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degree 2. We should note that our result holds also in the case where multiplication is commutative. On the other hand, in the case where it is possible to eliminate common subexpressions, the problem can be solved in polynomial time ([GJ]).

Other related problems in graph theory and arithmetic complexity are also shown to be NP-complete.

We now define precisely our general problem. Let Σ be a countable set of variable names and let $\theta = \{+, *\}$ be the set of binary operators on Σ such that the following laws hold:

(i) + and * are associative, i.e.,
$(a + b) + c = a + (b + c)$
$(a * b) * c = a * (b * c)$, for all $a, b, c \in \Sigma$

(ii) + is commutative, i.e.,
$a + b = b + a$, for all $a, b \in \Sigma$

(iii) * is distributive with respect to +, i.e.,
$a * (b + c) = a * b + a * c$
$(b + c) * a = b * a + c * a$, for all $a, b, c \in \Sigma$.

The main reason we have not assumed that * is commutative is that the same techniques can be applied to a matrix expression (in parallel computation) to reduce the number of arithmetic operations. We should note that the results of section 2 will also hold when * is commutative.

A <u>σ-dag</u> is a dag with a single root whose interior nodes are either + or * from θ and whose leaves are from Σ. Our problem can now be stated as follows: given a σ-dag D, find an equivalent dag D' with the fewest number of interior nodes.

2. NP-completeness result for expression dags

It is easy to check that if the given dag is a tree, the corresponding problem is trivial. The next simplest class of dags is that of leaf dags. Moreover, any arithmetic expression which involves both operators + and * got to be of degree at least 2. Therefore, the simplest type of σ-dags beyond trees is the class of leaf dags whose corresponding expressions are of degree 2. We define a subclass of these dags, namely those corresponding to <u>bilinear arithmetic expressions</u>. An arithmetic expression B is <u>bilinear</u> if it is of the form

$$B = x_{i_1} * y_{j_1} + x_{i_2} * y_{j_2} + \ldots + x_{i_k} * y_{j_k},$$

where $\{x_i\}$ and $\{y_j\}$ are nonoverlapping sets of variables and $(i_\ell, j_\ell) \neq (i_{\ell'}, j_{\ell'})$.

A bilinear expression can also be represented as $B = x^T R y$, where $r_{ij} = 1$ iff $x_i * y_j$ appears in B, otherwise, $r_{ij} = 0$.

Theorem 2.1: Let $B = x^T R y$ be an $n \times m$ bilinear arithmetic expression. The fewest number of multiplications needed to compute B is equal to the smallest r such that $R = XY$, where X and Y are $n \times r$ and $r \times m$ matrices with 0, 1 entries.

Let $B = x^T R y$ be a bilinear arithmetic expression. We can associate with B the bipartite graph† $G(B) = (V_1, V_2, E)$ defined as follows: $V_1 = \{v_i\}_{i=1}^p$ and $V_2 = \{w_j\}_{j=1}^q$ are two sets of distinct nodes corresponding respectively to the indeterminates $\{x_i\}_{i=1}^p$ and $\{y_j\}_{j=1}^q$; an edge $e = \{v_i, w_j\}$ is in E iff $r_{ij} = 1$. A decomposition of $G(B)$ consists of a set of Kuratowski (complete) subgraphs $G_i = (V_i, W_i, E_i)$, $1 \leq i \leq r$, such that $\bigcup_{i=1}^{r} V_i = V$, $\bigcup_{i=1}^{r} W_i = W$, $\bigcup_{i=1}^{r} E_i = E$ and $E_i \cap E_j = \phi$ for $i \neq j$; r is called the length of the decomposition.

We need several results before proving our main result. We start with the following definitions:

3-colorability problem: Given an undirected graph $G = (N, E)$, does there exist three disjoint sets of vertices (S_1, S_2, S_3) such that $\bigcup_{i=1}^{3} S_i = N$ and if $\{v_i, v_j\} \varepsilon E$, then v_i and v_j are in different sets?

3-m colorability problem: Given an undirected connected graph $G = (N, E)$ such that the degree of each node is at least 4 and $|E| \geq 2|N| + 1$, is G 3 colorable?

