Midterm CS290A: Quantum Information & Quantum Computation

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Question 1. $(10+10$ points) Take the 3 dimensional state space $\mathcal{A} = \{1,2,3\}$ and its *qutrits* $\alpha|1\rangle + \beta|2\rangle + \gamma|3\rangle$, with amplitudes $\alpha,\beta,\gamma \in \mathbb{C}$.
(a) Define the two qutrits $|\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{2}|2\rangle + \frac{\mathrm{i}}{2}|3\rangle$ and $|\phi\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{2}|2\rangle - \frac{1}{2}|3\rangle$. What is the inner product $\langle\psi|\phi\rangle$?
Answer:

$$\langle\psi|\phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{\mathrm{i}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \frac{1+\mathrm{i}}{4}.$$

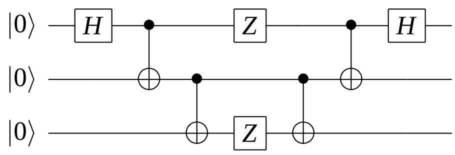
(b) Take the unitary transformation U that has

$$U : |1\rangle \mapsto |3\rangle$$
$$U : |2\rangle \mapsto \tfrac{3}{5}|1\rangle + \tfrac{4}{5}|2\rangle.$$

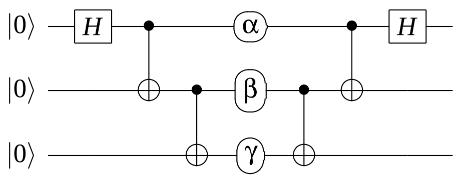
What can you say about the state $U|3\rangle$?
Answer: Because U is unitary, the vector $U|3\rangle$ has to have unit length and it has to be orthogonal to $U|1\rangle$ and $U|2\rangle$. Hence $U|3\rangle = \mathrm{e}^{\mathrm{i}\phi}(\frac{4}{5}|1\rangle - \frac{3}{5}|2\rangle)$ with $\phi \in [0,2\pi)$.

Question 2. $(10+10+10$ points) Take a state $|1,\ldots,1\rangle$ of n qubits that are all "1" and apply k Hadamard gates H to this string at k distinct places. Afterwards, we observe the n qubits in the computational basis $\{0,1\}^n$.
(a) Take $n=3$ and $k=2$, what is the probability of observing the 'all ones' state $|1,1,1\rangle$?
Answer: $\frac{1}{4}$.
(b) As a function of $n \in \mathbb{Z}^+$ and $0 \le k \le n$, what is the probability of observing the 'all ones' state $|1,\ldots,1\rangle$?
Answer: $\frac{1}{2^k}$.
(c) As a function of $n \in \mathbb{Z}^+$ and $0 \le k \le n$, what is the probability of observing the 'all zeros' state $|0,\ldots,0\rangle$?
Answer: If $k<n$ the probability is 0; if $k=n$ then the probability is $\frac{1}{2^n}$.

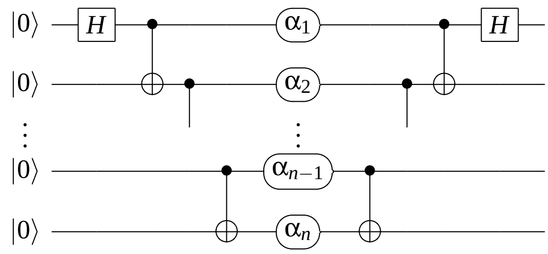
Question 3. $(10+15+10+15$ points)
(a) Consider the following 3 qubit circuit:



With its input state $|0,0,0\rangle$, what will the output state be?
Answer: The output state will be $|0,0,0\rangle$.
(b) Consider the more general circuit



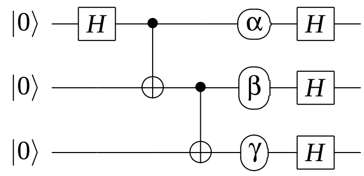
with general angles $\alpha,\beta,\gamma \in [0,2\pi)$ for the phase rotation gates. How does the output state depend on these angles?
Answer: The angles α,β,γ will only effect the first qubit of the output state, which will be $\frac{1}{2}(1+\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)})|0\rangle + \frac{1}{2}(1-\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)})|1\rangle \otimes |0,0\rangle$. Hence the output state is uniquely determined by the sum $\alpha+\beta+\gamma \bmod 2\pi$, and the probability of observing "0,0,0" is $\cos^2(\frac{1}{2}(\alpha+\beta+\gamma))$, while the probability of observing "1,0,0" is $\sin^2(\frac{1}{2}(\alpha+\beta+\gamma))$.
(c) How can you generalize the circuit and the result of the previous question (b) to n qubits?
Answer: The generalized n qubit circuit is



The output of this circuit depends on the sum $\sum_{j=1}^{n} \alpha_j \bmod 2\pi$ and will be

$$\frac{1}{2}(1+\mathrm{e}^{\mathrm{i}(\alpha_1+\cdots+\alpha_n)})|0\rangle + \frac{1}{2}(1-\mathrm{e}^{\mathrm{i}(\alpha_1+\cdots+\alpha_n)})|1\rangle \otimes |0,\ldots,0\rangle.$$

Hence, this time the probability of observing "$0,0,\ldots,0$" is $\cos^2(\frac{1}{2}(\alpha_1+\cdots+\alpha_n))$, while the probability of observing "$1,0,\ldots,0$" is $\sin^2(\frac{1}{2}(\alpha_1+\cdots+\alpha_n))$.
(d) (Save this question for last.) Change the circuit into



How do the output bits depend on the angles $\alpha,\beta,\gamma \in [0,2\pi)$?

Answer: The output state depends on the sum $\alpha+\beta+\gamma \bmod 2\pi$ and will be $\frac{1}{4}(1+\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)})(|0,0,0\rangle + |0,1,1\rangle + |1,0,1\rangle + |1,1,0\rangle) + \frac{1}{4}(1-\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)})(|0,0,1\rangle + |0,1,0\rangle + |1,0,0\rangle + |1,1,1\rangle)$. Hence the probability of observing one of the 3-bit strings with even parity ($\in \{000,011,101,111\}$) is $\cos^2(\frac{1}{2}(\alpha+\beta+\gamma))$, while the probability of observing one of the odd parity, 3-bit strings ($\in \{001,010,100,111\}$) is $\sin^2(\frac{1}{2}(\alpha+\beta+\gamma))$.