PHYS 105 A

comments to

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SOLUTIONS HW 6

(1) $\alpha \cos(\omega t) + \beta \sin(\omega t) = \gamma \cos(\omega t - \delta)$

we have to ~~fix~~ ~~fix~~ give γ and δ in terms of α and β

$$\gamma \cos(\omega t - \delta) = \gamma \left(\cos(\omega t) \cos(\delta) + \sin(\omega t) \sin(\delta) \right)$$

$$\Rightarrow \quad \alpha = \gamma \cos(\delta)$$

$$\beta = \gamma \sin \delta$$

$$\Rightarrow \quad \delta = \arctan\left(\frac{\beta}{\alpha}\right)$$

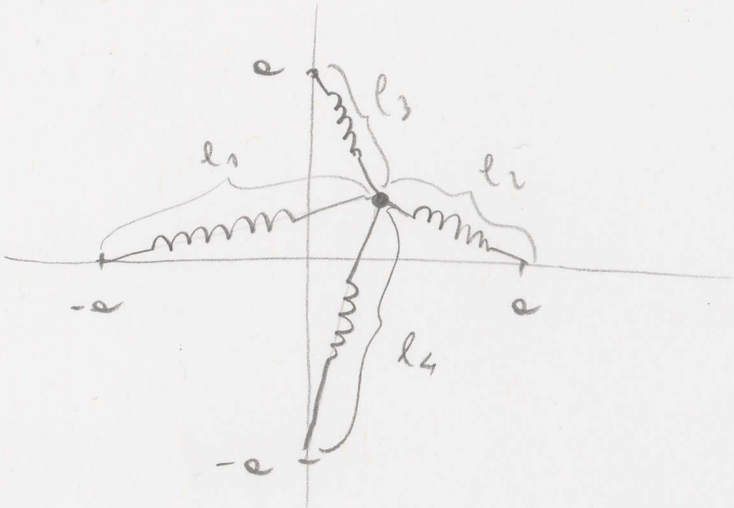
Now $\alpha^2 = \gamma^2 \cos^2 \delta$

$$\beta^2 = \gamma^2 \sin^2 \delta$$

$$\alpha^2 + \beta^2 = \gamma^2 (\cos^2 \delta + \sin^2 \delta$$

$$\Rightarrow \quad \gamma = \sqrt{\alpha^2 + \beta^2}$$

2



$$l_1 = \sqrt{(a+x)^2+y^2} = a\left(1+\overbrace{\boxed{\frac{2x}{a}+\frac{x^2+y^2}{a^2}}}^{\text{small}}\right)^{\frac{1}{2}}$$

$$l_1 \sim a\left[1+\left(\frac{2x}{a}+\frac{x^2+y^2}{a^2}\right)-\frac{1}{8}\left(\frac{2x}{a}\right)^2\right]+O\left(x^3,y^3\right)$$

$$l_1 \sim a\left[1+\frac{x}{a}+\frac{y^2}{2a^2}\right]$$

$$U_1 = \text{potential energy} = \frac{1}{2}K\left(l_1-l_0\right)^2 = \frac{1}{2}K\left(a+x+\frac{y^2}{2a}-l_0\right)^2$$

$$= \frac{1}{2}K\left[a^2+x^2+2ax+\overbrace{\cancel{\left(\frac{y^2}{2a}\right)^2}}^{O(y^4)}+l_0^2-2\frac{y^2}{2a}l_0+2(a+x)\left(\frac{y^2}{2a}-l_0\right)\right]$$

$$\sim \frac{1}{2}K\left[a^2+x^2+2ax+l_0^2-\frac{y^2}{a}l_0+y^2-2al_0+\overbrace{\cancel{\frac{xy^2}{a}}}^{O(xy^2)}-2xl_0\right]$$

$$\sim \frac{1}{2}K\left[(a-l_0)^2+2(a-l_0)x+x^2+\left(1-\frac{l_0}{a}\right)y^2\right]$$

to find U_2 we have to replace x by $-x$

U_3 we have to replace $x \leftrightarrow y$

U_4 we have to replace $x \leftrightarrow -y$

$$U_1 + U_2 = \frac{1}{2}\left[K\,(a-b)^2 + \cancel{2(a-b)}\,x + x^2 + \left(1-\frac{b}{a}\right)y^2\right] +$$

$$+ \frac{1}{2}\,K\,(a-b)^2 - \cancel{2(a-b)}\,x + x^2 + \left(1-\frac{b}{a}\right)y^2\Big]$$

$$= K\left[x^2 + \left(1-\frac{b}{a}\right)y^2\right] + \cancel{\text{const}}$$

we can ignore these

$$U_3 + U_4 = K\left[y^2 + \left(1-\frac{b}{a}\right)x^2\right] + \cancel{\text{const}}$$

$$U_1 + U_2 + U_3 + U_4 = K\left(x^2 + y^2 + \left(1-\frac{b}{a}\right)x^2 + y^2\right)$$

$$= K\left[\left(1 + 1 - \frac{b}{a}\right)\left(x^2+y^2\right)\right]$$

$$= K\left[\left(2-\frac{b}{a}\right)\left(x^2+y^4\right)\right]$$

$$= \frac{1}{2}\,\boxed{2K\left(2-\frac{b}{a}\right)}\,r^2$$

effective spring constant K'

