SOLUTION HW 4 PHYS 105 A

comment to

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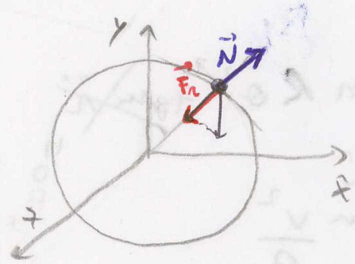
Problem 4.8



R radius of sphere

$\vec{N}$ = normal force

$\vec{F}_r$ = radial component of Force of gravity

The puck start at $\theta = 0$ and roll down on the sphere because of force of gravity.

In the starting point the energy E has contribute from potential energy U. So $E = mgR$

The puck will leave the sphere at an angle θ and the energy E has contribute from Kinetic energy (T) and potential energy (U).

$$\text{So } E = \frac{1}{2} m v^2 + mgR\cos\theta$$

For conservation of energy we get that

$$\frac{1}{2} m v^2 = mgR(1-\cos\theta)$$

with $v^2 = (R\dot{\theta})^2$ $\left(\vec{v} = \underbrace{\cancel{\dot{r}}}_{\text{the radius will not vary when we are on the sphere}} \hat{r} + r\dot{\theta}\hat{\theta}\right)$

~~[illegible]~~

The radial component of net force F_r

$F_r = N - mg\cos\theta$ where N is the normal force from the sphere to the puck.

and F_r can be written as $F_r = -mR\dot{\theta}^2 + \cancel{m\ddot{r}}$

$$= -m\frac{v^2}{R}$$

$\ddot{r} = 0$ the radius will not vary when we are on the sphere

So we have

$$-m\frac{v^2}{R} = N - mg\cos\theta$$

$$-2mg R(1-\cos\theta) = N - mg\cos\theta$$

$$N = mg(3\cos\theta - 2)$$

The puck will leave the sphere when the Normal force N is zero

so when $3\cos\theta - 2 = 0$ $\boxed{\theta = \arccos\frac{2}{3}}$

Problem 4.12

Calculate the gradient ∇f of the following functions

a) $f(x,y,z) = x^2 + z^3$

$$\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

$$= \hat{x}\, 2x + \hat{z}\, 3z^2$$

b) $f(x,y,z) = Ky$

$$\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

$$= K\hat{y}$$

c) $f(x,y,z) = \sqrt{x^2+y^2+z^2} = r$

$$\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

$$= \hat{x}\left(\frac{2x}{2\sqrt{x^2+y^2+z^2}}\right) + \hat{y}\left(\frac{2y}{2\sqrt{x^2+y^2+z^2}}\right) + \hat{z}\left(\frac{2z}{2\sqrt{x^2+y^2+z^2}}\right)$$

$$= \hat{x}\frac{x}{r} + \hat{y}\frac{y}{r} + \hat{z}\frac{z}{r}$$

$$= \frac{\vec{r}}{|r|} = \hat{r}$$

d) $f(x,y,z) = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$

$$\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

$$= -\cancel{2}x\frac{1}{\cancel{2}}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\hat{x} - \cancel{2}y\frac{1}{\cancel{2}}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\hat{y} +$$

$$-2z\frac{1}{\cancel{2}}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

$$= -\frac{x}{r^3}\hat{x} - \frac{y}{r^3}\hat{y} - \frac{z}{r^3}\hat{z}$$

$$= -\frac{\vec{r}}{r^3}$$

$\boxed{\text{Problem 4.14}}$

Consider $\left[\vec{\nabla}(f\cdot g)\right]_x = \frac{\partial}{\partial x}(f\cdot g) = f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}$

$$= f\left(\vec{\nabla} g\right)_x + g\left(\vec{\nabla} f\right)_x$$

Since the other two components work the same way we conclude that $\vec{\nabla}(f\cdot g) = f\vec{\nabla}\cdot g + g\vec{\nabla} f$

Problem 4.16

We have a particle with potential Energy $U(\vec{r}) = K(x^2+y^2+z^2) \equiv Kr^2$

Since U is spherically symmetric depend only on r, the same is true for $\frac{\partial U}{\partial r}$, so also the force $F(\vec{r})$ is spherically symmetric

$$\vec{F}(\vec{r}) = -\hat{r}\frac{\partial U}{\partial r} = -\hat{r}\frac{\partial}{\partial r}(Kr^2) = -2rK\hat{r}$$

