A Linear Time Algorithm for Optimal Routing around a Rectangle

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Abstract. The problem of connecting a set of terminals that lie on the sides of a rectangle to minimize the total area is discussed. An $O(n)$ algorithm is presented to solve this problem when the set of n terminals is initially sorted. The strategy in this paper is to reduce the problem to several problems such that no matter what instance is started with, at least one of these problems can be solved optimally by a greedy method.

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1. *Introduction*

Let T be a rectangle and let S be a set of n points that lie on the sides of T. Each point in set S has to be connected to another point in S by a wire. The path followed by these wires consists of a finite number of horizontal and vertical line segments. These line segments are assigned to two different layers. All horizontal line segments are assigned to one layer and all the vertical ones are assigned to the other layer. Line segments on different layers can be connected at any given point z by a wire perpendicular to the layers if both line segments include point z on their respective layers (this is normally referred to as a via or contact cut). Every pair of distinct and parallel line segments must be at least $\lambda > 0$ units apart and

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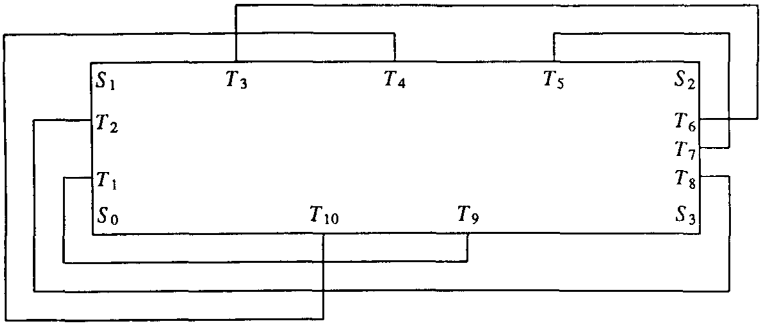


FIGURE 1

every line segment must be at least λ units from each side of T, except in the region where the path joins the point in S it connects. Also, no path is allowed inside of rectangle T on any of the layers (see Figure 1).

Problem 2-R1M (routing around one module "two terminal" nets) consists of specifying paths for all the wires in such a way that the total area is minimized. That is, to place T together with all the wires (that must satisfy the restrictions imposed above) inside a rectangle (with the same orientation as T) of least possible area. This problem has applications in the layout of integrated circuits [5, 8] and conforms to a set of design rules for VLSI systems [7].

Hashimoto and Stevens [4] present an $O(n \log n)$ algorithm to solve the 2-R1M problem for the case when all the points in S lie on one side of T and an $\Omega(n \log n)$ lower bound on the worst case time complexity for this problem was established in [3]. In [5], an $O(n^3)$ algorithm is presented to solve the 2-R1M problem. If more than two layers are allowed and wire overlap is permitted, then the problem becomes NP-hard [9]. Other generalizations of the 2-R1M problem have been shown to be NP-hard [6]. In this paper we present a linear time algorithm to solve the 2-R1M problem when the set of terminals is initially sorted. Since sorting the terminals can be done in $O(n \log n)$ time, the more general case can be solved in $O(n \log n)$ time.

The first few steps in our procedure are the initial steps in LaPaugh's algorithm [5]. These steps divide the 2-R1M problem into two subproblems and find the direction for the paths connecting local terminals (a terminal is said to be local if it is to be connected to another terminal located on the same side or on an adjacent side of T). After performing these operations it is only required to solve a restricted version of the 2-R1M problem. In this restricted version of the 2-R1M problem all the nonlocal terminals appear on the top and bottom sides of T. It is at this point that our algorithm will differ from the one given in [5].

To simplify the presentation of our results, first we show the existence of an optimal solution, D, that satisfies the following properties:

(a) *Balance.* The number of paths crossing two predefined vertical half-lines resting on the top side of T differs by at most one.
(b) *Minimum height.* The height on the top side of T for D (maximum number of paths crossing any vertical half-line resting on the top side of T) is minimum amongst all optimal solutions that satisfy (a).

Two suboptimal solutions satisfying (a) are defined. Both of these solutions can be easily generated. We show that an optimal solution satisfying (a) and (b) above differs from one of these suboptimal solutions by a set of connecting paths that can be easily characterized. Our algorithm generates these suboptimal solutions and interchanges the direction of several sets of paths. At least one of these feasible solutions is an optimal solution for our problem.

In Section 2 we present some initial definitions and the steps from LaPaugh's [5] algorithm that our procedure follows. In Section 3 we present a series of lemmas that show the existence of an optimal solution that can be generated by a greedy method. Our algorithm and complexity issues relating to the 2-R1M problem are discussed in Section 4.

2. *Definitions and Problem Transformations*

In this section, we redefine the 2-R1M problem, since our algorithm can be easily explained under this new definition. We also define some terms and present the steps from LaPaugh's algorithm that our procedure follows.

Let T be a rectangular component of size h by w (height by width). There are $2n$ terminals ($T_1, T_2, \ldots, T_{2n}$) on its sides. It is assumed that every pair of terminals is at least $\lambda > 0$ units apart and every terminal is located at least λ units from each of the corners of T. The *function* $C(i)$, for $1 \leq i \leq 2n$, indicates that terminal T_i is to be connected to terminal $T_{C(i)}$. If $C(i) = j$, then $C(j) = i$, that is, C is a symmetric function. Terminal T_i is to be connected to terminal $T_{C(i)}$ by a wire that follows a path beginning at point T_i and ending at point $T_{C(i)}$. Each of these paths can be partitioned into a finite number of straight line segments. These line segments must lie on the same plane as T, but cannot lie on the inside of rectangle T. Each line segment must be parallel to a side of T. Perpendicular line segments can intersect at any point, but parallel line segments must be at least λ units apart. Also, all line segments must be at least λ units away from every side of rectangle T except in the vicinity where a line segment connects a terminal. The *2-R1M problem* consists of specifying paths for all the interconnections subject to the rules memtioned above in such a way that the total area is minimized, that is, place the component together with all the interconnnecting paths inside a rectangle (with the same orientation as T) of least possible area.

Label the sides of the component (in the obvious way) left, top, right, and bottom. Starting in the bottom-left corner of T, traverse the sides of the rectangle clockwise. The ith corner to be visited is labeled S_{i-1}. Assume that terminal T_i is the ith terminal visited. The *closed interval* $[x, y]$, where x and y are the corners of T or the terminals T_i, consists of all the points on the sides of T that are visited while traversing the sides of T in the clockwise direction starting at point x and ending at point y. Note that interval $[x, x]$ includes only one point. Parentheses are used instead of square brackets when it is desired to specify an open interval. We use $[S_0, S_1]$, $[S_1, S_2]$, $[S_2, S_3]$, and $[S_3, S_0]$ to represent the left, top, right, and bottom sides of T, respectively. Terminal T_i is said to belong to side l, $S(i) = l$, if T_i is located on the interval $[S_l, S_{(l+1) mod(4)}]$.

Set $D = \{d_1, d_2, \ldots, d_n\}$ is said to be an *assignment* if $\{d_i, C(d_i) \mid 1 \leq i \leq n\} = \{1, 2, \ldots, 2n\}$. Any subset of an assignment is said to be a *partial assignment*. An assignment D indicates the starting point for each path connecting a pair of terminals. The direction of all the paths given by D is the clockwise direction. For any $i \in D$, the wire that connects terminal T_i to terminal $T_{C(i)}$ starts at terminal T_i moving perpendicular to side $S(i)$ and then continues in the clockwise direction with respect to T until it can be joined to a wire (all of it on the outside of T) perpendicular to $S(C(i))$ that ends at terminal $T_{C(i)}$. In a partial assignment, the starting point for some of the connecting paths might not be specified. The assignment for the layout given by Figure 1 is {3, 5, 8, 9, 10}. For any $l \in D$, we say that the path connecting terminal T_l *crosses* point z if $z \in [T_l, T_{C(l)}]$.

For any assignment (or partial assignment) D we define the *height function* H_D for $x, y \in \{T_1, T_2, \ldots, T_{2n}\} \cup \{S_0, S_1, S_2, S_3\}$ as follows:

$$H_D(x, y) = \mathbf{max}\{\text{number of paths given by } D \text{ that cross point } z \mid z \in [x, y]\}.$$

We refer to $H_D(x, y)$ as the *height of assignment D on the interval* $[x, y]$. For example: $H_D(S_0, S_1)$ is 3, $H_D(T_5, T_5)$ is 2, and $H_D(S_2, S_3)$ is 2, for the assignment, D, whose layout appears in Figure 1.

The next two lemmas establish that the 2-R1M problem reduces to the problem of finding an assignment D with least $(h + (H_D(S_1, S_2) + H_D(S_3, S_0)) \cdot \lambda) \cdot (w + (H_D(S_0, S_1) + H_D(S_2, S_3)) \cdot \lambda)$ and then in $O(n \log n)$ time ($O(n)$ time if the set of terminals is initially sorted) one may obtain an optimal area layout for it. This layout is an optimal solution to our problem. The proof of Lemma 2.1 is constructive. The construction process begins by finding physical routes for the wires on each of the four sides of the rectangle separately. This procedure is carried out by the algorithm given in [5]. These four layouts are combined to form the final layout by making the appropriate wire connections in each of the four corners of the rectangle.

