}}[\mathrm{top}]| \leqslant |N-(O^{\mathrm{h}} \cup O^{\mathrm{hh}})|$ and $|S_{11}^{\mathrm{hv}}[\mathrm{bottom}]| \leqslant |N-(O^{\mathrm{h}} \cup O^{\mathrm{hh}})|$. Combining these two inequalities with Lemma 3.1, we know that $|S_{11}^{\mathrm{hv}}[\mathrm{top}] \cup S_{11}^{\mathrm{vv}}| \leqslant 2(h-1)$ and $|S_{11}^{\mathrm{hv}}[\mathrm{bottom}]| \cup S_{11}^{\mathrm{vv}}| \leqslant 2(h-1)$.

The height h^{f} of the rectangle R^{f} with smallest area that includes the final layout satisfies

$$\begin{aligned} h^f \leqslant\ & d_t + d_b \\ & + \max\{2(h-1), |S_{11}^{hv}[top]| + |S_{11}^{hv}[bottom]| + |S_{11}^{vv}| + |S_{11}^{hh}|\}. \end{aligned}$$

Substituting the previous bound, we know that $h^{\mathrm{f}} \leqslant 2(h-1) + d_{\mathrm{t}} + d_{\mathrm{b}} + \min\{|S_{11}^{\mathrm{hv}}[\mathrm{top}]|, |S_{11}^{\mathrm{hv}}[\mathrm{bottom}]\} + |S_{11}^{\mathrm{hh}}|$. Since every net in set $S_{11}^{\mathrm{hv}}[\mathrm{top}] \cup S_{11}^{\mathrm{hv}}[\mathrm{bottom}] \cup S_{11}^{\mathrm{hh}}$, has a terminal point on the top and/or bottom side of the rectangle and for each of the d_{t} (d_{b}) tracks there is a boundary point on the top (bottom) of R'' without a terminal point, we know that $\min\{|S_{11}^{\mathrm{hv}}[\mathrm{top}]|, |S_{11}^{\mathrm{hv}}[\mathrm{bottom}]|\} + |S_{11}^{\mathrm{hh}}| \leqslant w - \max\{d_{\mathrm{t}}, d_{\mathrm{b}}\}$. Since $d_{\mathrm{t}} \leqslant (w-1)/2$ and $d_{\mathrm{b}} \leqslant (w-1)/2$, we know that $h^{\mathrm{f}} \leqslant 2h + (3/2)w - 3$. The total number of additional columns introduced is no more than four. This is because the transformation performed by

FIRST_TRANS and SECOND_TRANS increases the horizontal density by a most two, and two columns are used for the U wires and the wires introduced by the transformation rules. Compare the area of R and R^{f} and ignoring the constants, $A(R^{\mathrm{f}})/A(R) \leqslant (2hw + (3/2)w^2)/(hw) = 2 + (3/2)/r(R)$. Both FIRST_TRANS and SECOND_TRANS can be easily implemented to take $\mathrm{O}(n \log n)$ time. Routing the two-terminal-net instance can be carried out by the algorithm given in [5]. Therefore, algorithm ROUT_MULTINET requires only $\mathrm{O}(n \log n)$ time. □

Careful readers may notice that even if we are given an RRP instance $I = (R,N)$ which is not routable, i.e. $d^{\mathrm{v}}(N) > h + 1$, algorithm ROUT_MULTINET can still guarantee a layout solution in R^{f} such that asymptotically $A(R^{\mathrm{f}})/A(R) < 2 + (3/2)/r(R)$.