Theorem 2.2: The 3-m colorability problem is NP-complete.

Proof: We use a reduction from the 3-colorability problem which is known to be NP-complete [S]. □

Lemma 3.2: Given a graph $G = (N, E)$ $\deg v \geq 4$, $v \in N$, the elimination of k edges leaves at most $k/2$ nodes of degree zero.

Proof: The proof is simple and will be omitted. □

Our main result is to prove that the following problem (which we call the MP problem) is NP-complete: Given a $p \times q$ matrix with

†See [H] and [L] for definitions.

0, 1 entries and given a positive integer m, does there exist two matrices A and B such that $R = AB$, A and B are respectively $p \times m$ and $m \times q$ matrices with 0, 1 entries? We reduce the 3-m colorability problem [illegible] into an instance of the above problem.

<u>Theorem 2.3</u>: The MP problem is NP-complete.

<u>Proof</u>: It is straightforward to check that MP is in NP. We now show how to reduce the 3-m colorability problem to MP in polynomial time.

Let $G = (N, E)$ be an undirected graph in which each vertex is of degree greater than or equal to 4 and $|E| \geq 2|N| + 1$. Let $N = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_r\}$, where $n = |N|$ and $r = |E|$. From G, we construct the following instance of the MP problem. Take $p = 6n + 3r + 1$, $q = 6r + n$ and $m = 3r + 6n$; clearly, $q > m$.

The set of constants $\{r_{ij}\}$ defining R is constructed as follows:

a1) for each vertex $v_i \varepsilon N$ and all edges $e_j \varepsilon E$ incident upon v_i, set

$$r_{ij} = r_{n+i,r+j} = r_{2n+i,2r+j} = r_{3n+3r+i,\,3r+n+j} = r_{3r+4n+i,4r+n+j} = r_{3r+5n+i,54+n+j} = 1.$$

a2) for each i, $1 \leq i \leq n$, set

$$r_{i,3r+i} = r_{n+i,3r+i} = r_{2n+i,3r+i} = 1.$$

a3) for each j, $1 \leq j \leq 3r$, set

$$r_{3n+j,j} = r_{3n+j,3r+n+j} = 1.$$

a4) for j, $1 \leq j \leq 6r + n$, set

$$r_{3r+6n+1,j} = 1.$$

a5) set all other r_{ij} to zero.

Figure 2.1 shows the matrix $R = (r_{ij})$. G_0 is the incidence matrix of the graph G, i.e., it is an $n \times r$ matrix such that the entry (i, j) is equal to 1 if and only if v_i is incident upon e_j. I_k represents the identity matrix of size k. Row x consists of a sequence of consecutive 1's.

We will prove that G is 3-colorable if, and only if, R can be expressed as $R = AB$, where A and B are $p \times m$ and $m \times q$ matrices with 0, 1 entries (recall that $m = 3r + 6n$).

1) Suppose that G is 3-colorable and let $\{S_1, S_2, S_3\}$ be the corresponding partition of the nodes of G. Let A and B be the following matrices.

	r	r	r	n	r	r	r
n	G_0	0	0	I_n	0	0	0
n	0	G_0	0	I_n	0	0	0
n	0	0	G_0	I_n	0	0	0
r	I_r	0	0	0	I_r	0	0
r	0	I_r	0	0	0	I_r	0
r	0	0	I_r	0	0	0	I_r
n	0	0	0	0	G_0	0	0
n	0	0	0	0	0	G_0	0
n	0	0	0	0	0	0	G_0
				x			

	m
	I_m
m	y
1	

A

Figure 2.1

	r	r	r	n	r	r	r
n	G_0	0	0	I_n	0	0	0
n	0	G_0	0	I_n	0	0	0
n	0	0	G_0	I_n	0	0	0
r	I_r	0	0	0	I_r	0	0
r	0	I_r	0	0	0	I_r	0
r	0	0	I_r	0	0	0	I_r
n	0	0	0	0	G_0	0	0
n	0	0	0	0	0	G_0	0
n	0	0	0	0	0	0	G_0

