WORKING PAPER:
The Weighted Variable Power Problem

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1 Variable Power (Hu and Engel)

A *traffic matrix* D is an n by n matrix with non-negative entries. Entry $d_{i,j}$ represents the traffic from site i to site j. To simplify our notation we assume that $0 \leq i < n$, and $0 \leq j < n$. The total amount of *power* available is given by a positive number p The number of *links* that can be active simultaneously is a positive integer c (channels). A (c, p) *switching matrix* (cpSM) is an n by n matrix of nonnegative numbers with at most one nonzero entry in each row or column and at most c nonzero entries altogether. The sum of the entries of a cpSM must not exceed p, i.e., $\sum d_{i,j} \leq p$. A cpSM switching matrix represents the traffic that can be carried during one time slot. The number of *time slots* is a positive integer t.

The *Variable Power* (VP) problem is defined as follows, given p, c, t, and D, to determine whether or not there exist cpSM switching matrices $S_1, S_2, \ldots S_t$ such that $D = \sum_{k=1}^{t} S_k$. Hu and Engel showed that the VP problem is NP-Complete, by reducing 3-partition to it.

3-partition
INSTANCE: A finite set A of $3m$ elements, a bound $B \in Z^+$, and a size $s(a) \in Z^+$ for each $a \in A$, such that, $B/4 < s(a) < B/2$, and $\sum s(a) = mB$
QUESTION: Can A be partitioned into m disjoint sets $A_i, A_2, \ldots, A_m$ such that $\sum_{a \in A_i} s(a) = B > 2$ for $1 \leq i \leq m$? Note that each of the sets in a 3-partition must have exactly there elements.

Theorem 1.1 *The VP problem is NP-complete.*

Proof: It is simple to show that VP is in NP. We now give Hu and Engel's polynomial reduction from 3-partition to VP. Given an instance of 3-partition I3P we construct the instance IVP of VP as follows.

Let D be a $3m$ by $3m$ diagonal matrix with diagonal entries $s(a_1), s(a_2), \ldots, s(a_{3m})$, let $p = B$ be the power constraint, let $t = m$ be the time constant, and let $c = 3$ be the channel constraint. The correspondence of these two problems is obvious.

□

Definition 1.1 *A star network is a network configuration in which a single site (the hub) communicates with all other sites, and each site communicates with the hub.*

A *star variable power* (SVP) problem is the VP problem restricted to all entries in the D matrix being zero except of the ones in row 0 and in column 0.

Theorem 1.2 *The star variable power problem is NP-complete.*

Proof: It is simple to show that SVP is in NP. We now give Hu and Engel's polynomial reduction from 3-partition to SVP.

Given an instance of 3-partition I3P we construct the instance ISVP of SVP as follows.

Let D be a $3m+1$ by $3m+1$ matrix whose entries $d_{i,j}$ are all zero except:

$$d_{0,j} = B \text{ for } 1 \leq j \leq m$$

$$d_{i,0} = B - s(a_i) \text{for } 1 \leq i \leq 3m$$

So, row 0 of D has exactly m nonzero entries, all equal to B. Column 0 of D has exactly $3m$ nonzero entries. All other entries are zeroes, so D is the traffic matrix of a star network. Let $p = B$ be the power constraint. Let $t = 3m$ be the time constant. Let $c = 2$ be the channel constraint.

We now show that ISVP can be scheduled in t time slots iff I3P has a 3-partition. The proof is in two parts.

(a) If I3P has a 3-partition then ISVP has a schedule.

Let $A_1, A_2, \ldots, A_m$ be a 3-partition for I3P, i.e., $A_1, A_2, \ldots, A_m$ is a partition of A into three elements subsets such that the sum of the sizes of the element in each set sums up to exactly B. For each set A_i we construct three switching matrices. For $A_i = \{a, b, c\}$ the first switching matrix has nonzero entries $d_{a,0} = B - s(a)$ and $d_{0,i} = s(a)$, the second switching matrix has nonzero entries $d_{b,0} = B - s(b)$ and $d_{0,i} = s(b)$, and the third switching matrix has nonzero entries $d_{c,0} = B - s(c)$ and $d_{0,i} = s(c)$, Clearly each of the three switching matrices satisfies the power constraint, and the channel constraint. Therefore the concatenation of the m switching matrices constructed from the A_is also satisfies the time constraint and forms a schedule for ISVP.

(b) If ISVP has a schedule, then the I3P has a 3-partition.