4. Concluding remarks

We presented approximation algorithms for the unrestricted RRP and the three-terminal-net RRP problems. For the unstricted RRP, our algorithm ROUT_MULTINET guarantees that a layout can be constructed in a rectangle R^{f} extended from R such that asymptotically $A(R^{\mathrm{f}})/A(R) < 3.5$, which is smaller than the previous approximation bound of four [5]. However, the price paid for this is an increase of the aspect ratio. For the layout constructed by the algorithm of [5], $r(R^{\mathrm{f}}) = r(R)$. The aspect ratio of the layout constructed by our algorithms is different since we only add a constant number of columns, rather than doubling number of rows and columns. For the RRP of degree three, our algorithms ROUTE3 guarantees that a layout can be constructed in a rectangle R^{f} extended from R such that asymptotically $A(R^{\mathrm{f}})/A(R) < 2$. Algorithm ROUTE3_ALT guarantees that asymptotically $A(R^{\mathrm{f}})/A(R) < 24/13$. In terms of layout area, algorithm ROUTE3_ALT is asymptotically superior. There are classes of problem instances for which our algorithms ROUT_ALT and ROUT_MULTINET generate solutions with near optimal area. One of such classes for ROUT_MULTINET is the class of problems with large aspect ratio. In practice w and h are large. Because of these properties and to simplify our analyses, we only derived asymptotic bounds for the approximation constant, i.e., our bounds hold when w is larger than some fixed constant. For all h and w, the exact approximation bound for our algorithm is equal to the asymptotic one plus $\mathrm{O}(h + w)/(hw))$. It is important to note that the exact approximation bounds for our algorithms differ from the asymptotic ones, because there are several (at most five) additional rows and columns that are not accounted for in the analyses.

Our area bounds are not small. This is mainly because the combinatorial properties of RRP are still not well understood. This is reflected in the lower bound for the layout area we used and the approach of splitting multiterminal nets into two-terminal nets that we adopted. One way to improve our results is to develop better lower bounds for the area of an optimal solution. This does not seem to be a simple problem.

In practical situations one should run the algorithm two times, the second time after rotating the rectangle 90 degrees. The final solution is the best of the two solutions. When h and w have similar values this strategy will pay off. In an RRP problem with the majority of the nets having degree no more than three, the nets with no more than three terminals and the nets of larger degrees can be treated separately. By splitting three-terminal nets as in algorithm ROUTE3 or ROUTE3_ALT, and splitting other multiterminal nets by some other method, the area of the layout can be expected to be small.

One of the main problems with the layouts for the RRP problem generated by our algorithms and the algorithm given in [5] is that the wires connecting the nets may be long. Our algorithm will suffer from this problem even if we use Frank's [2] algorithm instead of the one in [5]. At this time there is no way around this problem. The problem of minimizing wire length and area seems very interesting and deserves careful study.

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Appendix. Transformation rules

In this appendix we present the rules for splitting a pair p_i of three-terminal nets into four two-terminal nets. The two three-terminal nets will be referred to as net A and net B. Assume that the middle terminals (with respect to x) of A and B are located on the top or bottom side of rectangle R. The terminals in net A (B) are labeled and ordered as follows: $a_{\mathrm{b}} < a_{\mathrm{m}} < a_{\mathrm{e}}$ ($b_{\mathrm{b}} < b_{\mathrm{m}} < b_{\mathrm{e}}$). The ordering is with respect to their x coordinate values (see Section 2). Exactly two new terminal points, which are indicated by a dashed line in our figures, are

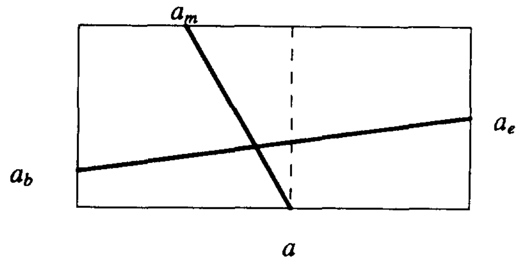

Fig. A.1. Transformation T_1.