B

Figure 2.2

Row y of A is constructed as follows:

n	n	n	r	r	r	n	n	n
S_1	S_2	S_3	(S_2, S_3)	(S_1, S_3)	(S_1, S_2)	S_1	S_2	S_3

where

b1) for all $v_i \varepsilon S_k$, $y[(k-1)n+i] = y[3r+(k-1)n+i] = 1$,

b2) for all edges e_j incident upon a vertex in S_2 and a vertex in S_3, set $y[3n+j] = 1$,

b3) for all edges e_j incident upon a vertex in S_1 and a vertex in S_3, set $y[3n+r+j] = 1$,

b4) for all edges e_j incident upon a vertex in S_1 and a vertex in S_2, set $y[3n+2r+j] = 1$.

To prove that $R = AB$, it is clear that we only have to verify that $yB = x$ whose proof is given by the following lemma.

<u>Lemma 1</u>: Let y and B be as defined in figure 2.2 and let x be a row vector consisting of 1's. Then we have $yB = x$.

<u>Proof of lemma 1</u>: The equation $yB = x$ is equivalent to

$$\sum_{\ell=1}^{m} y_\ell b_{j\ell} = 1, \text{ for all } j = 1, 2, \ldots, 6r+n. \qquad (*)$$

We distinguish several cases.

<u>Case 1</u>: $1 \le j \le r$.

Let $e_j = \{v_i, v_k\}$. It is easy to see from the construction of B that

$b_{ij} = b_{kj} = b_{3n+j,j} = 1$ and $b_{ij} = 0$ otherwise.

Either one of v_i or v_k belongs to S_1 or $v_i \varepsilon S_2$ and $v_k \varepsilon S_3$ (say). In the first case, precisely one of $y[i]$ or $y[k]$ is equal to 1 and $y[3n+j] = 0$; in the second case, $y[3n+j] = 1$ and $y[i] = y[k] = 0$. In either case (*) is satisfied.

<u>Case 2</u>: $r+1 \le j \le 3r$ or $3r+n+1 \le j \le 6r+n$.

The proof is similar to that of case 1.

<u>Case 3</u>: $3r+1 \le j \le 3r+n$.

The only nonzero elements in row j of matrix B are $b_{j,3r+j}$, $b_{n+j,3r+j}$ and $b_{2n+j,3r+j}$. If $v_j \varepsilon S_k$, then $y[(k-1)n+j] = 1$ and $y[(k'-1)n+j] \ne 1$ for all $k' \ne k$. Thus

$\sum_{\ell=1}^{m} y_\ell b_{j\ell} = 1.$ □

<u>Proof of Theorem 2.3 continued</u>: The above lemma completes the proof that, if G is 3-colorable, then $R = AB$, where A and B are $p \times (3r+6n)$ and $(3r+6n) \times q$ matrices with 0, 1 entries.

2) Suppose that $R = AB$ with $m = 3r + 6n$. We will prove that G is 3-colorable. The main proof is contained in the following lemma.

Lemma 2: Let R be as given in figure 2.1 and let A and B be any two $p \times m$ and $m \times q$ matrices of 0's and 1's such that $R = AB$. Then A and B must be of the form given in figure 2.2.

Proof of lemma 2: We actually prove that if $\bar{R} = AB$, where $\bar{R}$ is the same as R without the last row (i.e., row x) and A and B are $(p-1) \times m$ and $m \times q$ matrices, then $A = I_m$ and $B = \bar{R}$. The proof ~~is based upon the characterization~~ uses the graph formulation given ~~in~~ after theorem 2.1.

The bipartite graph $G(\bar{R})$ corresponding to $\bar{R}$ is given in figure 2.3 where there are two types of edges:

a) edges which represent the incidence matrix and which exist among the following sets of nodes:

$\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_r\}$,
$\{x_{n+1}, \ldots, x_{2n}\}$ and $\{y_{r+1}, \ldots, y_{2r}\}$,
$\{x_{2n+1}, \ldots, x_{3n}\}$ and $\{y_{2r+1}, \ldots, y_{3r}\}$,
$\{x_{3n+3r+1}, \ldots, x_{4n+3r}\}$ and $\{y_{3r+n+1}, \ldots, y_{4r}\}$,
$\{x_{4n+3r+1}, \ldots, x_{5n+3r}\}$ and $\{y_{4r+n+1}, \ldots, y_{5r+n}\}$,
$\{x_{5n+3r+1}, \ldots, x_{6n+3r}\}$ and $\{x_{5r+n+1}, \ldots, y_{6r+n}\}$.