LEMMA 2.1. *For every assignment D, there is a rectangle Q of size h_Q by w_Q, where*

$$h_Q = h + (H_D(S_1, S_2) + H_D(S_3, S_0)) \cdot \lambda$$

and

$$w_Q = w + (H_D(S_0, S_1) + H_D(S_2, S_3)) \cdot \lambda,$$

with the property that rectangle T together with the interconnecting paths defined by D can be made to fit inside Q.

PROOF. The proof appears in [6]. □

LEMMA 2.2. *For any assignment D, a layout with the area given by Lemma* 2.1 *can be obtained in $O(n \log n)$ time ($O(n)$ time if the set of terminals is initially sorted).*

PROOF. The proof of this lemma appears in [6]. □

In what follows we shall refer to an assignment as an *optimal assignment* when it is the assignment for an optimal area layout. Note that an optimal area layout for an optimal assignment can be obtained from the constructive proof for Lemma 2.1.

Terminal T_i is said to be a *global terminal* if $|S(i) - S(C(i))| = 2$, that is, terminal T_i is global if it is to be connected to a terminal located on the opposite side of the rectangle. Terminal T_i is said to be *local* otherwise, that is, if it is to be connected to a terminal located on the same side or on an adjacent side of the rectangle T. Terminals T_2, T_8, T_4, and T_{10} are the only global terminals for the problem depicted in Figure 1. For assignment D we define the *area function*, $A(D)$, as

$$(h + (H_D(S_1, S_2) + H_D(S_3, S_0)) \cdot \lambda) \cdot (w + (H_D(S_0, S_1) + H_D(S_2, S_3)) \cdot \lambda),$$

that is, the total area required by an optimal layout for T together with all the interconnections specified by D.

Definition 2.1. *Partial assignment D'.* Let D' be the partial assignment in which each local terminal is connected by a path that crosses the least number of corners of T.

LEMMA 2.3. *There is an optimal assignment, D, such that $D' \subseteq D$.*

PROOF. The proof appears in [6]. The proof is based on an interchange argument. Given any optimal assignment that does not include D', one constructs another assignment that includes D' (by connecting all local terminals by the paths given in D') without increasing the total layout area. □

Lemma 2.3 shows that given any instance of the 2-R1M problem there exists an optimal assignment in which all local terminals are connected by paths that cross at most one corner of T. The 2-R1M problem has been reduced to the problem of finding the starting point for the paths connecting the global terminals in the presence of the partial assignment D'. The next lemma partitions the 2-R1M problem into two separate problems: the problem of finding an optimal assignment for the 2-R1M problem in which all global terminals appear on the top and bottom sides of T (P_1) and the one in which all global terminals appear on the left and right sides of T (P_2). In both of these subproblems local terminals are connected by the paths given by D'.

LEMMA 2.4. *The assignment $D_1 \cup D_2$ is an optimal assignment, where D_i is an optimal assignment for problem P_i ($1 \le i \le 2$).*

PROOF. The proof of this lemma appears in [6] and is based on the fact that if $D_1 \cup D_2$ is not an optimal assignment to the original problem, then either D_1 is not an optimal assignment for P_1 or D_2 is not an optimal assignment for P_2. □

3. *The Restricted Problem*

In this section we show that given any instance of the restricted 2-R1M problem, it is always possible to obtain an optimal assignment by solving one of several problems using a greedy method. Hereafter, we restrict our attention to the solution of the 2-R1M problem in which all global terminals are located on the top and bottom sides of T and all local terminals are connected by the paths specified in D'. Note that problem P_2, defined in the previous section, can be transformed to this one by rotating the rectangle 90 degrees. If the number of global terminals located on the top side of T is zero, then D' is an optimal assignment (Lemma 2.3). In what follows we assume that there is at least one global terminal located on the top side of T.

First we define two points, T_α and T_β. Then we show the existence of an optimal assignment in which the global terminals located on the interval $[S_1, T_\alpha]$ ($[T_\beta, S_2]$) are connected by paths that cross the left (right) side of T.

Definition 3.1. *α and β.* Let

$L = D' \cup \{$all global terminals are connected by a path crossing the left side of $T\}$;
$l = \mathbf{min}\{k \mid H_L(T_k, T_k) = H_L(S_1, S_2)$ and $T_k \in [S_1, S_2]\}$;

$$\alpha = \begin{cases} l & \text{if } T_l \text{ is a local terminal;} \\ l - 0.5 & \text{otherwise;} \end{cases}$$

$R = D' \cup \{$all global terminals are connected by a path crossing the right side of $T\}$;
$r = \mathbf{max}\{k \mid H_R(T_k, T_k) = H_R(S_1, S_2)$ and $T_k \in [S_1, S_2]\}$; and

$$\beta = \begin{cases} r & \text{if } T_r \text{ is a local terminal;} \\ r + 0.5 & \text{otherwise.} \end{cases}$$

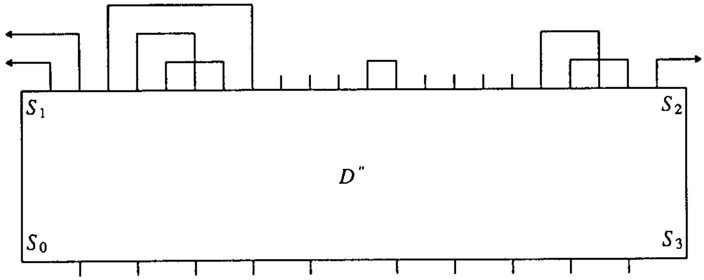


FIG. 2. Assignment D''.

Note that α and β do not always correspond to terminal indices. Sometimes they correspond to a terminal index $\pm$ 0.5. In what follows we corrupt our notation by making reference to T_α and T_β. When $T_\alpha(T_\beta)$ does not correspond to a terminal we mean a point located $\epsilon < \lambda$ units to the left (right) of terminal $T_{\lceil\alpha\rceil}(T_{\lceil\beta\rceil})$. In both of these cases such a point is located on the top side of T. We also assume that the definition of an interval and the height function have been extended to include T_α and T_β. For the problem given in Figure 1, α is 3 and β is 5 (if T_3 and T_5 were global terminals then $\alpha = 2.5$ and $\beta = 5.5$). The values for α and β are integers for the problems depicted in Figures 6 and 7. Note that when α (or β) is an integer it corresponds to the index of a local terminal. From the definition of α and β it is simple to prove that $\alpha \leq \beta$.

Definition 3.2. *Partial assignment D''.* Let $D'' = D' \cup$ {all global terminals located on the interval $[S_1, T_\alpha]$ connected by a path crossing the left side of T} $\cup$ {all global terminals located on the interval $[T_\beta, S_2]$ connected by a path crossing the right side of T} (see Figure 2).

Definition 3.3. *h_α and h_β.* Let $h_\alpha = H''_D(T_\alpha, T_\alpha)$ and $h_\beta = H''_D(T_\beta, T_\beta)$.

We now show that there is an optimal assignment, D, such that $D'' \subseteq D$. Before proving this, we prove the following lemma, which will be useful in the proof of subsequent lemmas.

LEMMA 3.1. *Given any assignment D such that $D' \subseteq D$,*

(i) $H_D(S_1, T_k) \leq H_D(T_k, S_2)$ *for every T_k in $[S_1, T_\alpha]$, and*
(ii) $H_D(S_1, T_k) \geq H_D(T_k, S_2)$ *for every T_k in $[T_\beta, S_2]$.*

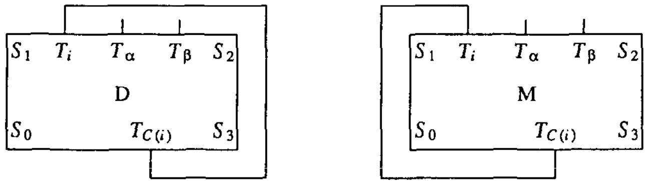
PROOF. Since the proof for (ii) is similar to the proof for (i), we only prove (i). Clearly, if we can prove that $H_D(T_k, T_k) \leq H_D(T_\alpha, T_\alpha)$ for every T_k in $[S_1, T_\alpha]$, then (i) holds. Let T_k be any global terminal located on the interval $[S_1, T_\alpha]$ and let W be the set of indices of the terminals that are connected differently in D and L, where L is the assignment in Definition 3.1. Note that all the terminals whose index is in W are global since $D' \subseteq D$ and $D' \subseteq L$. Let $W_1 = W \cap \{l \mid T_l \text{ is in } [S_1, T_k]\}$ and $W_2 = W \cap \{l \mid T_l \text{ is in } [T_\alpha, S_2]\}$. At this point it is important to remember that T_α is not a global terminal and thus $W_1 \cap W_2 = \emptyset$. Since each terminal whose index is in W is connected by a path that crosses the right side of T in D but not in L, we know that

$$H_D(T_k, T_k) \leq H_L(T_k, T_k) + |W_1| - |W_2|,$$

and

$$H_D(T_\alpha, T_\alpha) \geq H_L(T_\alpha, T_\alpha) + |W_1| - |W_2|.$$

FIGURE 3



From the definition of α we know that $H_L(T_k, T_k) \leq H_L(T_\alpha, T_\alpha)$ for any T_k in $[S_1, T_\alpha]$. Hence, $H_D(T_k, T_k) \leq H_D(T_\alpha, T_\alpha)$ for any T_k in $[S_1, T_\alpha]$. Thus, as established before it must be that (i) holds. This completes the proof of the lemma. □

LEMMA 3.2. *Let D'' be as defined above. There is an optimal assignment, D, such that $D'' \subseteq D$.*

PROOF. Lemma 2.3 guarantees the existence of at least one optimal assignment that includes D'. We show that at least one of such assignments includes D''. The proof is by contradiction. Suppose every optimal assignment that includes D' has the property that D'' is not a subset of it. Let D be an optimal assignment that includes D' and has the least disagreement (in terms of the number of connecting paths) with D''. There are two cases depending on where the disagreement occurs.