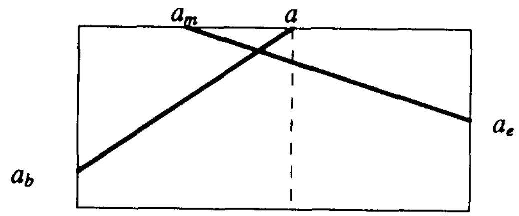

Fig. A.2. Transformation T_2.

introduced to split nets A and B. These two terminal points are labeled a and b. Each of the two two-terminal nets generated from net A (B) is defined by two terminals in $\{a_b, a_m, a_e, a\}$ ($\{b_b, b_m, b_e, b\}$) joined by a thick line. The number of columns after the transformation increases by one. Without loss of generality, we assume that $a_m \leqslant b_m$. The connectivity of two new two-terminal nets representing an original three-terminal net is enforced as follows. The terminals of these two nets may be connected by a fixed wire, which is represented by a zig-zag solid line in the figure. If the zig-zag line is not present, the two two-terminal nets generated from a three-terminal net have the property that wires connecting the two new nets in any layout always intersect. At the point they intersect the wires will be made electrically common by introducing a via. As we defined before, the x and y coordinate values for terminal t are

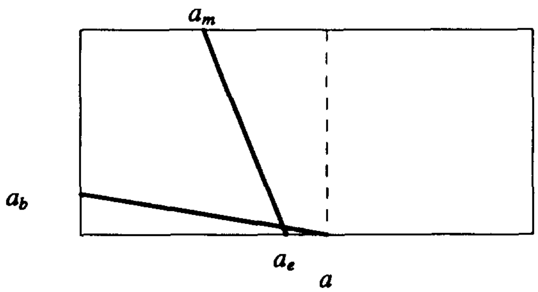

Fig. A.3. Transformation T_3.

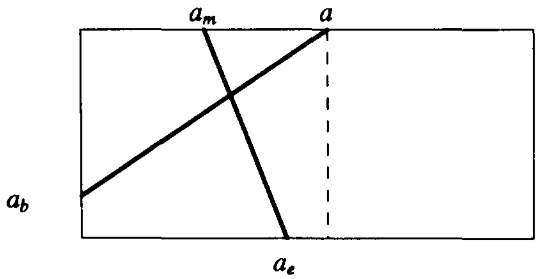

Fig. A.4. Transformation T_4.

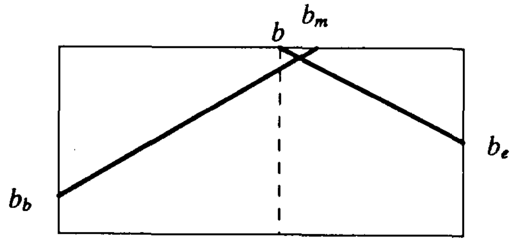

Fig. A.5. Transformation T_5.

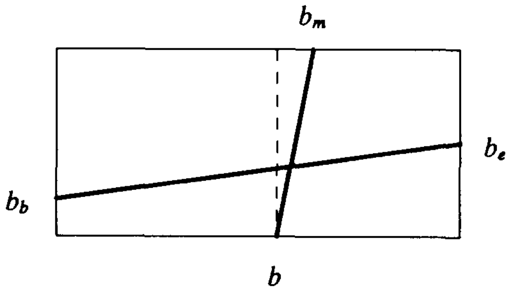

Fig. A.6. Transformation T_6.

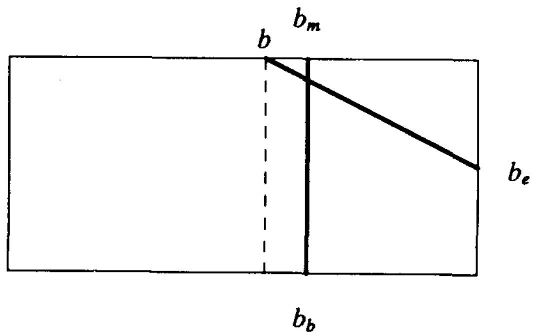

Fig. A.7. Transformation T_7.

