CS290A, Spring 2005:

Quantum Information & Quantum Computation

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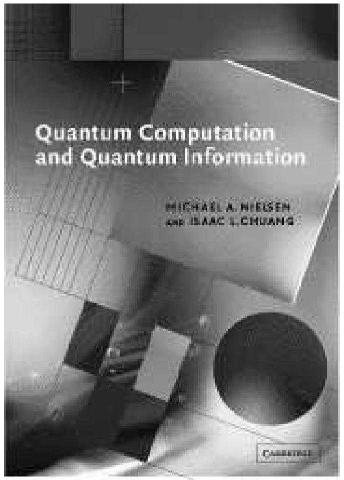
Administrivia



- Who has the book already?
- Office hours: Wednesday 13:30–15:30, otherwise by appointment.
- Midterm: 1/3, Final Project 2/3
- From now on: bring pen & paper:
  We will do calculations in class
- Extra notes will be posted before Thursday.
- Questions?

Loose Ends

What about two slit interference of bullets?

The wave length of a bullet is $\lambda \approx h/mv$ with Planck's constant $h \approx 6.6\times10^{-34}$ J s, and mv the mass$\times$speed of the bullet. Take m = 0.004 kg and v = 1000 m/s, then $\lambda \approx 1.65\times10^{-34}$ meter.

This means that the distance between the slits has to be of the order of $10^3\ \lambda \approx 10^{-31}$ m to have a noticeable effect.

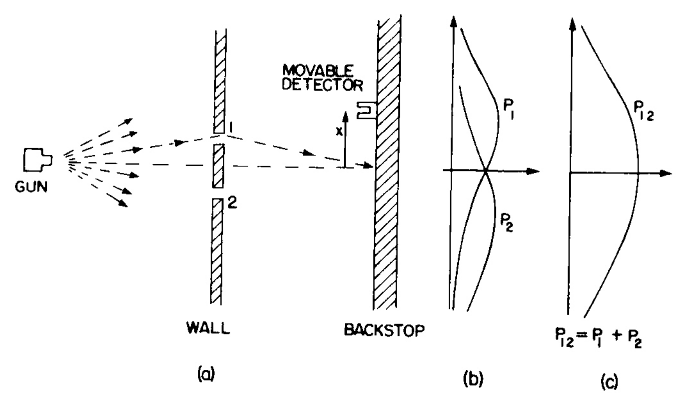
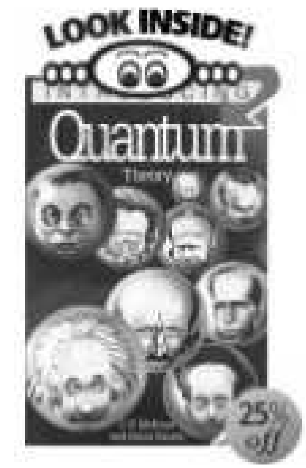


Fig. 1–1. Interference experiment with bullets.

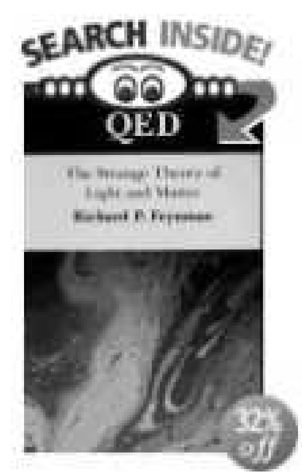
Bed Time Reading

“Painless learning” about quantum physics:

“Introducing Quantum Theory”,
J.P. McEvoy & O. Zarate ($10).



“QED: The Strange Theory of Light and Matter”,
R.P. Feynman ($11).



This Week

Mathematics of Quantum Mechanics:

- (Finite) Hilbert space formalism: vectors, lengths, inner products, tensor products.
- Finite dimensional unitary transformations.
- Projection Operators.

Circuit Model of Quantum Computation:

- Small dimensional unitary transformations as elementary quantum gates.
- Examples of important gates.
- Composing quantum gates into quantum circuits.
- Examples of simple circuits.

Quantum Mechanics

A system with D basis states is in a superposition of all these states, which we can label by {1,…,D}.

Associated with each state is a complex valued amplitude; the overall state is a vector $(\alpha_1,\ldots,\alpha_D)\in\mathbb{C}^D$.