Case 1. There is a terminal, T_i, located on the interval $[S_1, T_\alpha]$ connected differently in D'' and D.

We establish a contradiction by constructing another optimal assignment, $M = (D - \{i\}) \cup \{C(i)\}$, that includes D' and has less disagreement with D'' than D. Since $H_M(S_3, S_0) \leq H_D(S_3, S_0) + 1$, $D' \subseteq M$, and M has less disagreement with D'' than D, a contradiction can be obtained by proving that $H_M(S_1, S_2) \leq H_D(S_1, S_2) - 1$ (see Figure 3).

Since T_i is connected by a path that crosses the right side of T in D but not in M and T_i is not T_α (remember that T_i is a global terminal and T_α cannot be a global terminal), we know that $H_M(T_\alpha, S_2) = H_D(T_\alpha, S_2) - 1 \leq H_D(S_1, S_2) - 1$. Since $D' \subseteq M$, then from Lemma 3.1 we know that $H_M(S_1, T_\alpha) \leq H_M(T_\alpha, S_2)$. Hence, $H_M(S_1, S_2) = H_M(T_\alpha, S_2) \leq H_D(S_1, S_2) - 1$. A contradiction.

Case 2. There is a terminal located on the interval $[T_\beta, S_2]$ connected differently in D'' and D.

A contradiction for this case can be obtained by using a proof similar to the one for Case 1.

This completes the proof of the lemma. □

Definition 3.4. t. Let t be the number of global terminals located on the interval (T_α, T_β).

If $t = 0$ then partial assignment D'' is an assignment, that is, we have specified the starting point (and by convention, the direction) for the connection of all terminals, and by Lemma 3.2 we conclude that D'' is an optimal assignment. In what follows we assume that $t > 0$. A lower bound for the height on the top side of T for any assignment that includes D'' is given by Δ.

Definition 3.5. Δ. Let $\Delta = \lceil h_\alpha + h_\beta + t)/2\rceil$.

From the above definitions one can easily prove the following claim.

CLAIM 1. *$H_D(S_1, S_2) \geq \Delta$ for any assignment D such that $D'' \subseteq D$.*

In what follows we show that there is an optimal assignment that includes D'', with height on the top side of T is equal to Δ or $\Delta + 1$, and with the property that the heights at points T_α and T_β differs by at most one.

LEMMA 3.3. *There is an optimal assignment, D, such that*

(*a*) $D'' \subseteq D_i$,
(*b*) $|H_D(T_\alpha, T_\alpha) - H_D(T_\beta, T_\beta)| \leq 1$, *and*
(*c*) $\Delta \leq H_D(S_1, S_2) \leq \Delta + 1$.

PROOF. Using the following claim together with the fact that $H_D(T_\alpha, T_\alpha) + H_D(T_\beta, T_\beta) = h_\alpha + h_\beta + t$ one can easily prove the existence of an optimal assignment D satisfying (a)–(c). Hence, the proof of the lemma follows from the proof of the following claim.

CLAIM. *There is an optimal assignment, D, such that*

(*i*) $D'' \subseteq D$,
(*ii*) $H_D(S_1, S_2) - H_D(T_\alpha, T_\alpha) \leq 1$, *and*
(*iii*) $H_D(S_1, S_2) - H_D(T_\beta, T_\beta) \leq 1$.

PROOF. From Lemma 3.2 we know that there is at least one optimal assignment satisfying (i). We now show that one of such assignments also satisfies (ii) and (iii). This is shown by contradiction. Suppose all optimal assignments satisfying (i) do not satisfy (ii) or (iii). Let D be an optimal assignment such that $D'' \subseteq D$ and D has the smallest height on the top side of T (i.e., least $H_D(S_1, S_2)$) amongst all optimal assignments that satisfy (i). There are three cases depending on which of the inequalities is violated.

Case 1. $H_D(S_1, S_2) - H_D(T_\alpha, T_\alpha) \leq 1$ and $H_D(S_1, S_2) - H_D(T_\beta, T_\beta) > 1$.

If every global terminal located on the interval (T_α, T_β) is connected by a path crossing the right side of T in assignment D, then from the definition of β one can prove that $H_D(T_\beta, T_\beta) = H_D(T_\alpha, S_2)$. Since $D' \subseteq D$, then from Lemma 3.1 we know $H_D(S_1, T_\alpha) \leq H_D(T_\alpha, S_2) = H_D(T_\beta, T_\beta) = H_D(S_1, S_2)$. But, since (iii) is violated we know that $H_D(T_\beta, T_\beta) < H_D(S_1, S_2)$. Therefore, it must be that there is at least one global terminal located on the interval (T_α, T_β) connected by a path crossing the left side of T in D. Let T_r be the rightmost global terminal in (T_α, T_β) connected by a path crossing the left side of T. Let $M = (D - \{C(r)\}) \cup \{r\}$. If we prove that $D'' \subseteq M$, M is an optimal assignment and M has less height on top side of T than D, then there is a contradiction.

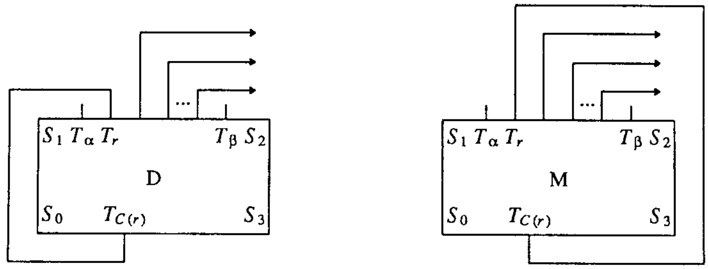
Since $T_r \in (T_\alpha, T_\beta)$, $D'' \subseteq M$ and only one connecting path differs in M and D, we know that $H_M(S_3, S_0) \leq H_D(S_3, S_0) + 1$. Hence, a contradiction can be obtained by proving that $H_M(S_1, S_2) \leq H_D(S_1, S_2) - 1$. From Figure 4 it is simple to observe that

$$H_M(S_1, T_k) = H_D(S_1, T_k) - 1 \leq H_D(S_1, S_2) - 1, \qquad \text{for every} \quad T_k \in [S_1, T_r).$$

Consequently, we only need to prove that $H_M(T_r, S_2) \leq H_D(S_1, S_2) - 1$.

Since all global terminals in $[T_r, S_2]$ are connected by paths that cross the right side of T in M, a proof similar to the one for Lemma 3.1 can be used to show that $H_M(T_r, S_2) = H_M(T_\beta, T_\beta)$. Clearly, $H_M(T_\beta, T_\beta) = H_D(T_\beta, T_\beta) + 1$. From these two equations and the inequality being violated in the conditions of Case 1, we know that $H_M(T_r, S_2) \leq H_D(S_1, S_2) - 1$, and as established before there is a contradiction.

FIGURE 4



Case 2. $H_D(S_1, S_2) - H_D(T_\alpha, T_\alpha) > 1$ and $H_D(S_1, S_2) - H_D(T_\beta, T_\beta) \le 1$.

Since the proof for this case is similar to the one for Case 1, it will be omitted.

Case 3. $H_D(S_1, S_2) - H_D(T_\alpha, T_\alpha) > 1$ and $H_D(S_1, S_2) - H_D(T_\beta, T_\beta) > 1$.

Let $T_l(T_r)$ be the leftmost (rightmost) global terminal located on the interval (T_α, T_β) connected by a path that crosses the right (left) side of T in D. One can prove the existence of these two terminals by using arguments similar to those in the proof for Case 1. If T_l is to the right of T_r, then a proof similar to the one for Lemma 3.1 can be used to prove that $H_D(S_1, T_r) = H_D(T_\alpha, T_\alpha)$, $H_D(T_l, S_2) = H_D(T_\beta, T_\beta)$, and since there are no global terminals in (T_r, T_l), we know that $H_D(T_r, T_l) \le H_D(T_\alpha, T_\alpha)$. Hence, it must be that $H_D(S_1, S_2) = \mathbf{max}\{H_D(T_\alpha, T_\alpha), H_D(T_\beta, T_\beta)\}$. This contradicts our assumption that (ii) and (iii) are violated. Therefore, T_l must be located to the left of T_r.

Let $M = (D - \{l, C(r)\}) \cup \{C(l), r\}$. We now show that assignment M has the following properties: $D'' \subseteq M$, M has less height on the top side of T than D, and M is an optimal assignment. Clearly, if these statements hold, then there is a contradiction (see Figure 5).

