Homework #3 Solution (Phys 105A)

1) a) $m\dot{\vec{V}} = q\vec{V}\times B - b\vec{V}$

$$\Rightarrow \begin{cases} \dot{V}_x = \omega V_y - cV_x \\ \dot{V}_y = -\omega V_x - cV_y \\ \dot{V}_z = -cV_z. \end{cases} \quad , \left(\text{where, } \omega = \frac{qB}{m}, \ c = \frac{b}{m}\right)$$

b) let. $\eta = V_x + iV_y$, then

$$\dot{\eta} = (-i\omega - c)\eta \Rightarrow \underline{\eta = A e^{-(c+i\omega)t}}, \ (A: \text{constant})$$

also. $\underline{V_z = V_z(0)\, e^{-ct}}$

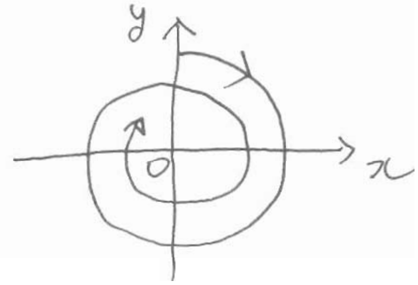
Now, plug in the initial condition; $\vec{V}(0) = V_0\hat{x}$

determine. A & $V_z(0)$ $\Rightarrow$ $A = V_0$. $V_z(0) = 0$

$$\therefore \begin{cases} V_x = V_0 e^{-ct}\cos\omega t \\ V_y = -V_0 e^{-ct}\sin\omega t \\ V_z = 0 \end{cases} \quad \& \quad \begin{cases} x = x_0 + \frac{V_0}{\omega^2+c^2}\left[-c\cdot\cos\omega t + \omega\sin\omega t\right]e^{-ct} \\ y = y_0 + \frac{V_0}{\omega^2+c^2}\left[\omega\cos\omega t + c\sin\omega t\right]e^{-ct} \\ z = z_0 \end{cases}$$

$$\Rightarrow \quad (x-x_0)^2 + (y-y_0)^2 = \frac{V_0^2}{\omega^2+c^2}e^{-2ct}, \quad z = z_0$$

the trajectory is a spiral.



2.

(a) $-i\tan iz = -i\dfrac{\dfrac{e^{-z}-e^{z}}{i}}{e^{-z}+e^{z}} = \dfrac{e^{z}-e^{-z}}{e^{z}+e^{-z}} = \tanh z$

(b) $(\tanh z)' = \left(\dfrac{\sinh z}{\cosh z}\right)' = \dfrac{\cosh^2 z-\sinh^2 z}{\cosh^2 z} = \dfrac{1}{\cosh^2 z} = \mathrm{sech}^2 z$

(c) $\int dz \cdot \tanh z = \int dz \cdot \dfrac{\sinh z}{\cosh z} = \int \dfrac{d(\cosh z)}{\cosh z} = \ln \cosh z$

(d) $\cosh^2 z - \sinh^2 z = \frac{1}{4}\{(e^{z}+e^{-z})^2 - (e^{z}-e^{-z})^2\} = 1$

$\Rightarrow$ multiplying $\frac{1}{\cosh^2 z}$ $\Rightarrow$ $1-\tanh^2 z = \mathrm{sech}^2 z$

(e) $\int \dfrac{dx}{1-x^2} = \int dz = \mathrm{arctanh}\, x$

(let $x = \tanh z$, $z = \mathrm{arctanh}\, x$)

3) (a) $m\dot{V} = -\dot{m} V_{ex} + F^{ext}$, where $F^{ext} = -bV$

also. $V_{ex} \Rightarrow$ const. $\dot{m} = -k$. (const)

$\Rightarrow$ $m\dot{V} = -\dot{m} V_{ex} - bV = kV_{ex} - bV$

$\Rightarrow$ $\dfrac{dV}{kV_{ex}-bV} = \dfrac{dt}{m} = \dfrac{dt}{m_0-kt}$ ($\because$ separation of variable)

