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A simple LP-free approximation algorithm for the minimum weight vertex cover problem

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Abstract

We present a simple LP-free (i.e., not requiring linear programming) approximation algorithm for the minimum weight vertex cover problem. Our new approximation algorithm does not need to solve a linear programming problem, nor such a formulation is needed to establish its approximation bound. The algorithm takes linear time with respect to the number of nodes and edges in the graph, and generates solutions that are within twice the weight of a minimum weight vertex cover. Both the algorithm and its proof of correctness are elegant and simple.

Keywords: Approximation algorithms; Analysis of algorithms

1. Introduction

We present a simple LP-free (i.e., not requiring linear programming) approximation algorithm for the minimum weight vertex cover problem. The most important feature of the algorithm is that linear programming theory is not needed to establish its approximation bound. Our proof is based on simple intuitive arguments.

A *vertex cover* for an undirected graph is a set of vertices such that all the edges in the graph are incident upon at least one vertex in the cover. The *minimum cardinality vertex cover* for a graph is a vertex cover with the least number of vertices.

The simplest approximation algorithm for the minimum cardinality vertex cover problem was developed by Gavril [2]. The approximation algorithm takes as input a graph G, and finds a maximal (maximal not maximum) matching, M, for G. The endpoints (vertices) of the edges in the matching M form the vertex cover generated by the algorithm.

Clearly, the number of vertices in the vertex cover generated by the algorithm is $\hat{f} = 2|M|$. Since for every edge in M at least one of its two endpoints must be in any vertex cover, we know that the cardinality of an optimal vertex cover, denoted by f^*, must be greater than or equal to $|M|$. Therefore, given any instance of the minimum cardinality vertex cover problem, Gavril's algorithm generates a vertex cover with cardinality $\hat{f} \leqslant 2f^*$. The approximation bound is best possible, i.e., there are problem instances (complete bipartite graphs, for example) for which equality holds. However, one can add a simple postprocessing procedure so that the new algorithm generates solutions such that $\hat{f} < 2f^*$. The postprocessing procedure deletes a maximal subset of vertices from the previous solution while maintaining a vertex cover. Note that if at least one vertex is deleted, then the new ap-

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proximation bound holds since the new $\hat{f}$ is less than $2|M|$. On the other hand, if none of the vertices can be deleted, then one can establish that $f^* > |M|$.

A maximal matching can be constructed in $O(n+m)$ time (where n is the number of nodes, and m is the number of edges in G) by simply beginning with an empty matching (one with zero edges), and repeatedly adding an edge (which is not adjacent to an edge in the current matching) until no further addition is possible. Therefore, Gavril's algorithm can be easily implemented to take $O(n+m)$ time. The post-processing procedure can also be implemented to take linear time.

For the weighted version of the vertex cover problem the graph has a weight assigned to each vertex, and the problem consists of finding a vertex cover with least total weight, i.e., find a vertex cover whose sum of the weight of the vertices in it is least possible.

Several approximation algorithms for this more general problem were developed in the 80's [1,3-5], all having an approximation bound of two, or asymptotic to two. The only one of these algorithms that takes linear time is the one by Bar-Yehuda, and Even [1]. However, the proof to establish the approximation bound is not simple, and it is based on linear programming theory. We present a new approximation algorithm. Our algorithm also takes linear time, and has the approximation bound of two. Our algorithm is very simple, and one can establish its approximation bound in a very simple and elegant way without linear programming theory. One can view our algorithm as a generalization of Gavril's method, and a modified version of Bar-Yehuda and Even's procedure.

2. Approximation algorithm for minimum weight vertex cover

Our approximation algorithm constructs a special type of matching (maximal generalized matching) for G and then selects the set of vertices (called saturated) in it as the cover for G. Let us now define precisely these terms.

Let $G = (V, E, W)$ be a vertex weighted (non-negative real weights) graph. An edge weighted (non-negative real weights) graph $G' = (V, E, W')$ is said to be a *generalized matching* for G if for every node v_i in V the sum of the weights (W') of the edges incident to vertex v_i is less than or equal to the weight (W) of vertex v_i in G. A generalized matching G' is said to be *maximal* if increasing the weight of any edge by any $\delta > 0$ results in a graph that is not a generalized matching for G.

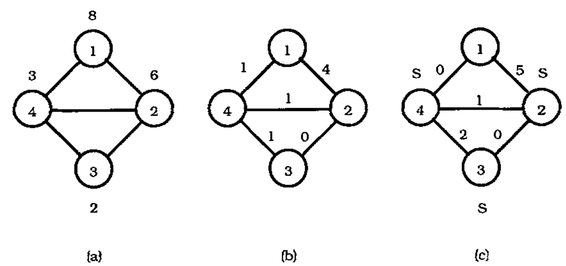

Fig. 1. (a) Graph G. (b) Non-maximal generalized matching for G. (c) Maximal generalized matching for G.

Vertex v_i in G' is said to be *saturated* if the sum of the weights of the edges incident to v_i in G' is equal to the weight of v_i in G; otherwise, the vertex is *unsaturated*. It is simple to show that if G' is a generalized maximal matching for G, then no edge in E can join two unsaturated vertices. Therefore, the set of saturated vertices in G' is a vertex cover for G.