Straightforward arguments can be used to show that a contradiction can be obtained by proving that

$$H_M(S_1, T_l) \le H_D(S_1, S_2) - 2 \quad \text{and} \quad H_M(T_r, S_2) \le H_D(S_1, S_2) - 2.$$

Since all global terminals located on the interval $[S_1, T_l]$ are connected by a path that crosses the left side of T in M, a proof similar to the one of Lemma 3.1 can be used to show that $H_M(S_1, T_l) = H_M(T_\alpha, T_\alpha)$. Following similar arguments we know $H_M(T_r, S_2) = H_M(T_\beta, T_\beta)$. Clearly, $H_M(T_\alpha, T_\alpha) = H_D(T_\alpha, T_\alpha)$ and $H_M(T_\beta, T_\beta) = H_D(T_\beta, T_\beta)$. From these equations and the inequalities violated in the conditions of Case 3, we know that $H_M(S_1, T_l) \le H_D(S_1, S_2) - 2$ and $H_M(T_r, S_2) \le H_D(S_1, S_2) - 2$. Thus, as it was established before there is a contradiction.

This completes the proof of the claim and the lemma. □

The remaining proofs and the algorithm could be greatly simplified if one could prove Lemma 3.3 with condition (c) replaced by either $H_D(S_1, S_2) = \Delta$ or $H_D(S_1, S_2) = \Delta + 1$. However, such lemmas cannot be proved. In what follows we give counterexamples to both of these proposed lemmas. In Figure 6 we give a problem instance that does not have an optimal assignment satisfying (a) and (b) in Lemma 3.3 and with height on the top side of T equal to Δ. Let us prove this claim. Since there are four global terminals located on the bottom side of T, the height on the bottom side of T for any assignment must be at least two. Therefore, if we can prove that there is no assignment with area $\le (h + 7\lambda) \cdot (w + 4\lambda)$

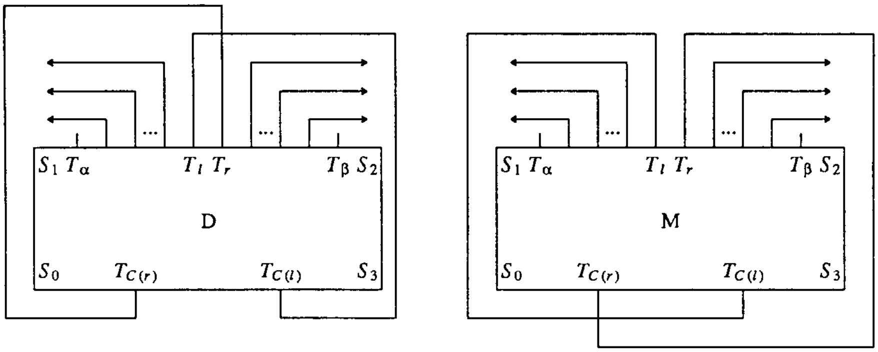


FIGURE 5

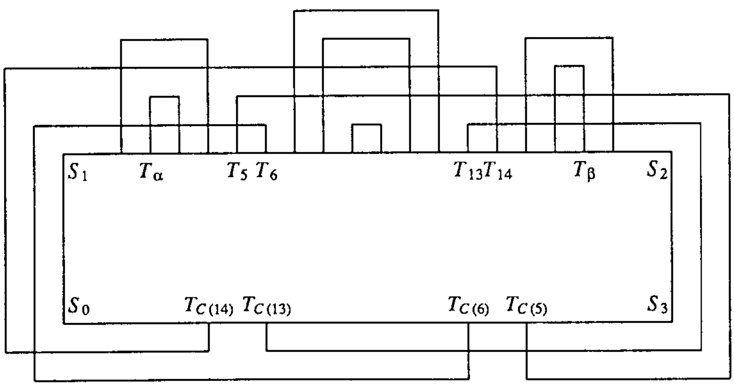
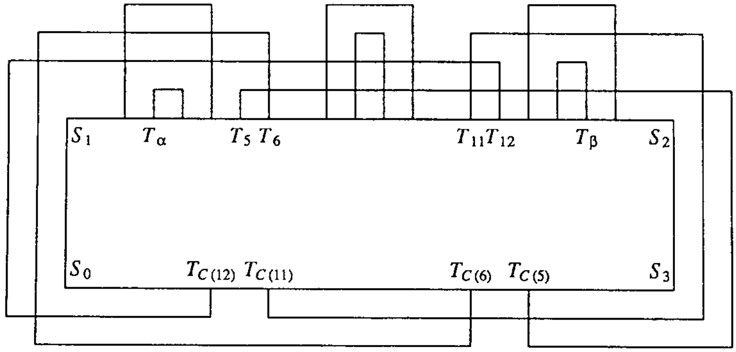


FIGURE 6

satisfying (a) and (b) in Lemma 3.3 and with height on the top side of T equal to $\Delta = 4$, then the assignment for the layout given in Figure 6 is optimal and there is no optimal assignment satisfying the above properties. The proof is by contradiction. Suppose there is an assignment, D, with area less than or equal to $(h + 7\lambda) \cdot (w + 4\lambda)$, satisfying (a) and (b) in Lemma 3.3 and height on the top side of T equal to Δ. Since $D' \subseteq D$ ((a) in Lemma 3.3), we know that all local terminals are connected as in Figure 6. Since the height of D on the top side of T is Δ, then at most one global terminal can be connected by a path that crosses the interval (T_6, T_{13}) on the top side of T. But if there is exactly one of such paths, then D does not satisfy (b) in Lemma 3.3. Hence, it must be that T_5 and T_6 (T_{13} and T_{14}) are connected by paths that cross the left (right) side of T. But then assignment D has height on the bottom side equal to 4 and on the top side equal to $\Delta = 4$. Hence, D has area greater than $(h + 7\lambda) \cdot (w + 4\lambda)$. A contradiction.

In Figure 7 we give an instance that does not have an optimal assignment satisfying (a) and (b) in Lemma 3.3 and whose height on the top side of T is $\Delta + 1$. Let us prove this claim. Clearly, there cannot be an assignment that satisfies (a) and (b) in Lemma 3.3 with height on the top side of T less than $\Delta = 4$ and since there are four global terminals located on the bottom side of T then any feasible assignment must have a height on the bottom side of T of at least two. Hence, the layout depicted in Figure 7 is optimal and there is no optimal assignment with height on the top side of T equal to $\Delta + 1$.

FIGURE 7



Note that in both of these examples $h_\alpha + h_\beta + t$ is even. When this value is odd, one can show that there always exists an optimal assignment whose height on the top side of T is Δ. The proof of this fact is similar to the proof of Lemma 3.3. In the next two lemmas we do not take advantage of this fact; however, it will be used by the final algorithm. A skeptical reader can ignore this fact and obtain another algorithm that is slower than ours.

We now define assignments E_{left} and E_{right}. Later on we show there is an optimal assignment that differs from E_{left} or E_{right} by a set of connecting paths that can be easily characterized. If $h_\alpha + h_\beta + t$ is even, then $E_{\text{left}} = E_{\text{right}} = D'' \cup$ {the leftmost y global terminals located on the bottom side of T that have not yet been included in D'' are connected by a path that crosses the left side of T} $\cup$ {the remaining global terminals (those terminals not included in D'' and not included in the previous partial assignment) are connected by a path that crosses the right side of T}, where y is such that the height at points T_α and T_β for this assignment is identical. When $h_\alpha + h_\beta + t$ is odd, E_{left} is defined similarly, except that the value for y is such that $H_{E_{\text{left}}}(T_\alpha, T_\alpha) = H_{E_{\text{left}}}(T_\beta, T_\beta) + 1$ (see Figure 9). Assignment E_{right} is defined similarly, except that the value for y is such that $H_{E_{\text{right}}}(T_\alpha, T_\alpha) + 1 = H_{E_{\text{right}}}(T_\beta, T_\beta)$. These assignments are formally defined below.

Definition 3.6. *E_{left} and E_{right}.* Let

$M = \{l \mid T_l \text{ is a global terminal located on the bottom side of } T \text{ and } l, C(l) \notin D''\}$;
$M_1 = \{l \mid l \text{ is in } M, l \text{ is the } i\text{th largest value in } M \text{ and } i \le \lceil (h_\alpha + h_\beta + t)/2 \rceil - h_\alpha\}$;
$M_2 = \{l \mid l \text{ is in } M, l \text{ is the } i\text{th largest value in } M \text{ and } i \le \lfloor (h_\alpha + h_\beta + t)/2 \rfloor - h_\alpha\}$;
$E_{\text{left}} = D'' \cup \{l \mid l \in M_1\} \cup \{C(l) \mid l \in (M - M_1)\}$;
$E_{\text{right}} = D'' \cup \{l \mid l \in M_2\} \cup \{C(l) \mid l \in (M - M_2)\}$.

Clearly, if $h_\alpha + h_\beta + t$ is odd, then E_{left} and E_{right} will differ only in the starting point for one path (see Figure 8). From the above definitions one can prove the following claim.

CLAIM 2. *For $E \in \{E_{\text{left}}, E_{\text{right}}\}$, E is an assignment for T; $D'' \subseteq E$; and $|H_E(T_\alpha, T_\alpha) - H_E(T_\beta, T_\beta)| \le 1$.*

In what follows we define some terms for assignment E, which can be either E_{left} or E_{right}. When we make use of the terms defined this way, it will be explicitly indicated which of E_{left} or E_{right} was used in the definition.

Definition 3.7. *R, T_l' and T_l''.* Let $R = H_E(S_1, S_2) - \Delta$. For $l = 1, 2, \ldots, R$, let T_l' (T_l'') represent the rightmost (leftmost) terminal located on the interval $[S_1, S_2]$ whose height is $\Delta + l$ and let T_0' (T_0'') represent T_β (T_α).

