CS138: Automata and Formal Languages
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UC Santa Barbara
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Homework 8

Posted: Wednesday, May 30, 2018 – 11:59pm
Due: Friday, June 8, 2018 – 2pm (Gradescope only!)

Task 1 – Countable and Uncountable Sets   (6 points)

Throughout this task, let $\mathbb{N} = \{0, 1, 2 \ldots\}$ be the set of natural numbers. Remember that a set S is *countable* if there exists a surjective function $\phi : \mathbb{N} \to S$, i.e., for every $x \in S$ there exists some $i \in \mathbb{N}$ such that $\phi(i) = x$.

a) Show that the union of two countable sets is countable.

b) Show that the set $\{S \subseteq \mathbb{N} : S \text{ is finite}\}$ is countable.

c) Is the set $\{S \subseteq \mathbb{N} : S \text{ is not finite}\}$ countable?

   Hint: Use **b)**. Moreover, is the power set $2^{\mathbb{N}}$ countable?

Task 2 – Halting Problem   (4 points)

Let $\Sigma = \{0, 1\}$ and $\Gamma = \{0, 1, a, b, \square\}$. Consider the following problem: Given a TM $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ and an input $w \in \Sigma^*$, we want to decide whether M ever writes the symbol a on its tape when run on input w.

Explain why the above problem is undecidable!

Hint: Show how given a TM M^* that decides the above problem, one can solve the halting problem for Turing Machines with input alphabet Σ and tape alphabet $\Gamma = \{0, 1, \square\}$. You can assume that the latter problem is undecidable.

Task 3 – Rice's Theorem   (6 points)

Let Σ be an alphabet, $\mathcal{L} \subseteq 2^{\Sigma^*}$ be a set of languages, and define some procedure enc to encode TMs with input alphabet Σ as binary strings in $\{0, 1\}^*$. Define

$$L_{\mathcal{L}} = \{\text{enc}(M) : L(M) \in \mathcal{L}\} \ .$$

Then Rice's theorem says that $L_{\mathcal{L}}$ is either (1) empty, (2) it contains the encoding of all TMs, or (3) it is not recursive.

Use Rice's theorem to show that the following problems are undecidable:

a) Given enc(M), decide whether $L(M) = \Sigma^*$.

b) Given enc(M), decide whether $L(M)$ is regular.

Task 4 – The Post Correspondence Problem (6 points)

a) Is the following PCP instance a YES-instance? Explain your answer!

a	ab	b	bba
bb	a	bba	a

b) Is the following PCP instance a YES-instance? Explain your answer!

aab	baa	aa
abb	b	bb

c) Show that the problem of deciding whether two CFGs G_1 and G_2 are such that $L(G_1) \cap L(G_2) \neq \emptyset$ (i.e., they generate at least one common string) is undecidable.

In particular, given a PCP instance $u_1, \ldots, u_n, v_1, \ldots, v_n \in \Sigma^*$, build two grammars G_1 and G_2 which generate a common string iff the instance is a YES-instance, and such that $L(G_1)$ and $L(G_2)$ are disjoint if it is a NO-instance.

Task 5 – Closure Properties of Recursive Languages (8 points)

a) Show that if L_1 and L_2 are recursive, then $L_1 \cup L_2$ is recursive. It is also true that if L_1, L_2 are only recursively enumerable, then $L_1 \cup L_2$ is recursively enumerable?

b) Show that if L is recursive, then its complement $\overline{L}$ is recursive.

c) Show that if both L and $\overline{L}$ are recursively enumerable, then L is recursive.

Hint: Give a high-level explanation (i.e., no low-level details) of how you would build a TM for the desired language given TMs for the given languages. Be careful to make sure the resulting TM is valid, i.e., no transition leaves the final state. In some cases, it may be easier to show that a two-tape TM exists (discussed in Tuesday's class), and use the fact that there exists a corresponding single-tape TM.