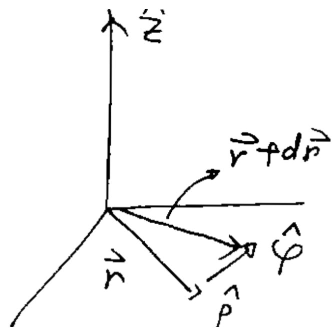
$\Rightarrow$ $\int \dfrac{dV}{kV_{ex}-bV} = \int \dfrac{dt}{m_0-kt} = -\frac{1}{k}\ln(m_0-kt)\Big|_0^t$

$\Rightarrow$ $= -\frac{1}{b}\ln|kV_{ex}-bV|\Big|_{V_0}^{V}$, $V_0 = 0$ ($\because$ initial condition)

$\Rightarrow$ $1 - \dfrac{b}{kV_{ex}}V = \left(\dfrac{m}{m_0}\right)^{b/k}$ ($\because m(t) = m_0 - kt$)

$$\therefore \quad V = \frac{k}{b} V_{ex} \left[1 - \left(\frac{m}{m_0} \right)^{b/k} \right]$$

4) (a) $\vec{r} = \rho \hat{\rho} + z \hat{z}$



$$d\vec{r} = \rho \, d\varphi \, \hat{\varphi} = \rho \omega \, dt \, \hat{\varphi}$$

($\because$ Since it is a rigid-body. $d\rho = dz = 0$, and it is rotating w/ angular velocity ω)

$$\Rightarrow \quad \vec{V} = \rho \omega \hat{\varphi}$$

or for the αth particle, $\vec{V}_\alpha = \rho_\alpha \omega \hat{\varphi}$

(b) $\vec{l}_\alpha = \vec{r}_\alpha \times \vec{P}_\alpha = (\rho_\alpha \hat{\rho} + z_\alpha \hat{z}) \times (m_\alpha \rho_\alpha \omega \hat{\varphi})$

$\Rightarrow$ z-component of the angular momentum is

$l_{z_\alpha} = m_\alpha \rho_\alpha^2 \omega \qquad (\because \hat{\rho} \times \hat{\varphi} = \hat{z}, \; \hat{\varphi} \times \hat{z} = \hat{\rho})$

(c) $L_z = \sum_\alpha l_{\alpha z} = \sum_\alpha m_\alpha \rho_\alpha^2 \omega = \left(\sum_\alpha m_\alpha \rho_\alpha^2 \right) \omega = I \omega$

5) (a) $\sum_\alpha m_\alpha \vec{r}_\alpha{}' = \sum_\alpha m_\alpha (\vec{r}_\alpha - \vec{R})$

$= \sum_\alpha m_\alpha \vec{r}_\alpha - \left(\sum_\alpha m_\alpha \right) \vec{R} = 0 \qquad \left(\because \vec{R} = \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{\sum_\alpha m_\alpha} \right)$

(b) $$L_{CM} = \sum_\alpha m_\alpha r'_\alpha \times \dot{r}'_\alpha$$

$$\dot{L}_{CM} = \cancelto{0}{\sum_\alpha m_\alpha \dot{r}'_\alpha \times \dot{r}'_\alpha} + \sum_\alpha m_\alpha r'_\alpha \times \ddot{r}'_\alpha$$

$$(\because \dot{r}'_\alpha \parallel \dot{r}'_\alpha)$$

$$= \sum_\alpha r'_\alpha \times m_\alpha \ddot{r}'_\alpha = \sum_\alpha r'_\alpha \times (m_\alpha \ddot{r}_\alpha - m_\alpha \ddot{R})$$

$$= \sum r'_\alpha \times F_\alpha - \underbrace{\sum_\alpha m_\alpha r'_\alpha}_{=0} \times \ddot{R}$$

$$= \sum r'_\alpha \times F_\alpha^{ext} = \Gamma_{CM}$$