Note that, for example, an edge between x_i and y_j, $1 \leq i \leq n$, $1 \leq j \leq r$, exists if and only if the node v_i of G is incident upon e_j.

b) edges which represent I_n or I_r and which exist among the following set of nodes:

$\{x_1, \ldots, x_n\}$ and $\{y_{3r+1}, \ldots, y_{3r+n}\}$,
$\{x_{n+1}, \ldots, x_{2n}\}$ and $\{y_{3r+1}, \ldots, y_{3r+n}\}$,
$\{x_{2n+1}, \ldots, x_{3n}\}$ and $\{y_{3r+1}, \ldots, y_{3r+n}\}$,
$\{x_{3n+1}, \ldots, x_{3n+r}\}$ and $\{y_1, \ldots, y_r\}$,
$\{x_{3n+r+1}, \ldots, x_{3n+2r}\}$ and $\{y_{r+1}, \ldots, y_{2r}\}$,
$\{x_{3n+2r+1}, \ldots, x_{3n+3r}\}$ and $\{y_{2r+1}, \ldots, y_{3r}\}$,
$\{x_{3n+1}, \ldots, x_{3n+r}\}$ and $\{y_{3r+n+1}, \ldots, y_{4r+n}\}$,
$\{x_{3n+r+1}, \ldots, x_{3n+2r}\}$ and $\{y_{4r+n+1}, \ldots, y_{5r+n}\}$,
$\{x_{3n+2r+1}, \ldots, x_{3n+2r}\}$ and $\{y_{5r+n+1}, \ldots, y_{6r+n}\}$.

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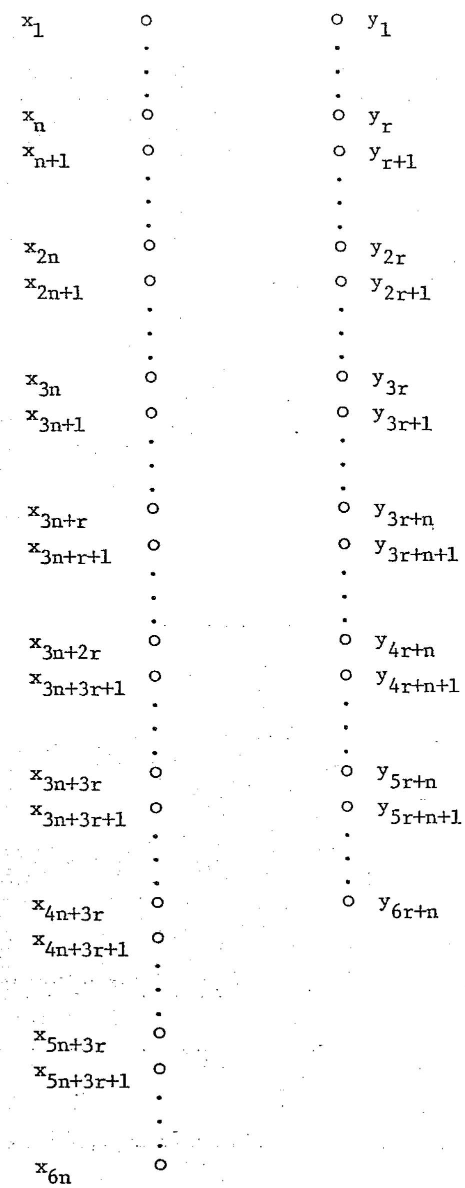

Figure 2.3

Notice that $G(\bar{R})$ has only two types of complete subgraphs $K_{1,\ell}$ and $K_{r,1}$*, where r, $\ell \geq 1$. The statement of the lemma can be reformulated as follows: $G(\bar{R})$ has only one decomposition of length $3r + 6n$ and this decomposition is obtained by taking each x_i and constructing the complete subgraph consisting of all edges incident upon x_i. The main idea

*Note that $K_{m,n}$ is the complete graph based on m nodes among the x_i's and n nodes among the y_j's.

of the proof is to show that any decomposition of $G(\bar{R})$ which contains complete subgraphs of the type $K_{r,\ell}$, $r > 1$, has length greater than $3r + 6n$. We now prove this fact.