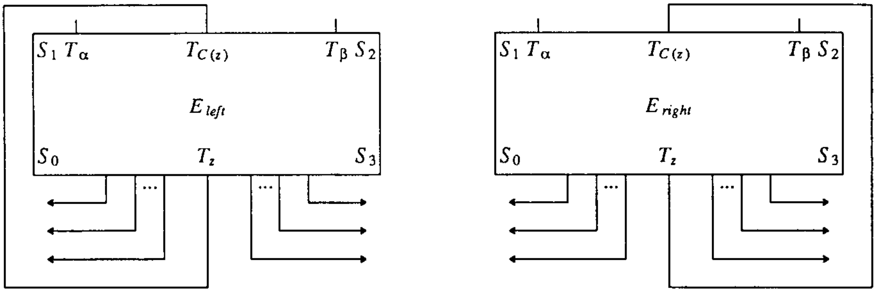


FIG. 8. $E_{\text{left}} \neq E_{\text{right}}$.

If for assignment E the value of R is 0 then T'_R is T_α and T''_R is T_β, and as we show in the next two lemmas if there is an optimal assignment with height equal to Δ then assignment E is optimal. An assignment D that includes D'' is said to *conform* to assignment $E \in \{E_{\text{left}}, E_{\text{right}}\}$ (and E *conform* to D) if $H_D(T_\alpha, T_\alpha) = H_E(T_\alpha, T_\alpha)$, that is, the number of global terminals connected by a path crossing the left side of T in D is the same as the number of global terminals connected by a path crossing the left side of T in E.

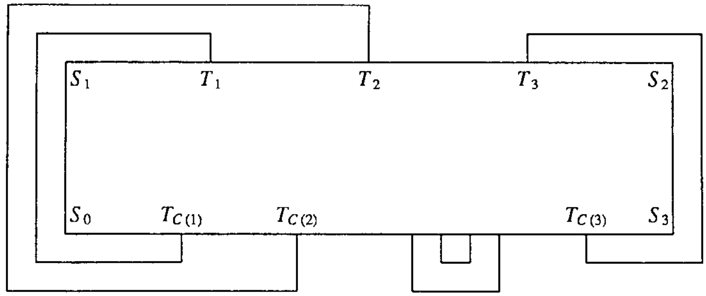
CLAIM 3. *Any optimal assignment, D, that satisfies conditions* (*a*) *and* (*b*) *in Lemma* 3.3 *conforms to the assignment* E_{left} *or* E_{right}.

PROOF. Since D'' is included in D, E_{left} and E_{right}, we know that for each of these assignments the sum of the heights at points T_α and T_β is equal to $h_\alpha + h_\beta + t$. When this value is even then from condition (b) in Lemma 3.3 we know that the heights at points T_α and T_β is the same. This property is also satisfied by $E_{\text{left}} = E_{\text{right}}$. Hence, D conforms to E_{left} and to E_{right}. When $h_\alpha + h_\beta + t$ is odd then we know that in assignment D the heights at points T_α and T_β differs by 1. Hence, D conforms to E_{left} if $H_D(T_\alpha, T_\alpha) > H_D(T_\beta, T_\beta)$, and D conforms to E_{right}, otherwise. □

We define E_{left} and E_{right} because there are instances for which there is no optimal assignment that conforms to E_{left}. This is also true for E_{right}. The problem depicted in Figure 9 does not have an optimal solution that conforms to E_{right}. Note that this happens only when $h_\alpha + h_\beta + t$ is odd.

Example. Let D be an optimal assignment that satisfies (a)–(c) in Lemma 3.3 and let $E \in \{E_{\text{left}}, E_{\text{right}}\}$ conform to D. A connecting path is said to be an *RDLE*(*LDRE*) *path* if it crosses the right (left) side of T in D but not in E. Clearly, the only difference between assignments D and E is because of RDLE and LDRE paths. In Lemma 3.4 we show that there is an optimal assignment D that satisfies Lemma 3.3 with the same number of RDLE and LDRE paths and the number of such paths is either $\lfloor R/2 \rfloor$ or $\lceil R/2 \rceil$. Furthermore, we know the region on the top side of T where the terminals, that these paths connect, are located. In Lemma 3.5 we show that there is an optimal assignment, D, such that in the assignment $E \in \{E_{\text{left}}, E_{\text{right}}\}$ that conforms to D we can precisely identify the terminals that the RDLE and LDRE paths connect. The identification of these terminals can be carried out by a simple greedy method. The main idea behind our algorithm is to construct assignments E_{left} and E_{right}, and then identify the RDLE and LDRE paths in them. Once this process is completed we reverse the direction of the RDLE and LDRE paths to obtain an optimal assignment.

FIGURE 9



LEMMA 3.4. *There is an optimal assignment D such that $E \in \{E_{left}, E_{right}\}$ conforms to it and*

(*a*) $D'' \subseteq D$;
(*b*) $|H_D(T_\alpha, T_\alpha) - H_D(T_\beta, T_\beta)| \leq 1$;
(*c*) $\Delta \leq H_D(S_1, S_2) \leq \Delta + 1$;
(*d*) *Each RDLE (LDRE) path connects a terminal located on the interval* $[T'_R\ T_\beta)$ $((T_\alpha, T''_R])$; *and*
(*e*) *The number of RDLE and LDRE paths is* $\lceil R/2 \rceil$ *if* $H_D(S_1, S_2) = \Delta$ *and* $\lfloor R/2 \rfloor$, *otherwise.*

PROOF. From Lemma 3.3 we know there is an optimal assignment that satisfies (a), (b), and (c); and from Claim 3 we know that any of such assignments conforms to $E \in \{E_{\text{left}}, E_{\text{right}}\}$. We now show that at least one of these optimal assignments satisfies (d) and (e). This will be shown by contradiction. Suppose all optimal assignments satisfying (a), (b), and (c) do not satisfy (d) or (e). Let D be one of these assignments that differs the least with E (the assignment that conforms to it). There are two cases depending on which of (d) or (e) is violated.

Case 1. (d) is violated. There are two subcases depending on how (d) is violated.

Subcase 1.1. There is an LDRE path connecting a terminal located on the interval (T''_R, T_β).

Let T_l (T_r) be the rightmost (leftmost) terminal located on the top side of T connected by an LDRE (RDLE) path. By assumption the LDRE path exists. The existence of an RDLE path is guaranteed by the fact that if there are more LDRE paths than RDLE paths, then (b) does not hold since we know E conforms to D. Let $M = (D - \{C(l), r\}) \cup \{l, C(r)\}$. We now show that M is an optimal assignment satisfying (a)–(c) and that it has less disagreement with E and D. It is simple to show that M satisfies (a) and (b) and that it has less disagreement with E and D. Now, since the connecting paths for T_l and T_r overlap on the bottom side of T in D (this can be seen from the definition of assignment E), $H_D(S_3, S_0) \geq H_M(S_3, S_0)$. Therefore, to establish a contradiction we only need to prove that $H_D(S_1, S_2) \geq H_M(S_1, S_2)$.

A contradiction can be easily obtained if it is the case that T_r is located to the left of T_l. So let us assume that T_r is located to the right of T_l (see Figure 10).

Again, since E conforms to D and D satisfies (b), we know that there are the same number of RDLE and LDRE paths. Let f be this number. Since there are at most $f - 1$ LDRE paths connecting a terminal in $[S_1, T''_R]$ and all the f RDLE paths connect a terminal located on the interval $(T''_R, S_2]$, we know that

$$H_D(T''_R, T''_R) \geq H_E(T''_R, T''_R) - (f-1) + 1 - f = H_E(T''_R, T''_R) - 2f + 2.$$

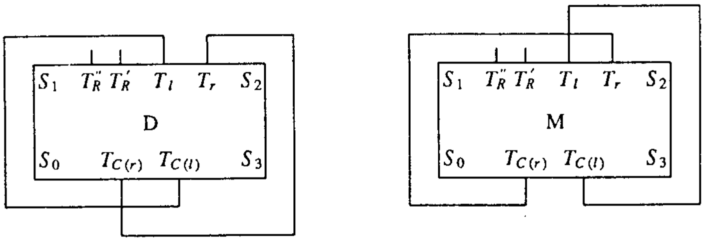


FIGURE 10

With respect to M and E there are only $f - 1$ LDRE and RDLE paths. Since all the LDRE paths (with respect to M and E) are located on the interval $[S_1, T_l)$ and all the RDLE paths (with respect to M and E) are located on the interval $(T_r, S_2]$, we know that $H_M(T_k, T_k) = H_E(T_k, T_k) - 2(f - 1)$ for every $T_k \in [T_l, T_r]$. Now, by the definition of T_R'' we know $H_E(T_k, T_k) \leq H_E(T_R'', T_R'')$ for every $T_k \in [T_l, T_r]$. Hence, $H_M(T_k, T_k) \leq H_D(T_R'', T_R'')$ for every $T_k \in [T_l, T_r]$. Clearly, $H_M(S_1, T_k) = H_D(S_1, T_k)$ for every $T_k \in [S_1, T_l)$ and $H_M(T_k, S_2) = H_D(T_k, S_2)$ for every $T_k \in (T_r, S_2]$. From these inequalities we know $H_M(S_1, S_2) \leq H_D(S_1, S_2)$, which as established before is a contradiction.

