Problem 1

SOLUTION HW 2

PHYS 105 A

$\vec{v}(t)$ = velocity of a particle $\qquad |\vec{v}|$ = magnitude of $\vec{v}$

$\frac{d\vec{v}}{dt}$ = acceleration

ⓐ We need to prove that $\frac{d\vec{v}}{dt} \perp \vec{v} \Rightarrow |\vec{v}| = \text{constant}$

We know

$$\frac{d\vec{v}}{dt} \perp \vec{v} \Rightarrow \frac{d\vec{v}}{dt} \cdot \vec{v} = 0$$

We consider $\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt}$

$$\frac{d}{dt}(\vec{v}\cdot\vec{v}) = 2\,\frac{d\vec{v}}{dt}\cdot\vec{v}$$

$$\frac{d}{dt}(\vec{v}\cdot\vec{v}) = 0$$

$$\frac{d}{dt}|\vec{v}|^2 = 0 \Rightarrow |\vec{v}| \text{ constant.}$$

ⓑ We need to prove that $|\vec{v}| = \text{constant} \Rightarrow \frac{d\vec{v}}{dt} \perp \vec{v}$

We know $|\vec{v}| = \text{constant}$

and also $\vec{v}\cdot\vec{v} = \text{constant}$

taking derivative $\frac{d}{dt}(\vec{v}\cdot\vec{v}) = 0$

②

$$\frac{d\vec{v}}{dt}\cdot\vec{v}+\vec{v}\cdot\frac{d\vec{v}}{dt}=0$$

$$2\frac{d\vec{v}}{dt}\cdot\vec{v}=0$$

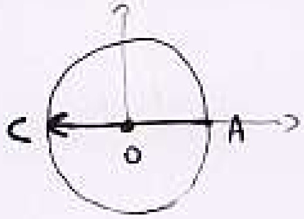
$$\vec{a}\cdot\vec{v}=0$$

so $\vec{a}\perp\vec{v}$

Problem 2

SOLUTION HW 2

(a) The puck as seen from the ground travels in a straight line through the center.

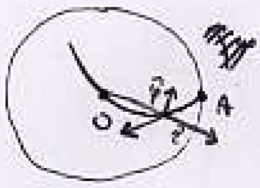


It start from the point A at $t=0$

$$\begin{cases} \phi(t) = 0 \\ r(t) = R - v_0 t \end{cases}$$ with v_0 constant speed

It fall off at point C after $T = \frac{2R}{v_0}$

(b) Imagine an observer sitting on the rotating turntable near A. As seen from the ground he is travelling north with a speed ωR. As seen by the observer, the puck's initial velocity has a components ωR south.

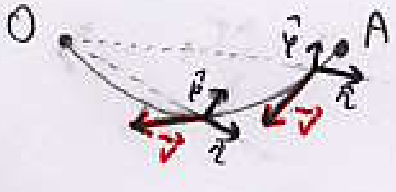


$$\begin{cases} r(t) = R - v_0 t \\ \phi(t) = \phi(0) - \omega t = -\omega T \end{cases}$$

As the puck moves to a smaller radius r, the component of velocity ωr gets less. After $t = \frac{R}{v_0}$ the puck is on the center.

$$\dot{r}(t) = -v_0$$

$$\dot{\phi}(t) = -\omega r$$



Homework #2. Part 2. Solution.

Q3)

(a) $m\dot{V}_x = -bV_x$, $m\dot{V}_y = mg - bV_y$

$$\Rightarrow V_x(t) = V_x(0)\, e^{-t\cdot\frac{b}{m}}$$

$$= V_x(0)\left[1 - t\cdot\frac{b}{m} + \frac{b^2}{2m^2}t^2 + \cdots\right]$$

as $b \to 0$, $V_x(t) \to V_x(0)$ — ①

$$V_y(t) = \frac{mg}{b} + \left(V_y(0) - \frac{mg}{b}\right) e^{-t\cdot\frac{b}{m}}$$

$$= \frac{mg}{b} + \left(V_y(0) - \frac{mg}{b}\right)\left(1 - t\cdot\frac{b}{m} + \frac{1}{2}\left(\frac{b}{m}\right)^2 t^2 + \cdots\right)$$

$$= \frac{mg}{b} + \left(V_y(0) - \frac{mg}{b}\right) + \left(-\frac{b}{m}V_y(0)t + gt\right) + \cdots$$

$$= \underbrace{\left(\frac{mg}{b} - \frac{mg}{b}\right)}_{O(\frac{1}{b})} + \underbrace{\left(V_y(0) + gt\right)}_{O(1)} + \underbrace{\left(-\frac{b}{m}V_y(0)t - \frac{1}{2}g\left(\frac{b}{m}\right)t^2\right)}_{O(b)} + \cdots$$