Consider any decomposition D of $G(\bar{R})$ of length $3r + 6n$ and suppose it contains α complete subgraphs of the type $K_{r,1}$, $r > 1$. Each such $K_{r,1}$ has one vertex among the y_j's, say y_{j_r}. Therefore α can be expressed as $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$, where α_i is the number of $K_{r,1}$ subgraphs with j_r belonging to the i^{th} set of nodes which form the y_j's.

We now remove the edges corresponding to the above $K_{r,1}$ subgraphs and try to determine the number of the x_i's nodes whose degrees are nonzero. Removing the first $\alpha_1 + \alpha_2 + \alpha_3$ subgraphs destroys no x_i's. If we next remove the α_4 subgraphs, then, at most, $\min(\frac{\alpha_1}{2}, \alpha_4) + \min(\frac{\alpha_2}{2}, \alpha_4) + \min(\frac{\alpha_3}{2}, \alpha_4)$ of the x_i's will disappear completely (Lemma 3.2). Deleting the next α_5 subgraphs can cause at most $\min(\alpha_5, \alpha_1) + \frac{\alpha_5}{2}$ x_i nodes to disappear. Similarly, taking out the remaining subgraphs can result in the removal of at most $\min(\alpha_6, \alpha_2) + \frac{\alpha_6}{2} + \min(\alpha_7, \alpha_5) + \frac{\alpha_7}{2}$ x_i nodes.

It follows that the maximum number of x_i nodes which could disappear is given by

$$\mu = \min(\frac{\alpha_1}{2}, \alpha_4) + \min(\frac{\alpha_2}{2}, \alpha_4) + \min(\frac{\alpha_3}{2}, \alpha_4) + \min(\alpha_5, \alpha_1) + \frac{\alpha_5}{2} + \min(\alpha_6, \alpha_2) + \frac{\alpha_6}{2} + \min(\alpha_7, \alpha_3) + \frac{\alpha_7}{2}.$$

Three cases arise:

(i) $\alpha_4 \geq 1$. Using the fact that $\min(k_1, k_2) \leq \frac{k_1 + k_2}{2}$ and $\min(k_1, k_2) \leq k_1$ or k_2, we obtain the following

$$\mu \leq (\frac{\alpha_1}{2} + \frac{\alpha_2}{2} + \frac{\alpha_3}{2}) + (\frac{\alpha_5 + \alpha_1}{2}) + \frac{\alpha_5}{2} + (\frac{\alpha_6 + \alpha_2}{2}) + \frac{\alpha_6}{2} + (\frac{\alpha_7 + \alpha_3}{2}) + \frac{\alpha_7}{2},$$

$$\mu \leq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 = \alpha - \alpha_4.$$

But since all the remaining subgraphs of $\mathcal{D}$ are of the type $K_{1,\ell}$, then $\mathcal{D}$ must have at least

$6n+3r-(\alpha-\alpha_4)$ such complete subgraphs.
Therefore, the length of $\mathcal{D}$ is at least

$$\alpha+(6n+3r-(\alpha-\alpha_4)) = 6n+3r+\alpha_4 > 6n+3r$$

which contradicts the assumption that the length of $\mathcal{D}$ is $6n+3r$.

(ii) $\alpha_4 = 0$ and $\alpha_1+\alpha_2+\alpha_3 \geq 1$. In this case,

$$\mu = \min(\alpha_5,\ \alpha_1)+\frac{\alpha_5}{2}+\min(\alpha_6,\ \alpha_2)+\frac{\alpha_6}{2}+$$
$$\min(\alpha_7,\ \alpha_3)+\frac{\alpha_7}{2}.$$

Thus $\mu \leq \alpha_5+\alpha_6+\alpha_7+\frac{\alpha_1}{2}+\frac{\alpha_2}{2}+\frac{\alpha_3}{2}$.