You can also use cartesian coordinates

$$\vec{F} = -\vec{\nabla}U = -2K(x,y,z) = -2Kr\hat{r}$$

Problem 4.22

Coulomb force is $\vec{F} = \gamma\frac{\hat{r}}{r^2}$

We see that

- The component F_θ and F_ϕ are zero.
- The component F_r is independent from θ and ϕ.

$$\vec{\nabla}\times\vec{F} = \hat{r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta \overset{=0}{F_\phi}) - \frac{\partial}{\partial\phi}\overset{=0}{F_\theta}\right] +$$
$$+\hat{\theta}\left[\frac{1}{r\sin\theta}\overset{=0}{\frac{\partial}{\partial\phi}F_r} - \frac{1}{r}\frac{\partial}{\partial r}(r\overset{=0}{F_\phi})\right] +$$
$$+\hat{\varphi}\frac{1}{r}\left[\frac{\partial}{\partial r}(r\overset{=0}{F_\theta}) - \overset{=0}{\frac{\partial}{\partial\theta}F_r}\right]$$

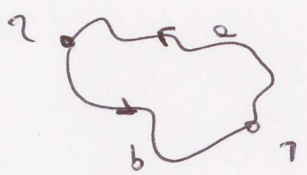
Each term in $\vec{\nabla} \times \vec{F}$ is zero

So we can conclude that $\vec{\nabla} \times \vec{F} = 0$

and the Force the Coulomb is conservative

Problem 4.25

We can consider two any points 1 and 2 and two any path connecting this two point. For example Path a start at point 1 goes to point 2 and Path b return from point 2 to point 1.



$$\oint_{\Gamma} \vec{F} \cdot d\vec{r} = \int_{1\,a}^{2} \vec{F} \cdot d\vec{r} + \int_{2\,b}^{1} \vec{F} \cdot d\vec{r}$$

$$= \int_{1\,a}^{2} \vec{F} \cdot d\vec{r} - \int_{1\,b}^{2} \vec{F} \cdot d\vec{r}$$

Now if $\displaystyle\int_{1\,a}^{2} \vec{F} \cdot d\vec{r} = \int_{1\,b}^{2} \vec{F} \cdot d\vec{r} \Rightarrow \oint_{\Gamma} \vec{F} \cdot d\vec{r} = 0$

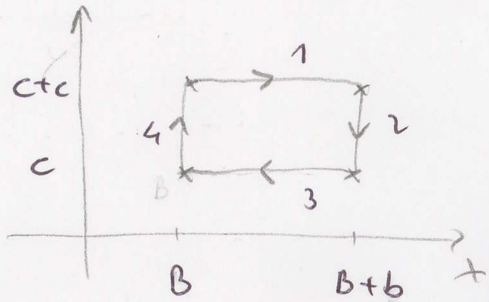
if $\displaystyle\oint_{\Gamma} \vec{F} \cdot d\vec{r} = 0 \Longrightarrow \int_{1\,a}^{2} \vec{F} \cdot d\vec{r} = \int_{1\,b}^{2} \vec{F} \cdot d\vec{r}$

So we have that the path independence of $\int_1^2 \vec{F} \cdot d\vec{r}$ is equivalent to say that the $\oint_{\Gamma} \vec{F} \cdot d\vec{r}$ around any closed line Γ is zero.

(b) Stokes theorem say $\oint_\Gamma \vec{F}\cdot d\vec{n} = \int (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, dA$

From this follows immediately that

(c)



$$\oint_\Gamma \vec{F}\cdot d\vec{n} = \int_1 + \int_2 + \int_3 + \int_4 \vec{F}\cdot d\vec{n}$$

Now

$$\int_1 + \int_3 = \int_B^{B+b} dx\, F_x(x, C+c, z) - \int_B^{B+b} F_x(x, C, z)\, dx$$

and

$$F_x(x, C+c, z)\, dx - F_x(x, C, z) = \int_C^{C+c} dy\, \frac{\partial F_x(x,y,z)}{\partial y}$$

$$\Rightarrow \int_1 + \int_3 = \int_B^{B+b} dx \int_C^{C+c} dy\, \frac{\partial F_x}{\partial y}(x,y,z) = \int \frac{\partial F_x}{\partial y}\, dA$$

Similarly for $\int_2 + \int_4$

$$\oint \vec{F}\cdot d\vec{n} = \int \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_x}{\partial x}\right) dA = \int (\vec{\nabla}\cdot\vec{F})\cdot\hat{n}\, dA$$