The probability of observing state j is $|\alpha_j|^2$.

When combining states/events you have to add or multiply the amplitudes involved.

Examples of Interference:
Constructive: $\alpha_1=½$, $\alpha_2=½$, such that $|\alpha_1+\alpha_2|^2 = 1$
Destructive: $\alpha_1=½$, $\alpha_2=–½$, such that $|\alpha_1+\alpha_2|^2 = 0$

(Probabilities are similar but with $\mathbb{R}$ instead of $\mathbb{C}$.)

Quantum Bits (Qubits)

- A single quantum bit is a linear combination of a two level quantum system: {“zero”, “one”}.

- Hence we represent that state of a qubit by a two dimensional vector $(\alpha,\beta)\in\mathbb{C}^2$.

- When observing the qubit, we see “0” with probability $|\alpha|^2$, and “1” with probability $|\beta|^2$.
- Normalization: $|\alpha|^2+|\beta|^2=1$.

- Examples: “zero” = (1,0), “one” = (0,1), uniform superposition = $(1/\sqrt{2},1/\sqrt{2})$ another superposition = $(1/\sqrt{2}, i/\sqrt{2})$

Quantum Registers

- A string of n qubits has 2^n different **basis states** $|x\rangle$ with $x \in \{0,1\}^n$. The state of the quantum register $|\psi\rangle$ has thus $N=2^n$ complex amplitudes. In ket notation:

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle \in \mathbb{C}^{2^n}$$

- $|\psi\rangle$ is a column vector, with α_x *in alphabetical order.*
- The probability of observing $x \in \{0,1\}^n$ is $|\alpha_x|^2$.
- The amplitudes have to obey the normalization restriction: $\sum_{x \in \{0,1\}^n} |\alpha_x^2| = 1$

- What can we do with such a state?

Measuring is Disturbing

- If we measure the quantum state $|\psi\rangle$ in the computational basis $\{0,1\}^n$, then we will measure the outcome $x \in \{0,1\}^n$ with probability $|\alpha_x|^2$.

- For the rest, this outcome is fundamentally random. (Quantum physics predicts probabilities, not events.)

- Afterwards, the state has 'collapsed' according to the observed outcome: $|\psi\rangle \mapsto |x\rangle$, which is irreversible: all the prior amplitude values α_y are lost.

Time Evolution

- Given a quantum register $|\psi\rangle$, what else can we do besides measuring it? Answer: rotating it by T.

- Remember that $|\psi\rangle$ is a length one vector and if we change it, the outcome $|\psi'\rangle = T|\psi\rangle$ should also be length one: “T is a norm preserving transformation”.
- Experiments show that QM is linear: T has to be linear.

- Hence, if T acts on a D-dimensional state space, then T can be described by a $D \times D$ matrix $T \in \mathbb{C}^{D \times D}$.

- “T is norm-preserving: T is a (unitary) rotation.”

Classical Qubit Transformations

- Some simple qubit (D=2) transformations:

- Identity with Id:$|\psi\rangle \mapsto |\psi\rangle$ for all ψ: $\mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- NOT gate with NOT:$|0\rangle \mapsto |1\rangle$ and NOT:$|1\rangle \mapsto |0\rangle$;
by linearity we have NOT:$\alpha|0\rangle+\beta|1\rangle \mapsto \alpha|1\rangle+\beta|0\rangle$

- Note that NOT is norm-preserving:
If $|\psi\rangle$ has norm one, then so has NOT$|\psi\rangle$ $\mathrm{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- Also note: NOT:$(|0\rangle+|1\rangle)/\sqrt{2} \mapsto (|1\rangle+|0\rangle)/\sqrt{2}$:
the uniform superposition remains unchanged.

Hadamard Transfrom

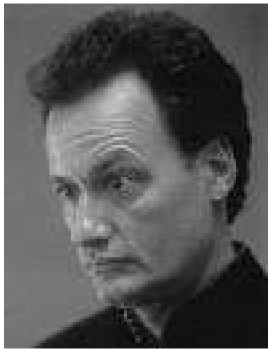
- Define the Hadamard transform: $$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
- We have for this H:

$$\begin{aligned} |0\rangle &\mapsto \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |1\rangle &\mapsto \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) &\mapsto |0\rangle \\ \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle) &\mapsto |1\rangle \end{aligned}$$

- Note: $H^2 = \mathrm{Id}$.
  It changes classical bits
  into superpositions
  and vice versa.
- It sees the difference between the uniform superpositions $(|0\rangle+|1\rangle)/\sqrt{2}$ and $(|0\rangle-|1\rangle)/\sqrt{2}$.