3 weakly damped

$$x(t) = A e^{-\beta t} \cos(\omega t - \delta)$$

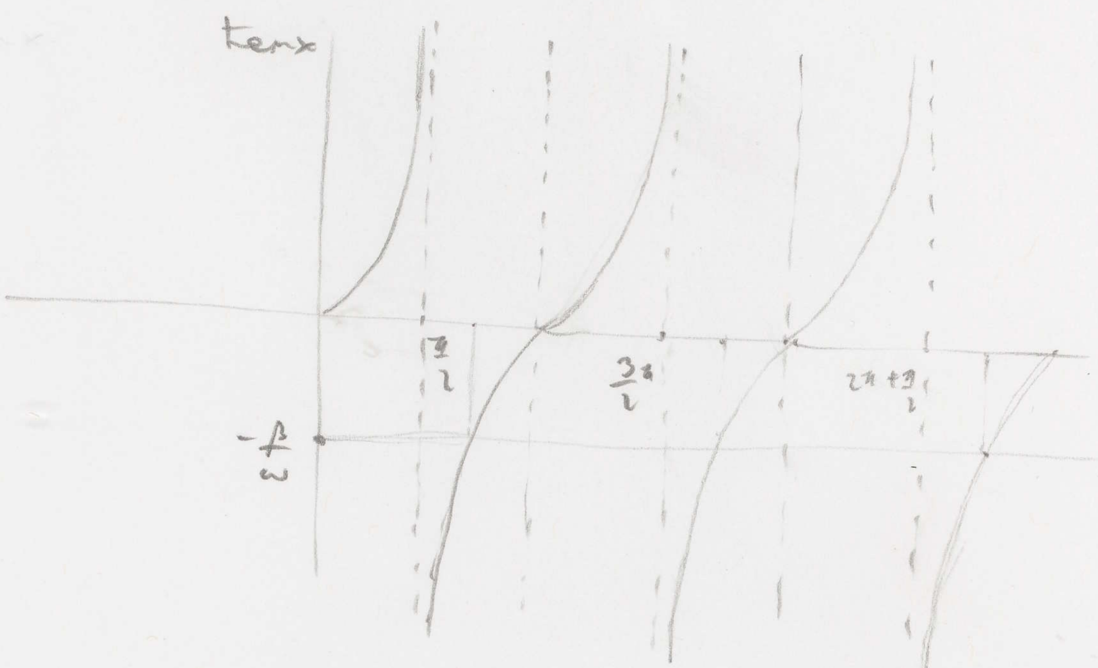
$$\frac{dx(t)}{dt} = A(-\beta) e^{-\beta t} \cos(\omega t - \delta) + A e^{-\beta t} \omega (-) \text{sen}(\omega t - \delta)$$

$$= - A e^{-\beta t} \left(\beta \cos(\omega t - \delta) + \omega \,\text{sen}(\omega t - \delta) \right) = 0 \quad \text{for maxima}$$

$$\beta \cos(\omega t - \delta) = - \omega \,\text{sen}(\omega t - \delta)$$

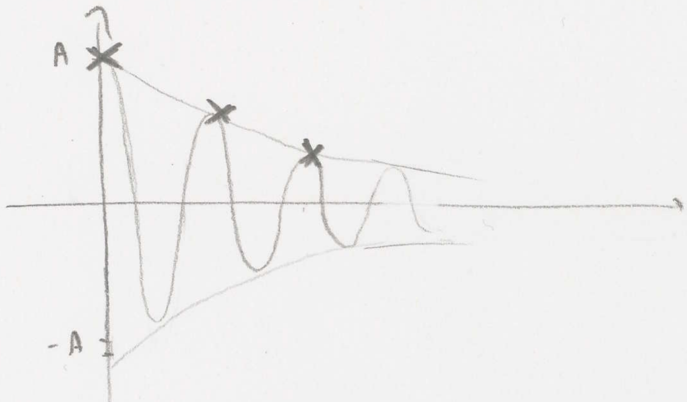
$$-\frac{\beta}{\omega} = \frac{\text{sen}(\omega t - \delta)}{\cos(\omega t - \delta)}$$

$$\tan(\omega t - \delta) = -\frac{\beta}{\omega}$$



Tan x is a periodic function Tan $x = -\frac{\beta}{\omega}$ when $x = x + \frac{\pi}{2} + n\pi$

with $n = 0, 1, 2, 3$ ---



maxima ~~or~~ at minima when $(\omega T - \delta) = \alpha + \frac{\pi}{2} + n\pi$

maxima at even interval.

