Answers to Exercises in Quantum Computation III

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Answer 1. (Deutsch-Jozsa on General Functions)
(a) —
(b) After the application of the U_F gate the first n qubits of the system are described by $\sum_x (-1)^{F(x)}|x\rangle/\sqrt{2^n}$, where the summation is over all strings $x \in \{0,1\}^n$. Applying n Hadamard gates to this state gives

$$\mapsto \frac{1}{2^n}\sum_{x\in\{0,1\}}(-1)^{F(x)}\sum_{y\in\{0,1\}^n}(-1)^{x_1y_1+\cdots+x_ny_n}|y\rangle.$$

As a result the inner product between this output state and $|0,\ldots,0\rangle$ is $2^{-n}\sum_x(-1)^{F(x)}$. Because $(-1)^{F(x)} = 1-2F(x)$, this gives the probability of observing the all zeros state

$$\begin{aligned}\Pr(0\cdots 0) &= \frac{1}{2^{2n}}\left(\sum_{x\in\{0,1\}^n}1-2F(x)\right)^2\\ &= \left(1-2\frac{\sum_x F(x)}{2^n}\right)^2.\end{aligned}$$

In other words, it depends quadratically on the Hamming weight $\sum_x F(x)$ of the string $F(0,\ldots,0),\ldots,F(1,\ldots,1)$. The probability is 1 if and only if $\sum_x F(x)$ is 0 or 2^n; the probability is 0 if and only if $\sum_x F(x) = 2^{n-1}$.

Answer 2. (Preparation for Midterm) —

Answer 3. (One-Out-of-Four) Throughout the algorithm the 3rd qubit is used to implement the phase flip trick, and hence remains unchanged. After the first two steps, the first two qubits will be $\frac{1}{2}\sum_{x,y}|x,y\rangle - |s,r\rangle$ where the summation is over all bit combinations $(x,y)\in\{0,1\}^2$. Because we can write this also as $H\otimes H|0,0\rangle - |s,r\rangle$, we see that after step 3 the state equals $|0,0\rangle - H\otimes H|s,r\rangle$. Regardless of the values s,r, the amplitude of the $|0,0\rangle$ component of this state is $\frac{1}{2}$, hence after step 4 (with its $\frac{1}{2}|0,0\rangle \mapsto -\frac{1}{2}|0,0\rangle$), the state will have changed to $-H\otimes H|s,r\rangle$. This shows that after the final transformation of step 5, the state of the system is $-|s,r\rangle$. Given the two unknown bits $s,r\in\{0,1\}$, the measurement will therefore reveal the values s,r with 100% accuracy.
(a) This circuit searches a database of size 4 (encoded by $U_{s,r}$) in a single query. It is a small example of a quantum search algorithms that is more efficient than is possible classically.

Answer 4. (Summing Phases)
(a) Note that

$$\zeta_n\cdot\left(\sum_{j=0}^{n-1}\zeta_n^j\right) = \sum_{j=1}^{n}\zeta_n^j = \sum_{j=0}^{n-1}\zeta_n^j.$$

Assuming $n\neq 1$, we have $\zeta_n \neq 1$, which implies $\sum_{j=0}^{n-1}\zeta_n^j = 0$. (For the degenerate case $n=1$, we have $\sum_{j=0}^{n-1}\zeta_n^j = n$.)

(b) Similarly, note that

$$\zeta_n^q\left(\sum_{j=0}^{p-1}\zeta_n^{qj}\right) = \sum_{j=0}^{p-1}\zeta_n^{qj}.$$

Hence, assuming that $q\neq n$, we have $\zeta_n^q \neq 1$, which gives again $\sum_{j=0}^{p-1}\zeta_n^{qj} = 0$. If $q=n$, then $p=1$ and the summation becomes $\sum_{j=0}^{0}\zeta_n^{nj} = 1$. (The degenerate case $n=1$ obviously gives $\sum_{j=0}^{p-1}\zeta_n^{qj} = p$.)
(c) For $n=1$ the sum will be m. Otherwise, use the standard *geometric sum* equality

$$\sum_{j=0}^{m-1}\zeta_n^j = \frac{1-\zeta_n^m}{1-\zeta_n}.$$