A maximal generalized matching for G can be generated as follows. Initially G' is a copy of G with the weight of all edges set to null. Initialize the vector *tot* to zeroes. The value of tot_i represents the sum of the weights of the edges incident upon vertex i in G', where the value of null is zero. Now consider each edge in G' at a time. Assign the edge a weight of zero if at least one of the vertices adjacent to it is saturated. On the other hand, when both the edge's endpoints are unsaturated vertices, assign to the edge a weight such that at least one of the vertices becomes saturated and update the value of *tot* of both vertices. It is simple to show that the above procedure can be implemented to take linear time with respect to the number of nodes and edges in the graph G. The approximation algorithm, which we shall refer to as *GMM*, is defined below (see Fig. 1).

procedure GMM
Find a maximal generalized matching, G',
  for graph G;
Output the set of saturated vertices in the
  maximal generalized matching;
end of procedure;

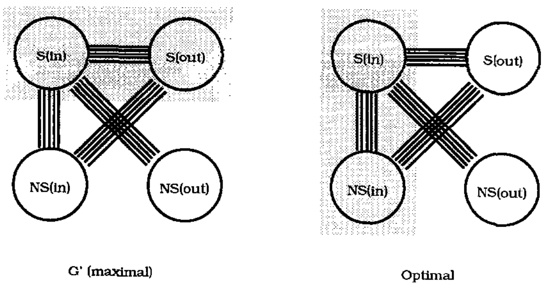

Fig. 2. Solution generated by the algorithm, and an optimal cover.

Clearly, the weight of the cover for G generated by procedure GMM is $\hat{f} = W(S)$, where S is the set of saturated vertices in G', and $W(S)$ is the sum of the weights of the vertices in S in G. We now claim that $W(S)/2$ is a lower bound for the weight of a minimum weight vertex cover for G, i.e., $f^* \geqslant W(S)/2$. The reason for this is simple. Let F^* be any optimal cover for G and let f^* be its objective function value (i.e., the sum of the weights of the vertices in it). Partition S into the set of vertices that are part of the optimal cover F^* ($S(in)$), and the ones that are not in it ($S(out)$). Let $NS(in)$ be the set of unsaturated vertices in G' that are part of the optimal cover F^*, and let $NS(out)$ be the remaining set of unsaturated vertices (see Fig. 2). Clearly, $f^* \geqslant W(S(in))$. Let us now establish that $W(S(out)) \leqslant f^*$. Since every vertex $v_i \in S(out)$ is a saturated vertex in G', we know that $W(S(out)) = W'(E(out))$, where $E(out)$ is the set of edges incident to a vertex in $S(out)$. Since each vertex $S(out)$ is not in F^*, and since F^* is a vertex cover, we know that $S(out)$ is an independent set and every vertex adjacent to a vertex in $S(out)$ must be in F^*. Since for every vertex $v_i \in V$ the sum of the weights of the edges in G' is no more than the weight of vertex v_i in G (because G' is a generalized matching for G), we know that $W'(E(out)) \leqslant W(S(in)) + W(NS(in)) = f^*$. Therefore, $W(S(out)) = W'(E(out)) \leqslant f^*$, and $\hat{f} = W(S(in)) + W(S(out)) \leqslant 2f^*$.

Theorem 1. *For any instance of the minimum weight vertex cover problem algorithm GMM generates solutions such that $\hat{f} \leqslant 2f^*$, where $\hat{f}$ and f^* are as defined above. Furthermore, algorithm GMM takes linear time with respect to the number of nodes and edges in G.*

Proof. The proof follows from the above arguments. □

As in the case of Gavril's procedure [2], there are problem instances for which the approximation bound of two can be achieved, but our algorithm can be easily modified so that $\hat{f} < 2f^*$. The idea is similar to the one used in the introduction for the minimum cardinality vertex cover. Delete a maximal set of saturated vertices while maintaining a vertex cover. If at least one vertex is deleted, then $\hat{f} < W(S)$. Otherwise, we know that either $\hat{f} = f^*$, or $NS(in) \neq \emptyset$. In the former case the result follows, and in the latter case one can establish that $f^* > W(S(in))$, and $f^* > W(S)/2$. Therefore, for the new algorithm $\hat{f} < 2f^*$. The new algorithm can be easily implemented to take linear time with respect to the number of nodes and edges in the graph.

3. Discussion

We have presented a simple approximation algorithm for the minimum weight vertex cover problem. We have established that its approximation bound is two, by using a simple and intuitive proof that does not require linear programming theory.

References

[1] R. Bar-Yehuda and S. Even, A linear time approximation algorithm for the weighted vertex cover problem, *J. Algorithms* **2** (1981) 198–203.

[2] Gavril, cited in [1].

[3] D.S. Hochbaum, Approximation algorithms for the set covering and vertex cover problems, *SIAM J. Comput.* **11** (3) (1982) 555–556.

[4] D.S. Hochbaum, Efficient bounds for the stable set, vertex cover and set packing problems, *Discrete Appl. Math.* **6** (1983) 243–254.

[5] B. Monien and E. Speckenmeyer, Ramsey numbers and an approximation algorithm for the vertex cover problem, *Acta Inform.* **22** (1985) 115–123.