Subcase 1.2. There is an RDLE path connecting a terminal located on the interval (T_α, T_R').

A contradiction for this case can be obtained by applying arguments similar to those in the previous subcase.

Hence if (d) is violated there is a contradiction.

Case 2. Assignment D satisfies (d) but not (e).

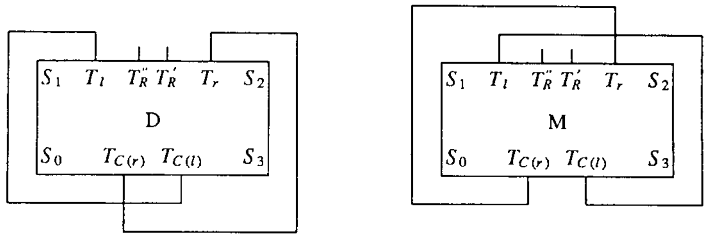
Again, since E conforms to D and D satisfies (b), we know that there are the same number of RDLE and LDRE paths. Let f be this number. Let T_k represent T_R'' if T_R'' is not connected by an LDRE or RDLE path, let T_k represent a point $\epsilon < \lambda$ units to the right of T_R'' if T_R'' is connected by an LDRE path and let T_k represent a point $\epsilon < \lambda$ to the left of T_R'' if T_R'' is connected by an RDLE path (note that this can happen only when $T_R'' = T_R'$). Assume that the definition of the height function has been extended to include T_k when T_k does not correspond to a terminal point. Clearly, $H_E(T_k, T_k) - H_D(T_k, T_k) = 2f$. Also, if $H_D(S_1, S_2) = \Delta + a$, where a is 0 or 1, then $H_E(T_k, T_k) - H_D(T_k, T_k) \geq R - a$. Hence, the value for f is at least as large as the bounds given in (e). If equality occurs, there is nothing to prove. So assume that f is larger than the bound in (e). Therefore, $R \geq 1$ and there is at least one RDLE and one LDRE path. Let T_r (T_l) be the leftmost (rightmost) terminal located on the top side of T connected by an RDLE (LDRE) path. Since D satisfies (d), we know that $l < r$. Let $M = (D - \{C(l), r\}) \cup \{l, C(r)\}$. We now show that M is an optimal assignment satisfying (a)–(d) and that it has less disagreement with E than D. It is simple to show that M satisfies (a), (b), and (d) and that it has less disagreement with E than D. Also, one can easily show that

$$H_D(S_3, S_0) \geq H_M(S_3, S_0) \quad \text{and} \quad H_M(T_k, T_k) = H_D(T_k, T_k)$$
$$\text{for every} \quad T_k \in [S_1, T_l) \cup (T_r, S_2].$$

Hence, a contradiction can be obtained by showing that $H_D(S_1, S_2) \geq H_M(T_k, T_k)$ for every $T_k \in [T_l, T_r]$ (see Figure 11). Clearly,

$$H_M(T_k, T_k) = H_E(T_k, T_k) - 2 \cdot f + 2 \quad \text{for} \quad \textit{every } T_k \in [T_l, T_r],$$

FIGURE 11



where f is the number of RDLE (or LDRE) paths for D and E,

$$H_D(T''_R, T''_R) = H_E(T''_R, T''_R) - 2 \cdot f, \qquad \text{(eq. 1)}$$

if the path connecting T''_R is the same in D and E, and

$$H_D(T''_R, T''_R) = H_E(T''_R, T''_R) - 2 \cdot f + 1, \qquad \text{(eq. 2)}$$

if the path connecting T''_R differs in D and E. From the definition of T''_R and T'_R we know $H_E(T_k, T_k) \le H_E(T''_R, T''_R) = H_E(T'_R, T'_R)$, for every $T_k \in [T_l, T_r]$. From these inequalities we know

$$H_M(T_k, T_k) \le H_D(T''_R, T''_R) + 2 \quad \text{for } \textit{every } T_k \in [T_l, T_r], \qquad \text{(eq. 3)}$$

if the path connecting T''_R is the same in D and E, and

$$H_M(T_k, T_k) \le H_D(T''_R, T''_R) + 1 \quad \text{for } \textit{every } T_k \in [T_l, T_r], \qquad \text{(eq. 4)}$$

if the path connecting T''_R differs in D and E. Now if $H_D(S_1, S_2) = \Delta$, then substituting $f > \lceil R/2 \rceil = \lceil (H_E(T''_R, T''_R) - \Delta)/2 \rceil$ in eq. 1 and eq. 2, we know that $H_D(T''_R, T''_R) \le H_E(T''_R, T''_R) - 2 \cdot (\lceil (H_E(T''_R, T''_R) - \Delta)/2 \rceil + 1)$, if the path connecting T''_R is the same in D and E, and $H_D(T''_R, T''_R) \le H_E(T''_R, T''_R) - 2 \cdot (\lceil (H_E(T''_R, T''_R) - \Delta)/2 \rceil + 1) + 1$, if the path connecting T''_R differs in D and E.
Simplifying,

$H_D(T''_R, T''_R) \le \Delta - 2$ if the path connecting T''_R is the same in D in E, and

$H_D(T''_R, T''_R) \le \Delta - 1$ if the path connecting T''_R differs in D and E.

Similarly, when $H_D(S_1, S_2) = \Delta + 1$, one can prove that

$H_D(T''_R, T''_R) \le \Delta - 1$ if the path connecting T''_R is the same in D and E, and

$H_D(T''_R, T''_R) \le \Delta$ if the path connecting T''_R differs in D and E.

Substituting the last two pairs of inequalities in (eq. 3) and (eq. 4), we know that for every $T_k \in [T_l, T_r]$,

$$H_M(T_k, T_k) \le \Delta \qquad (\text{if } H_D(S_1, S_2) = \Delta),$$

and

$$H_M(T_k, T_k) \le \Delta + 1 \quad (\text{if } H_D(S_2, S_2) = \Delta + 1).$$

Hence, $H_M(T_k, T_k) \le H_D(S_1, S_2)$ for every $T_k \in [T_l, T_r]$ and as it was established before, there is a contradiction. □

Sets of terminals will be defined and subsets of them will be labeled A', A'', B', and B''. These sets will be used in Lemma 3.5 where it will be shown that there is an optimal assignment that differs from E_{left} or E_{right} by the set of paths that connect the terminals labeled A' and A'', or B' and B''. Consequently in order to construct an optimal assignment it is only required to construct E_{left} and E_{right}, and then interchange some set of connecting paths. One of the assignments obtained this way will be an optimal assignment. In sets P'_l (P''_l) defined below we identify all the terminals that could possibly be connected by an RDLE (LDRE) path.

Definition 3.8. Sets P'_l and P''_l. For $l = 1$ to R, let P'_l (P''_l) be the set of global terminals located on the interval $[T'_l, T_\beta)((T_\alpha, T''_l])$ connected by a path crossing the left (right) side of T in assignment E.

CLAIM 4. *For $E \in \{E_{left}, E_{right}\}$ and $1 \le l \le R$, $|P'_l| \ge \lceil l/2 \rceil$ and $|P''_l| \ge \lceil l/2 \rceil$.*

PROOF. The proof is by contradiction. Suppose that for some problem instance an assignment $E \in \{E_{\text{left}}, E_{\text{right}}\}$ violates the above inequalities, that is, for some $1 \le l \le R$, either $|P'_l| < \lceil l/2 \rceil$ or $|P''_l| < \lceil l/2 \rceil$. Since a contradiction can be obtained by applying similar arguments in both cases, assume that for some $1 \le l \le R$, $|P'_l| < \lceil l/2 \rceil$. Let $W_1 = \{i \mid T_i$ is a global terminal located on the interval $[S_1, T'_l)$ that is connected by a path that crosses the left side of T in $E\}$. The assignment R in Definition 3.1 is assignment E after reversing the paths in $W_1 \cup P'_l$. Hence,

$$H_E(T'_l, T'_l) \le H_R(T'_l, T'_l) - |W_1| + |P'_l|$$

and

$$H_E(T_\beta, T_\beta) = H_R(T_\beta, T_\beta) - |W_1| - |P'_l|.$$

From the definition of β we know that $H_R(T'_l, T'_l) \le H_R(T_\beta, T_\beta)$. Combining these inequalities we know that

$$H_E(T_\beta, T_\beta) \ge H_E(T'_l, T'_l) - 2|P'_l|.$$

Subsituting $H_E(T'_l, T'_l) = \Delta + l$ and $|P'_l| < \lceil l/2 \rceil$ in the above inequality, we know that $H_E(T_\beta, T_\beta) \ge \Delta + 1$. But, from the definition of Δ and E we know that $H_E(T_\beta, T_\beta) \le \Delta$. A contradiction. □

Let us now explain the reason behind our labeling procedures. Let D be an optimal assignment with height on the top side of T equal to Δ that satisfies the conditions of Lemma 3.4. For simplicity, let us assume that $\lceil R/2 \rceil$ is odd. From Lemma 3.4 we know that on the interval $[T'_R, T_\beta)$ there are exactly $\lceil R/2 \rceil$ terminals connected by RDLE paths. For all k, the interval $[T'_{R-2k}, T_\beta)$ contains at least $\lceil R/2 \rceil - k$ terminals connected by RDLE paths as otherwise one can show that the height on the top side of T for assignment D is $> \Delta$. Hence, at least one RDLE path connects a terminal in P'_1, at least two RDLE paths connect terminals in P'_3, and so on. Let us now consider how we can determine which of the terminals in set P'_1 is connected by an RDLE path. Let i and j be any two indices of terminals in set P'_1. If $C(i) > C(j)$, T_i is connected by an RDLE path and T_j is not connected by an RDLE path, then one can obtain another optimal solution by reversing the paths that connect terminals T_i and T_j in assignment D. The optimality of this new assignment follows from the fact that the height on the bottom side of T did not increase with the interchange and the new assignment can be shown to have height on the top side of T equal to Δ. This suggests that there is an optimal solution such that the terminal with index i is connected by a path type RDLE if for all other indices j in P'_1 it is the case that $C(i) < C(j)$. The same arguments apply for set P'_3 and the remaining sets as well as for the sets P''_l's. In what follows we label terminals A' and A'' following the procedure just described and in the next lemma we show that if there is an optimal solution with height on the top side equal to Δ that satisfies the previous lemma, then there is an optimal assignment in which all terminals labeled A' (A'') are connected by RDLE (LDRE) paths. The labeling B' and B'' are used when there is an optimal assignment satisfying the previous Lemma and with height on the top side of T equal to $\Delta + 1$.