The ratio between two consecutive maxima is

$$\frac{x_{n+2}}{x_n} = \frac{A e^{-\beta\left(\alpha + \frac{\pi}{2} + (n+2)\pi\right)} \cos\left(\alpha + \frac{\pi}{2} + (n+2)\pi\right)}{A e^{-\beta\left(\alpha + \frac{\pi}{2} + n\pi\right)} \cos\left(\alpha + \frac{\pi}{2} + n\pi\right)}$$

$$\alpha + \frac{\pi}{2} + n\pi = \theta$$

$$= e^{-2\beta\pi} \cancel{\frac{\cos(\theta + 2\pi)}{\cos(\theta)}}$$

So the ratio between two consecutive maxima is constant.

4 The key relationships we need are

e

$$F(t) = F_0 \cos(\omega t)$$

$$x(t) = A \cos(\omega t - \delta)$$

$$A = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$\tan \delta = 2\beta\omega \frac{1}{\omega_0^2 - \omega^2}$$

(see paragraph 5.5 in Taylor)

The rate at wich the $F(t)$ does the work is

$$P = F \cdot \dot{x} = F_0 \cos \omega t \,(-A\omega)\, \text{sen}(\omega t - \delta)$$

$$= -A\omega F_0 \cos \omega t \left(\text{sen}(\omega T)\cos(-\delta) + \cos(\omega T)\,\text{sen}(-\delta) \right)$$

$$= -A\omega F_0 \left(\cos(\omega t)\,\text{sen}(\omega t)\cos(\delta) + A\omega F_0 \cos^2(\omega t)\,\text{sen}(\delta) \right)$$

$$\langle P \rangle = \frac{1}{\tau}\int_0^\tau P(t)\, dt \qquad \text{with } \tau = \frac{2\pi}{\omega}$$

$$= \omega A F_0 \frac{1}{\tau}\int_0^\tau \cos^2(\omega t)\,\text{sen}\,\delta\, dt + \cancel{A\omega F_0 \frac{1}{\tau}\int_0^\tau \cos(\omega t)\,\text{sen}(\omega t)\cos(\delta)\, dt} = 0$$

$$= \frac{1}{2}\omega A F_0\, \text{sen}\,\delta = \frac{1}{2}\omega A F_0 \frac{2\beta\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} = m\beta\omega^2 A^2$$

(b) The damping force has magnitude $2m\beta\dot{x}$ in the opposite direction to the velocity

$$P_{loss}(t) = F_{damp}\,\dot{x} = 2m\beta\dot{x}$$

$$= 2m\beta\,\omega^2 A^2 \sin^2(\omega t - \delta)$$

$$\langle P_{loss}\rangle = \frac{1}{\tau}\int_0^\tau P_{loss}(t) = 2m\beta\,\omega^2 A^2 \frac{1}{\tau}\int_0^\tau \sin^2(\omega t - \delta)$$

$$= 2m\beta\omega^2 A^2 \frac{1}{\tau}\int_0^{\tau = \frac{2\pi}{\omega}} \left(\frac{1}{2} - \cos(2(\omega t - \delta))\right) dt$$

$$= \cancel{2}m\beta\omega^2 A^2 \frac{1}{\cancel{\tau}}\frac{\cancel{\tau}}{\cancel{2}}$$

$$= m\beta\omega^2 A^2$$

(c) $$\langle P\rangle = \frac{\beta F_0}{m}\,\frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

we set $\frac{d}{d\omega}\langle P\rangle = 0$

$$\frac{d\langle P\rangle}{d\omega} = \frac{\beta F_0}{m}\left(\frac{2\omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} - \frac{\omega^2\left[2(\omega_0^2 - \omega^2)2\omega + 8\beta^2\omega\right]}{\left[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\right]^2}\right)$$

We can verify the maximum is $\omega = \omega_0$

for $\omega = \omega_0$ we have 0

$$\frac{d\langle P\rangle}{d\omega} = \frac{\beta F_0}{m}\left[\frac{2\omega_0}{4\beta^2\omega_0^2} - \frac{\omega_0^2(8\beta^2\omega_0)}{(4\beta^2\omega_0^2)^2}\right] = 0$$

5 The damping changes the frequency

$$\omega_1 = \sqrt{\omega_0^2 - \beta}$$

$$\Rightarrow \quad \beta = \omega_0 \sqrt{1 - \frac{\omega_1^2}{\omega_0^2}} = \omega_0 \sqrt{1 - \frac{\tau_0^2}{\tau_1^2}} = \omega_0 \sqrt{1 - \frac{1}{1{,}001}} = 0{,}281 \frac{1}{s}$$

After a time $t = 10\,\tau_1 \approx 10\,\tau_0$

the amplitude will have changed by a factor of

$$e^{-\beta t} = e^{-\beta 10 \tau_0} = e^{-\beta 10 \frac{2\pi}{\omega_0}} = \cancel{e} = 0{,}06$$

The amplitude will be diminished by a factor

$$\frac{A(t=0)}{A(t=10\tau_1)} = \frac{1}{0{,}06} = 17$$

A change in the amplitude will be more noticeable than the change of period that is $0{,}1\%$