It is easy to check that $\mu \leq \alpha_1+\alpha_2+\alpha_3+\alpha_5+\alpha_6+\alpha_7-1$ and the proof carries as before.

(iii) $\alpha_4 = 0$ and $\alpha_1+\alpha_2+\alpha_3 = 0$. It is clear that $\mu \leq \frac{\alpha_5+\alpha_6+\alpha_7}{2}$ and the proof is similar to the previous cases.

Therefore any decomposition of $G(\bar{R})$ which contains subgraphs of the type $K_{r,1}$, $r>1$, has to be of length greater than $6n+3r$. □

<u>Proof of Theorem 2.3 continued</u>: We now know that for any A and B such that $R = AB$, both A and B must be of the form given in figure 2.2. Note that row y of A has not been specified. Define the following three sets of nodes in G:

$$D_1 = \{v_j \mid y[j]=1\},$$
$$D_2 = \{v_j \mid y[n+j]=1\},$$
$$D_3 = \{v_j \mid y[2n+j]=1\}.$$

These sets are pairwise disjoint because if $v_K \varepsilon D_1 \cap D_2$, say, then multiplying y by the $(3r+k)^{th}$ column of B produces a sum of 2 which is not correct. Moreover, these sets exhaust all the nodes of G by the fact that

$$y \begin{bmatrix} I_n \\ I_n \\ I_n \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = [\overbrace{1 \quad 1 \quad \ldots \quad 1}^{n}].$$

We now prove that no edge has itw two nodes in one set D_i. Suppose $e_j = \{v_i, v_\ell\}$ is such that v_i and v_ℓ are in D_k. Multiplying y by the $((k-1)r+j)^{th}$ column of B results in a number greater than one since $y[(k-1)n+i] = y[(k-1)n+\ell] = 1$. It follows that the above partition of vertices defines a 3-coloration for G and the proof of the theorem is complete. □

3. Complexity of Related Problems

Another context where these results are relevant is that of computing a set of bilinear forms in algebraic complexity ([BD], [BM], [J], [W]). Note that it is not known whether the general problem with integer constants is decidable [M]. Let R be a commutative ring and let $K \subseteq R$ such that $0,1 \in K$. Suppose $x = (x_1, x_2, \ldots, x_p)^T$ and $y = (y_1, y_2, \ldots, y_q)^T$ are two column vectors of indeterminates; we have to compute m bilinear forms:

$$B_i = \sum_{j=1}^{p} \sum_{k=1}^{q} \alpha_{ijk} x_j y_k = x^T G_i y, \quad i = 1, 2, \ldots, m,$$

where G_i is a $p \times q$ matrix with elements in K.

Theorem 3.1: Given a bilinear form B over $K = \{0, 1\}$ and given a positive integer δ, the problem of determining whether or not B can be computed with δ multiplications is NP-complete

We have the following immediate corollary.

Corollary: Given a set of bilinear forms $\{B_i\}_{i=1}^m$ over $\{0, 1\}$ and given a positive integer δ, the problem of determining whether or not these bilinear forms can be computed with δ multiplications is NP-complete. □

The above results rely heavily on the fact that the constant set is $\{0, 1\} \subseteq Z$. A much more interesting case is when the constant set consists of $\{0, 1, -1\}$ as in most of the published algorithms ([St]). Finding the corresponding complexity seems to be harder in this case; however, we could not extend the above proofs to cover this case. It is worth mentioning that, for a given single bilinear form $B = \sum_{i,j} r_{ij} x_i y_j$, $r_{ij} = 0, 1$, the introduction of subtraction can reduce the number of multiplications.

As we have seen in section 2, the multiplicative complexity of a single bilinear arithmetic expression is related to the length of a decomposition of the associated bipartite graph G(B). In view of Theorem 2.3, we have the following immediate result.

Theorem 3.2: Given a bipartite graph G and a positive integer k, the problem of determining whether G has a decomposition of length k is NP-complete.

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