Definition 3.9. A', A'', B' and B'' labeling. We label terminals as follows:

A′ labeling

for $l = 1$ **to** R **by** 2 **do**
  Let $X = \{C(i) \mid T_i \in P'_l$ and T_i was not labeled A' when considering sets $P'_1, P'_3, \ldots, P'_{l-2}\}$;
  Label A' terminal T_i, where i is such that $C(i) = \mathbf{min}\{x \mid x \in X\}$;
endfor

A″ labeling

This labeling procedure is identical to the previous one, except for the sets P'_l being replaced by P''_l and "**min**" is replaced by "**max**".

B′ labeling

for $l = 2$ **to** R **by** 2 **do**
  Let $X = \{C(i) \mid T_i \in P'_l$ and T_i was not labeled B' when considering sets $P'_2, P'_4, \ldots, P'_{l-2}\}$;
  Label B' terminal T_i, where i is such that $C(i) = \mathbf{min}\{x \mid x \in X\}$;
endfor

B″ labeling

This labeling procedure is identical to the previous one, except for the sets P'_l being replaced by P''_l and

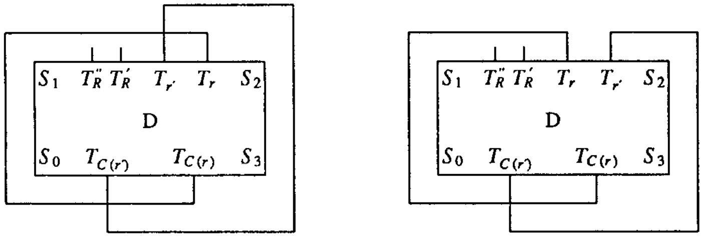
"**min**" is replaced by "**max**".

From Claim 4 we know that during each iteration of the above labeling procedures the set X contains at least one element and thus one terminal will be labeled. The next lemma establishes that there is an optimal assignment which can be obtained by starting from E_{left} or E_{right} and reversing the path connecting the terminals labeled (A' and A'') or (B' and B'').

LEMMA 3.5. *There is an optimal assignment D, such that $E \in \{E_{left}\ E_{right}\}$ conforms to it and*

(*a*) $D'' \subseteq D$;
(*b*) $|H_D(T_\alpha, T_\alpha) - H_D(T_\beta, T_\beta)| \leq 1$;
(*c*) $\Delta \leq H_D(S_1, S_2) \leq \Delta + 1$;
(*d*) *Each RDLE (LDRE) path connects a terminal located on the interval* $[T'_R, T_\beta)((T_\alpha, T''_R])$;
(*e*) *The number of RDLE and LDRE paths in D is* $\lceil R/2 \rceil$ *if* $H_D(S_1, S_2) = \Delta$ *and* $\lfloor R/2 \rfloor$ *otherwise*; *and*
(*f*) *If* $H_D(S_1, S_2) = \Delta$, *then a terminal is connected by an RDLE (LDRE) path in D iff the terminal is labeled* A' (A''). *If* $H_D(S_1, S_2) = \Delta + 1$, *then a terminal is connected by an RDLE (LDRE) path in D iff the terminal is labeled* B' (B'').

PROOF. From the previous lemma we know there is at least one optimal assignment satisfying (a)–(e) and from Claim 3 we know that any of such assignments conforms to assignment $E \in \{E_{\text{left}}, E_{\text{right}}\}$. We now show that at least one of these assignments satisfies (f). The proof is by contradiction. For each assignment, D, that satisfies (a)–(e) and with $H_D(S_1, S_2) = \Delta$, we define the function $K(D)$ as $l' + l''$, where l' (l'') is either the smallest integer such that terminal, T_k, is not connected by an RDLE (LDRE) path but it was the l' (l'') terminal to be labeled



(a) (b)

FIGURE 12

A' (A''), or $R + 1$ if all the terminals connected by RDLE (LDRE) paths are labeled A' (A''). The definition of $K(D)$ when $H_D(S_1, S_2) = \Delta + 1$ is similar, except that instead of using A' and A'', we use B' and B''. Let D be an optimal assignment satisfying (a)–(e) with the largest value for $K(D)$. Clearly, the previous lemma guarantees the existence of at least one of these assignments. Now, since there are the same number of RDLE and LDRE paths as the number of terminals labeled A' and A'' (B' and B'') if $H_D(S_1, S_2) = \Delta$ (if $H_D(S_1, S_2) = \Delta + 1$), we know that there must be a terminals labeled A' or A'' (B' or B'') that is not connected by an RDLE or LDRE path. If $H_D(S_1, S_2) = \Delta$ ($H_D(S_1, S_2) = \Delta + 1$), let T_r be a terminal such that it is neither connected by an RDLE or an LDRE path, but for the least value of l it was the lth terminal to be labeled A' or A'' (B' or B''). There are four cases depending on the label of T_r. Remember that if $H_D(S_1, S_2) = \Delta$, T_r is labeled A' or A'', otherwise T_r is labeled B' or B''.

Case 1. $H_D(S_1, S_2) = \Delta$, terminal T_r is labeled A' and T_r is not connected by an RDLE path in D.

Terminal T_r and l are defined above. Clearly, $T_r \in P'_l$. We now show there is a terminal, $T_{r'} \in P'_l$ connected by an RDLE path in D that was not labeled A' when labeling sets $P'_1, P'_3, \ldots, P'_l$. From the labeling procedure and Claim 4 we know that the number of terminals labeled A while considering sets $P'_1, P'_3, \ldots, P'_l$ is $\lceil l/2 \rceil$. Now if $T_{r'}$ does not exist then there are no more than $\lceil l/2 \rceil - 1$ terminals connected by RDLE paths in D for the interval $[T'_l, T_\beta)$. Hence, $H_D(T'_l, T'_l) \geq H_E(T'_l, T'_l) - 2 \cdot (\lceil (H_E(T'_l, T'_l) - \Delta)/2 \rceil - 1) > \Delta$. A contradiction. So, $T_{r'}$ exists.

Now, since both T_r and $T_{r'}$ belong to P'_l and the labeling procedure did not select $T_{r'}$, we know $T_{C(r)}$ is to the right of $T_{C(r')}$ on the bottom side of T. Figure 12 depicts the two possible cases for the relative location of $T_{r'}$ and T_r.

Let $M = (D - \{r', C(r)\}) \cup \{C(r'), r\}$. Clearly, $K(M) > K(D)$. It is simple to show that if we prove that $H_M(S_1, S_2) = \Delta$, then we have established a contradiction. Clearly, if the situation shown in Figure 12a occurs we know there is a contradiction. So, the remaining case is depicted in Figure 12b.

One can easily show that if $H_M(T_k, T_k) \leq \Delta$ for every $T_k \in [T_r, T_{r'}]$ then we obtain a contradiction. This is equivalent to proving that $H_D(T_k, T_k) \leq \Delta - 2$ for every $T_k \in (T_r, T_{r'})$, since this also implies that $H_D(T_r, T_r) \leq \Delta - 1$, and $H_D(T_{r'}, T_{r'}) \leq \Delta - 1$. Let us prove this bound. Let I_z be the interval (T'_z, T_β) for $1 \leq z \leq R$ (note that these intervals do not correspond to the P' intervals) and let T_k be any terminal located on the interval $(T_r, T_{r'})$. Let z be the smallest integer such that $T_k \in I_z$. Let w be the number of paths that are neither RDLE nor LDRE paths and cross T_k in D. Clearly, there are the same number of such paths that cross T_k in E. There are at least $\lfloor z/2 \rfloor$ terminals connected by RDLE paths that were labeled A' when considering sets $P_1', P_3', \ldots, P_l'$ located on the interval $[T_k, S_2]$, since by assumption the first $l-1$ terminals labeled A' are connected by RDLE paths and $z \le l$. Clearly, $T_{r'}$ is to the right of T_r and $T_{r'}$ is connected by an RDLE path not included in the previous bound. Hence at least $\lfloor z/2 \rfloor + 1$ RDLE paths cross T_k in E and since $T_k \in (T_r, T_{r'})$, all LDRE paths cross T_k in E. So, it must be that $H_E(T_k, T_k) \ge w + \lceil R/2 \rceil + \lfloor z/2 \rfloor + 1$. Now, since $T_k \in (T_r, T_{r'})$ then no LDRE path crosses point T_k in D and at least $\lfloor z/2 \rfloor + 1$ RDLE paths do not cross point T_k in D. Therefore, we know that

$$H_D(T_k, T_k) \le w + \lceil R/2 \rceil - \lfloor z/2 \rfloor - 1.$$

From these two inequalities and the fact that $H_E(T_k, T_k) \le \Delta + z - 1$, we know that $H_D(T_k, T_k) \le \Delta - 2$. Hence, $H_M(S_1, S_2) \le \Delta$, and as it was established before there is a contradiction. This completes the proof of this case. □

Case 2. $H_D(S_1, S_2) = \Delta$, terminal T_r is labeled A'' and T_r is not connected by an LDRE path in D.

