CS138: Automata and Formal Languages
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UC Santa Barbara
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Homework 1

Posted: Monday, April 02, 2018 – 11:59pm
Due: Wednesday, April 11, 2018 – 2pm (HFH 2108 or in class)

Instructions and Rules

- **Do attend your section!** While you may lack some background to start solving the homework, this will be discussed in detail in the section on Monday following the posting of the homework. Some more details may be discussed on the day before submission is due.
- Your solution **must be stapled**, and please do not forget to indicate on **every single sheet** (1) Your name and (2) The section you are enrolled in (either **Friday 9am** or **Friday 10am**). Your graded solution will be returned in the session you are enrolled in.
- Write your solutions clearly, with appropriate mathematical rigor. Justify **all steps** of your solution. Partially incorrect solutions can still be worth several points, but unjustified incorrect results will result in zero points for the corresponding question.
- You are not allowed to copy or transcribe answers to homework assignments from others or other sources.
- You are not allowed to post full solutions of your homework on the Piazza Q&A. Moreover, if you use facts from the online discussion, you should provide your own justification in your solution.
- Although you are allowed to discuss homework assignments with others, you must write your answers *independently*. You should always be able to argue and explain your answers when asked for clarifications.
- **If you are unable to hand in the homework in time you must report this to the lecturer (ST)** ***as soon as possible*****, and always before the deadline.** No matter the reason, you will always be asked to present documentation.

Task 1 – Proofs by Induction **(5 points)**

Prove by induction that for all integers $n \geq 1$, we have

$$\sum_{i=1}^{n} (2i-1) = n^2 .$$

Task 2 – Strings and Languages **(7 points)**

a) Let $u = abaabaab$ and $v = bab$ be strings over alphabet $\Sigma = \{a, b\}$. Compute the following strings:

$$\text{(i)}\ \ uv \quad \text{(ii)}\ \ uv^2u \quad \text{(iii)}\ \ v^R u^R \quad \text{(iv)}\ \ ((v)^2 u^R)^2$$

b) Let Σ be an alphabet with $k = |\Sigma|$ elements, and let $\ell \geq 0$. How many strings $u \in \Sigma^*$ are there such that $|u| = \ell$?

c) Let Σ be an alphabet with $k = |\Sigma|$ elements, and let $\ell \geq 0$. How many strings $u \in \Sigma^*$ are there such that $|u| \leq \ell$?

Task 3 – Encodings **(5 points)**

Very often in this class, we will need encode objects into strings. In particular, an encoding of a (possibly infinite) set S with alphabet Σ is a mapping $e : S \to \Sigma^*$ which is *injective*, i.e., for every two $x, x' \in S$, where $x \neq x'$, we have $e(x) \neq e(x')$.

Generally, we would like this encoding to be as "compact" as possible. Given two encodings $e, e' : S \to \Sigma^*$, we say that e is *more compact than* e' if for $|e(x)| \leq |e'(x)|$ for all $x \in S$, and there exists *at least one* $x \in S$ such that $|e(x)| < |e'(x)|$.

a) For the set $S = \{a, b, c, d\}$ and alphabet $\Sigma = \{0, 1\}$, consider the following two encodings e and e':

$$\begin{array}{llll} e(a) = 00 & e(b) = 01 & e(c) = 10 & e(d) = 11 \\ e'(a) = \lambda & e'(b) = 0 & e'(c) = 1 & e'(d) = 10 . \end{array}$$

Which encoding is more compact? Justify your answer!

b) Give an encoding for $\mathbb{N} = \{0, 1, \ldots\}$ with alphabet $\Sigma = \{1\}$. Argue why this is a valid encoding.

c) Give an encoding for $\mathbb{N}$ with alphabet $\Sigma = \{0, 1\}$ which is more compact than the one from **b)**. Explain why this is the case, and why what you have given is a valid encoding.

Task 4 – Properties of Reversion (7 points)

We consider a formal recursive definition of the reverse operation for strings in Σ^*. Namely, the reverse operator R is such that:

- $\lambda^R = \lambda$
- $(ua)^R = au^R$ and $(au)^R = u^R a$ for all $u \in \Sigma^*$, $a \in \Sigma$.

a) Apply the recursive formula step-by-step to compute $(1011)^R$.

Describe every single step in detail.

b) Prove by induction (over the string length) that $(u^R)^R = u$ for all $u \in \Sigma^*$.

Hint: For the induction step, write u as $u'a$, where $a \in \Sigma$ (i.e., it is a single symbol from the alphabet) and $u' \in \Sigma^*$ (i.e., u' is a string which is necessarily shorter than u.)

Task 5 – Palindromes (6 points)

We define the languages $L, L' \subseteq \Sigma^*$ such that

$$L = \left\{ u \in \Sigma^* \ : \ u^R = u \right\} \ , \ \ L' = \left\{ uv \in \Sigma^* \ : \ u = v^R \right\} \ .$$

a) Prove that $L' \subseteq L$.

Hint: Show that every element in L' is also in L. Use that $(u^R)^R = u$ and $(uv)^R = v^R u^R$.

b) Prove that L' is a *proper* subset of L for any Σ, i.e., show that no matter what Σ is, there are elements in L which are not in L'.