Clearly, the sum of all the entries in D is $3mB$. Since $t = 3m$ and $p = B$ it must be that each switching matrix must sum up to B. Since there are $3m$ positive entries in column zero and one can use at most one such entry in each switching matrix, it follows that each entry in column zero must be in exactly one switching matrix. Reorder the switching matrices so that the one for that uses entry $d_{i,0}$ is the i^{th} one. We say that the i^{th} switching matrix represents element a_i. Since $p = B$, each switching matrix matrix must sum to B and one can only use one entry in row zero in each switching matrix, it then follows that exactly one entry in row 0 in the i_{th} switching matrix must be nonzero and has the value $s(a_i)$. Since $D = \sum_{k=1}^{t} S_k$, it must be that the switching matrices that contribute to $d_{0,j}$ must contribute exactly B units, and and the elements a_i these switching matrices represent sum to B. Therefore, A has a 3-partition.

□

2 Weighted Variable Power (New Result)

The Weighted Variable Power (WVP) Problem is a generalization of the VP problem. A *traffic matrix* D is an n by n matrix with non-negative entries. Entry $d_{i,j}$ represents the traffic from site i to site j. To simplify our notation we assume that $0 \leq i < n$, and $0 \leq j < n$. A *weight matrix* W is an n by n matrix with positive entries. Entry $w_{i,j}$ represents the amount of power needed to transfer one unit of traffic the from site i to site j. To simplify our notation we assume that $0 \leq i < n$, and $0 \leq j < n$. The total amount of *power* is given by a positive number p The number of *links* that can be active simultaneously is a positive integer c (channels). A (c, p) *switching matrix* (cpSM) is an n by n matrix of nonnegative numbers with at most FOUR nonzero entry in each row or column and at most c nonzero entries altogether. The total (weighted) amount of power used must not exceed p, i.e., $\sum w_{i,j} \cdot d_{i,j} \leq p$. A cpSM switching matrix represents the traffic that can be carried during one time slot. The number of time *slots* is a positive integer t.

The *Weighted Variable Power* (WVP) problem is, given p, c, t, W, and D, to determine whether or not there exist integer valued cpSM switching matrices $S_1, S_2, \ldots S_t$ such that $D = \sum_{k=1}^{t} S_k$.

The differences between the VP and the WVP problem are: the power constraint is weighted ($\sum w_{i,j} \cdot d_{i,j} \leq p$ rather than just $\sum d_{i,j} \leq p$), the switching matrices are integer valued (rather than real values), and there can be four nonzero entries in each row or column in each switching matrix rather than just one. We show that the WVP problem is NP-Complete. We reduce 3-partition to WVP.

Theorem 2.1 *The WVP problem is NP-complete.*

Proof: It is simple to show that WVP is in NP. We now give polynomial reduction (identical to the one by Hu and Engel) from 3-partition to WVP.

Given an instance of 3-partition I3P we construct the instance IWVP of WVP as follows. Let D be a $3m$ by $3m$ diagonal matrix with diagonal entries $s(a_1), s(a_2), \ldots, s(a_{3m})$, and all the entries in the W matrix have value 1. Let $p = B$ be the power constraint, let $t = m$ be the time constant, and let $c = 3$ be the channel constraint. The correspondence of these two problems is obvious.

□

We now show that the problem remains NP-complete even when the network configuration is a star network.

Theorem 2.2 *The star weighted variable power problem is NP-complete.*

Proof: It is simple to show that SWVP is in NP. We now give a polynomial reduction from 3-partition to SWVP similar to the one by Hu and Engel for the SVP.

Given an instance of 3-partition I3P we construct the instance ISWVP of SWVP as follows. Let D be a $12m + 1$ by $12m + 1$ matrix whose entries $d_{i,j}$ are all zero except:

$$d_{0,j} = B \text{ for } 1 \leq j \leq 10m$$

$$d_{i,0} = B - s(a_i) \text{ for } 1 \leq i \leq 3m$$

$$d_{i,0} = B \text{ for } 3m + 1 \leq i \leq 12m$$

Column zero and row zero for the weight matrix W have the following entry values and the rest are not important.

$$d_{0,j} = 1 \text{ for } 1 \leq j \leq m$$

$$d_{0,j} = B^2 \text{ for } m+1 \leq j \leq 10m$$

$$d_{i,0} = 1 \text{ for } 1 \leq i \leq 3m$$

$$d_{i,0} = B^6 \text{ for } 3m+1 \leq i \leq 12m$$

So, row 0 of D has exactly $10m$ nonzero entries, all equal to B. The first m entries have weight 1, and the remaining ones have weight B^2. Column 0 of D has exactly $12m$ nonzero entries. The first $3m$ entries have weight 1, and the remaining ones have weight B^6. All other entries are zeroes, so D is the traffic matrix of a star network. Let $p = 3B^7 + 3B^3 + B$ be the power constraint. Let $t = 3m$ be the time constant. Let $c = 8$ be the channel constraint (this will always be satisfied because there is only one row and column in D with nonzero values). We now show that ISWVP can be scheduled in t time slots iff the I3P has a 3-partition. The proof is in two parts.

(a) If I3P has a 3-partition than ISWVP has a schedule.