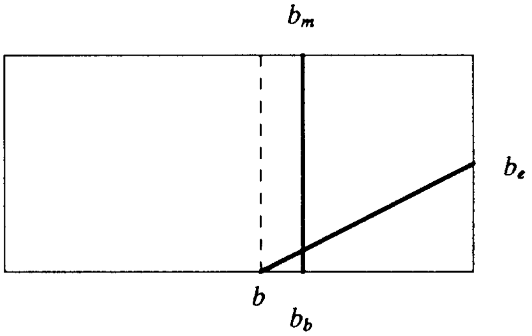

Fig. A.8. Transformation T_8.

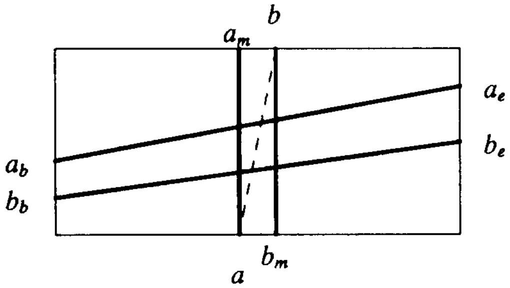

Fig. A.9. Transformation T_9.

referred to by $x(t)$ and $y(t)$. We use $[x(t), x(t')]$ ($[y(t), y(t')]$) to represent the set $\{z \mid y(t) \leqslant z \leqslant y(t')\}$ ($\{z \mid x(t) \leqslant z \leqslant x(t')\}$).

There are three cases that need to be considered. (Transformations T_1–T_{10} see Figs. A.1–A.10.)

Case 1. a_m and b_m are located on the same side of R.
Assume without loss of generality that a_m and b_m are located on the top side of R. If $x(b_m) \leqslant x(a_e)$ then let $i = 1$, otherwise let $i = 3$; and if $x(b_b) < x(b_m)$ then

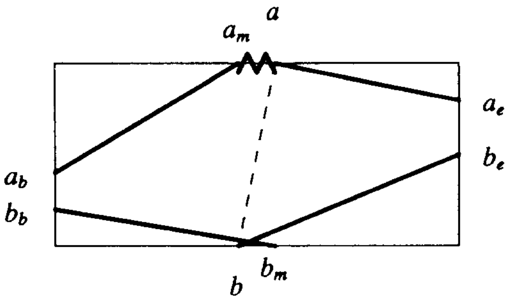

Fig. A.10. Transformation T_{10}.

let $j = 5$, otherwise let $j = 7$. Our procedure applies the transformation given in T_i to net A and the one in T_j to net B. If the horizontal density of the four new nets is 4, then the transformation applied to net A is T_{i+1} and the one for B is T_{j+1}. One can easily show that the vertical (horizontal) density of the new four nets at any horizontal (vertical) cut differs from the vertical (horizontal) density of the original two nets by a most one. Furthermore, the vertical density increases by a most one only in the interval $[x(a_m), x(b_m)]$.

Case 2. a_m and b_m are located on opposite sides of R and $x(a_m) \neq x(b_m)$. The transformation in this case is omitted since it is similar to the one in Case 1.

Case 3. a_m and b_m are located on opposite sides of R and $x(a_m) = x(b_m)$.

Depending on the locations of terminals of A and B, one of the transformations T_9 and T_{10} is applied. First, T_9 is applied. If the horizontal density of these four new nets is 4, then T_9 is replaced by T_{10}. Note that a fixed wire (zig-zag line) is introduced in T_{10}. One can easily show that the vertical (horizontal) density of the new four nets (without including the fixed wire) at any horizontal (vertical) cut differs from the vertical (horizontal) density of the original two nets by at most one. Furthermore, the vertical density increases by at most one only in the interval $[x(a), x(a_m)]$. Note that the number of columns increases by one and the number or rows remains unchanged.