Hadamard Norm Preserving?

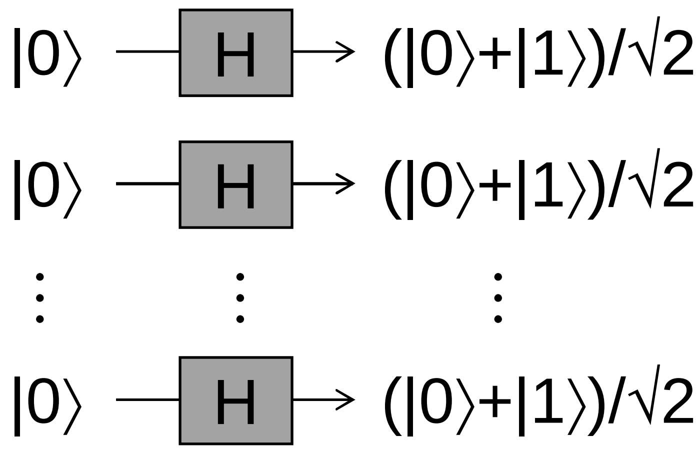
- Is H a proper quantum transformation?
  Linear: Yes, by definition, it is a matrix.
  Norm preserving? Hmmm….
- We have: $H : \alpha|0\rangle + \beta|1\rangle \quad \mapsto \quad \frac{(\alpha+\beta)}{\sqrt{2}}|0\rangle + \frac{(\alpha-\beta)}{\sqrt{2}}|1\rangle$
- Q: If $|\alpha|^2+|\beta|^2 = 1$, then also $½|\alpha+\beta|^2 + ½|\alpha-\beta|^2 = 1$?
- Use complex conjugates*: $|\alpha|^2 = \alpha\cdot\alpha^*$.
- Answer: Yes.



Hadamard as a Quantum Gate

- Often we will apply the H gate to several qubits.
- Take the n-zeros state $|0,\ldots,0\rangle$ and perform in parallel n Hadamard gates to the zeros, as a circuit:

Starting with the all-zero state and with only n elementary qubit gates we can create a uniform superposition of 2^n states.



Typically, a quantum algorithm will start with this state, then it will work in "quantum parallel" on all states at the same time.

As an equation:

$$|0,\ldots,0\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle$$

Combining Qubits

If we have a qubit $|x\rangle = (|0\rangle+|1\rangle)\sqrt{2}$, then 2 qubits $|x\rangle$ give the state $\frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle)$.

Tensor product notation for combining states $|x\rangle\in\mathbb{C}^N$ and $|y\rangle\in\mathbb{C}^M$: $|x\rangle\otimes|y\rangle = |x\rangle|y\rangle = |x,y\rangle \in \mathbb{C}^{NM}$.

Example for two qubits: $(\alpha_0|0\rangle + \alpha_1|1\rangle)\otimes(\beta_0|0\rangle + \beta_1|1\rangle)$

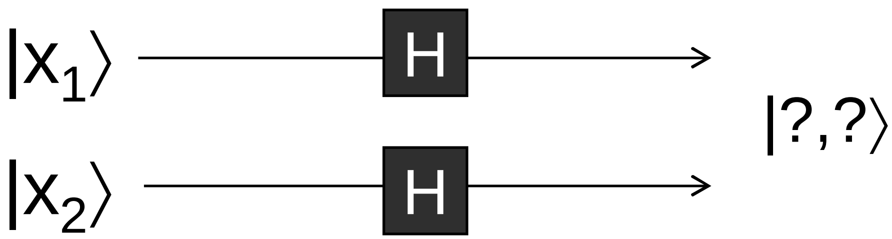
$$= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

Note that we *multiply* the amplitudes of the states.

Also note the exponential growth of the dimensions.

Two Hadamard Gates

What does this circuit do on {00,01,10,11}?



What is the $\mathbb{C}^{4\times4}$ rotation matrix of this operation?

What is the effect of n parallel Hadamard gates?

How does the corresponding $2^n\times2^n$ matrix look like?