Let $A_1, A_2, \ldots, A_m$ be a 3-partition for I3P, i.e., $A_1, A_2, \ldots, A_m$ is a partition of A into three elements subsets such that the sum of the sizes of the element in each set sums up to exactly B. For each set A_i we construct three switching matrices. For $A_i = \{a, b, c\}$ the first switching matrix has the following nonzero entries, and the corresponding weights are given below.

$$\begin{array}{cccc}
d_{a,0} = B - s(a) & d_{m+9(i-1)+1,0} = B & d_{m+9(i-1)+2,0} = B & d_{m+9(i-1)+3,0} = B \\
w_{a,0} = 1 & w_{m+9(i-1)+1,0} = B^6 & w_{m+9(i-1)+2,0} = B^6 & w_{m+9(i-1)+3,0} = B^6 \\
d_{0,i} = s(a) & d_{0,3m+9(i-1)+1} = B & d_{0,3m+9(i-1)+2} = B & d_{0,3m+9(i-1)+3} = B \\
w_{0,i} = 1 & w_{0,3m+9(i-1)+1} = B^2 & w_{0,3m+9(i-1)+2} = B^2 & w_{0,3m+9(i-1)+3} = B^2
\end{array}$$

the second switching matrix has nonzero entries

$$\begin{array}{cccc}
d_{b,0} = B - s(b) & d_{m+9(i-1)+4,0} = B & d_{m+9(i-1)+5,0} = B & d_{m+9(i-1)+6,0} = B \\
w_{b,0} = 1 & w_{m+9(i-1)+4,0} = B^6 & w_{m+9(i-1)+5,0} = B^6 & w_{m+9(i-1)+6,0} = B^6 \\
d_{0,i} = s(b) & d_{0,3m+9(i-1)+4} = B & d_{0,3m+9(i-1)+5} = B & d_{0,3m+9(i-1)+6} = B \\
w_{0,i} = 1 & w_{0,3m+9(i-1)+4} = B^2 & w_{0,3m+9(i-1)+5} = B^2 & w_{0,3m+9(i-1)+6} = B^2
\end{array}$$

the third switching matrix has nonzero entries

$$\begin{array}{cccc}
d_{c,0} = B - s(c) & d_{m+9(i-1)+7,0} = B & d_{m+9(i-1)+8,0} = B & d_{m+9(i-1)+9,0} = B \\
w_{c,0} = 1 & w_{m+9(i-1)+7,0} = B^6 & w_{m+9(i-1)+8,0} = B^6 & w_{m+9(i-1)+9,0} = B^6 \\
d_{0,i} = s(c) & d_{0,3m+9(i-1)+7} = B & d_{0,3m+9(i-1)+8} = B & d_{0,3m+9(i-1)+9} = B \\
w_{0,i} = 1 & w_{0,3m+9(i-1)+7} = B^2 & w_{0,3m+9(i-1)+8} = B^2 & w_{0,3m+9(i-1)+9} = B^2
\end{array}$$

Multiplying the weights by the transmission we know that the power used is $B + 3B^7 + 3B^3$ which is equal to p. So, each of the three switching matrices satisfies the power constraint, and the channel constraint. Therefore the concatenation of the $3m$ switching matrices also satisfies the time constraint and is a schedule for ISWVP.

(b) If ISWVP has a schedule, then the I3P has a 3-partition.

We now claim that a feasible schedule must have each of its switching matrices with at most $3B$ transmissions with weight B^6. Suppose not, suppose that one such switching matrix has more than $3B$ of such transmissions. Then the power consumed by those entries is at least $3B^7 + B^6$ but that is greater than $p = B + 3B^7 + 3B^3$ because B is at least 2. Therefore every switching matrix in a feasible schedule must have at most $3B$ entries with weight B^6. Since the sum of all the entries in D with weight B^6 is $9mB$, and $t = 3m$, it must be that all the switching matrices in a feasible schedule have exactly $3B$ transmissions with weights B^6.

At this point one can use similar arguments to show that all the switching matrices in a feasible schedule have exactly $3B$ transmissions with weights B^2. Similarly, one can show that all the switching matrices in a feasible schedule have exactly B transmissions with weights 1.

Since there can be at most four different transmissions in each row and in each column, it must then be that each switching matrix in a feasible schedule has exactly three transmissions of B units each with weight B^6, exactly three transmissions of B units each with weight B^2, and two transmissions (one in column zero and one in row zero) of a total of B units with weight 1. Reorder the switching matrices so that the one for that uses entry $d_{i,0}$ is the i^{th} one. We say that the i^{th} switching matrix represents element a_i. Since $D = \sum_{k=1}^{t} S_k$, it must be that the switching matrices that contribute to $d_{0,j}$ must contribute exactly B units, and the elements a_i that these switching matrices represent sum to B. Therefore, A has a 3-partition.

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