$$= V_y(0) + gt + O(b)$$

and as $b \to 0$. $O(b) \to 0$. $\Rightarrow \lim_{b\to 0} V_y(t) = V_y(0) + gt$

(b) For the case of quadratic air resistance,

For simplicity, we assume the following eqn. of motion

$$\begin{cases} m\dot{V}_x = -cV_x^2 \\ m\dot{V}_y = mg - cV_y^2 \end{cases}$$

$$\Rightarrow \quad V_x(t) = \frac{V_x(0)}{1 + \frac{cV_x(0)}{m}t} \quad \Rightarrow \quad \lim_{c\to 0} V_x(t) = V_x(0)$$

$$V_y(t) = \sqrt{\frac{mg}{c}}\tanh\left(g\sqrt{\frac{c}{mg}}\,t\right) \qquad (V_y(0)=0)$$

Since. $\tanh x = x - \frac{x^3}{3} + \cdots$

$$V_y(t) = \sqrt{\frac{mg}{c}}\left(g\sqrt{\frac{c}{mg}}\,t - \frac{1}{3}\left(\sqrt{\frac{c}{m}g}\right)^3 t^3 + \cdots\right.$$

$$= \underbrace{gt}_{O(1)} - \underbrace{\frac{1}{3}\frac{c}{m}g\,t^3}_{O(c^1)} + \underbrace{\cdots\cdots}_{O(c^n)}. \quad n \geq 2.$$

$$\Rightarrow \underline{\underline{\lim_{c\to 0} V_y(t) = gt}}$$

(Q4) $F = m\dot{V} \Rightarrow \quad F\cdot V = mV\cdot\dot{V} = \frac{m}{2}\frac{d}{dt}(V^2)$

$$= F\cdot\frac{dx}{dt}$$

integrate $\Rightarrow$ $\int_{x_0}^{x} F\cdot dx = \frac{m}{2}\int_{V_0}^{V} V^2$

$$\Rightarrow \boxed{V^2 = V_0^2 + \frac{2}{m}\int_{x_0}^{x} F\cdot dx}$$

if $F(x)$ is constant,

then $\frac{m}{2}V^2 - \underbrace{\int_{x_0}^{x} F\cdot dx}_{F\cdot(x-x_0)} = \frac{m}{2}V_0^2$

or $\frac{m}{2}V^2 - \frac{m}{2}V_0^2 = F\cdot(x-x_0)$

$$\Delta T = W$$

(change of kinetic energy) (Work done by the force)

Q5)

a) $\dot{V}_x = \omega V_y$
$\dot{V}_y = -\omega V_x$ $\Rightarrow$ $\ddot{V}_x = \omega \dot{V}_y = \omega(-\omega V_x)$

or $\ddot{V}_x + \omega^2 V_x = 0$

$$\Rightarrow \boxed{\begin{aligned} V_x &= A\sin\omega t + B\cos\omega t. \\ V_y &= \frac{\dot{V}_x}{\omega} = A\cos\omega t - B\sin\omega t. \end{aligned}}$$

b) $\eta = V_x + iV_y \Rightarrow \dot{\eta} = \omega(V_y - iV_x) = -i\omega\eta.$

$\eta = C \cdot e^{-i\omega t}$, $C \Rightarrow$ complex number

$= V_x + iV_y = A\sin\omega t + B\cos\omega t + i(A\cos\omega t - B\sin\omega t)$

$= iAe^{-i\omega t} + Be^{-i\omega t}$ $\quad (\because e^{-i\omega t} = \cos\omega t - i\sin\omega t)$

$= (B + iA)e^{-i\omega t}$

So, if we choose $C = B + iA$

then $\eta = (B + iA)e^{-i\omega t}$

$= V_x + iV_y$

Q6)

(a) $\dfrac{\cancel{m}\,dV}{V^3} = -\dfrac{c}{m}dt$

$$\left.\begin{aligned} \Rightarrow \int_{V_0}^{V} \frac{dV}{V^3} &= -\frac{c}{m}t \\ = -\frac{V^{-2}}{2}\Big|_{V_0}^{V} &= -\frac{1}{2}\left(\frac{1}{V^2} - \frac{1}{V_0^2}\right) \end{aligned}\right\} \Rightarrow \frac{1}{V^2} = \frac{1}{V_0^2} + \frac{2c}{m}t$$

or $V = V_0 \cdot \dfrac{1}{\sqrt{1 + \frac{2cV_0^2}{m}t}}$

(b) $$x = \int_0^t \frac{V_0}{\sqrt{1+\frac{2CV_0^2}{m}t}}\,dt$$

$$= V_0 \cdot \frac{m}{CV_0^2}\left[\sqrt{1+\frac{2CV_0^2}{m}t} - 1\right]$$

$$= \frac{m}{CV_0}\left[\sqrt{1+\frac{2CV_0^2}{m}t} - 1\right]$$