The proof of this case is similar to Case 1. □

Case 3. $H_D(S_1, S_2) = \Delta + 1$, terminal T_r is labeled B' and T_r is not connected by an RDLE path in D.

The proof of this case is similar to the one for Case 1. It will be included because we feel it explains the reason behind the two labeling procedures. Let T_r and l be as defined above. Clearly, $T_r \in P_l'$. Now we show that there exists a terminal, $T_{r'} \in P_l'$ connected by an RDLE path in D that was not labeled B' when labeling sets $P_2', P_4', \ldots, P_l'$. From the labeling procedure and claim 4 we know that the number of terminals labeled B' while considering sets $P_2', P_4', \ldots, P_l'$ is $l/2$. Now if $T_{r'}$ does not exist then there are no more than $(l/2) - 1$ terminals connected by RDLE paths in D for the interval $[T_l', T_\beta]$. Hence,

$$H_D(T_l', T_l') \ge H_E(T_l', T_l') - 2 \cdot \left(\left\lceil \frac{H_E(T_l', T_l') - \Delta}{2} \right\rceil - 1 \right) > \Delta + 1.$$

A contradiction.

Now, since both T_r and $T_{r'}$ belong to P_l' and the labeling procedure did not select $T_{r'}$, we know $T_{C(r)}$ is to the right of $T_{C(r')}$ on the bottom side of T. Figure 12 depicts the two possible cases for the location of $T_{r'}$ and T_r. Let $M = (D - \{r', C(r)\}) \cup \{C(r'), r\}$. Clearly, $K(M) > K(D)$. It is simple to show that if we prove that $H_M(S_1, S_2) = \Delta + 1$, then we will establish a contradiction. Clearly, if the situation depicted in Figure 12a occurs, we know there is a contradiction. So, the remaining case is depicted in Figure 12b.

One can easily show that if $H_M(T_k, T_k) \le \Delta + 1$ for every $T_k \in [T_r, T_{r'}]$, then we obtain a contradiction. This is equivalent to proving that $H_D(T_k, T_k) \le \Delta - 1$ for every $T_k \in (T_r, T_{r'})$, since this bound implies that $H_D(T_r, T_r) \le \Delta$, and $H_D(T_{r'}, T_{r'}) \le \Delta$. Let us now prove this bound. Let I_z be the interval (T_z', T_β) for $1 \le z \le R$ (remember that these intervals do not correspond to the sets P') and let T_k be any terminal located on the interval $(T_r, T_{r'})$. Let z be the smallest integer such that $T_k \in I_z$. Let w be the number of paths that are neither RDLE or LDRE paths and cross T_k in D. Clearly, there are the same number of such paths that cross T_k in E. There are $\lceil z/2 \rceil - 1$ terminals connected by RDLE paths and labeled B' when considering sets $P_2', P_4', \ldots, P_l'$ located on the interval $[T_k, S_2]$, since by assumption the first $l - 1$ terminals labeled B' are connected by RDLE paths and $z \le l$. Clearly $T_{r'}$ is to the right of T_r and $T_{r'}$ is connected by an RDLE path not included in the previous bound. Hence at least $\lceil z/2 \rceil$ RDLE paths cross T_k in E and since $T_k \in (T_r, T_{r'})$, all LDRE paths cross T_k in E. So, it must be that $H_E(T_k, T_k) \geq w + \lfloor R/2 \rfloor + \lceil z/2 \rceil$. Now, since $T_k \in [T_r, T_{r'})$, then no LDRE path crosses point T_k in D and at least $\lceil z/2 \rceil$ RDLE paths do not cross point T_k in D. Therefore, $H_D(T_k, T_k) \leq w + \lfloor R/2 \rfloor - \lceil z/2 \rceil$. From these two inequalities and the fact that $H_E(T_k, T_k) \leq \Delta + z - 1$, we know tht $H_D(T_k, T_k) \leq \Delta - 1$. Hence, $H_M(S_1, S_2) \leq \Delta$ and as it was established before there is a contradiction.

This completes the proof of this case. □

Case 4. $H_D(S_1, S_2) = \Delta + 1$, terminal T_r is labeled B'' and T_r is not connected by an LDRE path in D.

The proof for this case is similar to the one for Case 3. □

4. *Algorithm and Complexity Issues*

In this section we present our algorithm to solve the 2-R1M problem. The algorithm is based on the lemmas presented in Sections 2 and 3. Our algorithm has worst case time complexity $O(n \log n)$ and $O(n)$ when the set of terminals is initially sorted. In the last part of this section we discuss lower bounds for the worst case time complexity of decision tree algorithms for the 2-R1M problem. The algorithm is given below.

algorithm ROUTING
Rename the set of terminals in such a way that when traversing T in the clockwise direction starting at point S_0, the terminals are visited in the order $T_1, T_2, \ldots, T_{2n}$;
Label the terminals local and global following the definitions that appear after Lemma 2.2;
Construct D'; // def 2.1 //
Partition the problem into the following two subproblems:
  P_1 is the initial problem after deleting all global terminals located on the left and right sides of T, and
  P_2 is the initial problem after deleting all global terminals located on the top and bottom sides of T;
$D_1 \leftarrow$ SOLVE(P_1);
$D_2 \leftarrow$ SOLVE(P_2); // Assume that the rectangle is rotated 90 degrees //
Combine D_1 and D_2 into the final assignment D;
Construct and output the final layout for D using the procedure discussed in the proof of Lemma 2.2 [L];
end of algorithm ROUTING;

procedure SOLVE(P);
Construct D' for P; // def 2.1 //
if there are no global terminals **then return** (D') **endif**;
Compute α and β; // def 3.1 //
Construct D''; // def 3.2 //
Compute t; // def 3.4 //
if $t = 0$ **then** return(D'') **endif**;
Compute Δ; // def 3.5 //
Construct E_{left} and E_{right}; // def 3.6 //
Compute R for E_{left} and E_{right}; // def 3.7 //
Define $T_0', \ldots, T_R', T_0'', \ldots, T_R''$ for E_{left} and E_{right}; // def 3.7 //
Perform the A', A'', B', and B'' labelings for E_{left} and E_{right}; // def 3.9 //
if $h_\alpha + h_\beta + t$ is even **then** // $E_{\text{left}} = E_{\text{right}}$ //
  $D_1 \leftarrow$ MODIFY(A', A'', E_{left});
  $D_2 \leftarrow$ MODIFY(B', B'', E_{left});
  else // there is an optimal assignment with height on the top side of T is equal to Δ //
  $D_1 \leftarrow$ MODIFY(A', A'', E_{left});
  $D_2 \leftarrow$ MODIFY(A', A'', E_{right});
endif
return(D_1 if $A(D_1) \leq A(D_2)$ and D_2 otherwise);
end of procedure

procedure MODIFY(L, L', E)
$D \leftarrow E$ except that the paths connecting all terminals labeled L and L' is reversed;
return(D);
end of procedure

THEOREM 4.1. *Algorithm ROUTING solves the 2-R1M problem.*

PROOF. The proof is based on Lemmas 3.1, 3.2, 3.3, 3.4, and 3.5. □

THEOREM 4.2. *The time complexity of procedure ROUTING is $O(n \log n)$.*

PROOF. The first step in procedure ROUTING (sorting the terminals) takes $\Omega(n \log n)$ time. Once the terminals are sorted all other steps can be easily shown to take $O(n)$ time, except for labelings A', A'', B', and B''. The problem of labeling the terminals has been reduced to the offline min and offline max problems in which the set of elements is restricted to integers in the range of $[1, 2n]$ which can be solved by union and find operations [1]. For this special case Gabow and Tarjan [2] showed that the overall time complexity for these operations is bounded by $O(n)$. Therefore the time complexity for our algorithm is $O(n \log n)$. □

THEOREM 4.3. *The time complexity of procedure ROUTING is $O(n)$ when the set of terminals is initially sorted.*

PROOF. See the proof of Theorem 4.2. □

In [3], it was shown that $\Omega(n \log n)$ comparisons are required by any decision tree algorithm that solves the 1-dimensional 2-R1M problem. This result holds even when comparisons among linear functions are allowed. A similar result can also be proven for the case when the input to the 1-dimensional 2-R1M problem is restricted to terminals located at a distance of at least $\lambda > 0$ units from each other. Clearly, this result also holds for the 2-R1M problem. Hence, the worst case time complexity for the 2-R1M problem is $\Theta(n \log n)$.

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