Classical Mechanics

Phys105A, Winter 2007

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The Final

The final is scheduled as
Wednesday, March 21, 8:00 – 11:00 am, Broida 1640

Material:
Everything, including the last 2 weeks, with an emphasis on the post-midterm chapters 5, 6 and 7.

What to expect:
See the ★★ problems in the book.

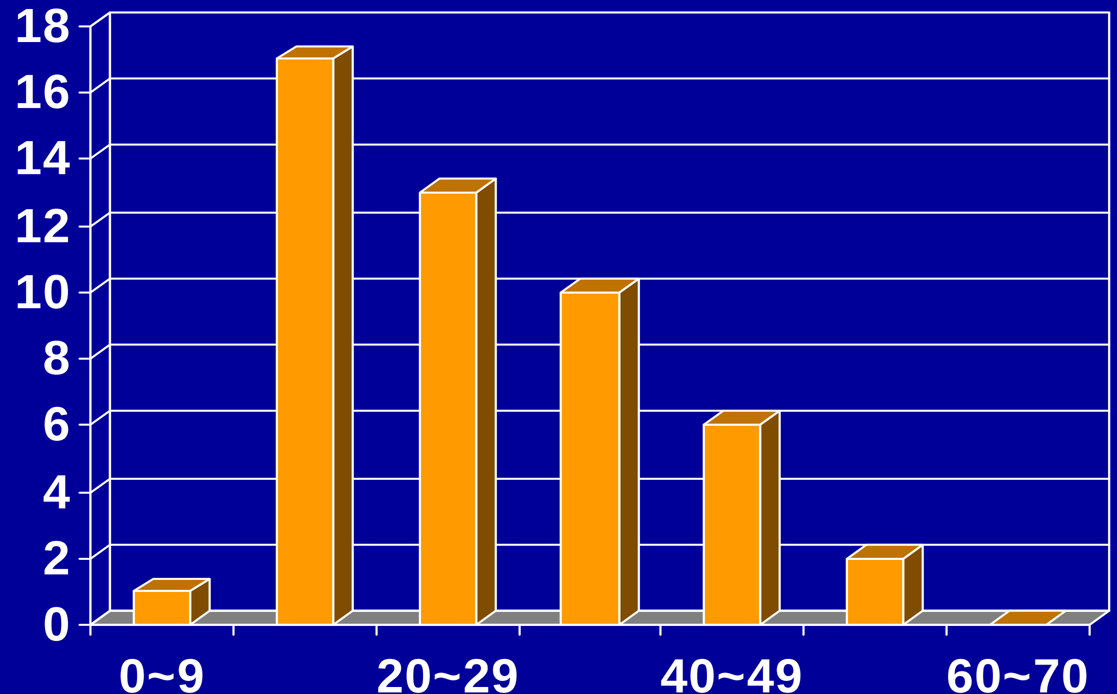
Questions?

Midterm Distribution

Total: 49 students

Average: 26
Stdev: 12

Median: 26



Chapter 7

Lagrange's Equations

Lagrangian

For an unconstrained system in 3 dimensions subject to the a conservative field with potential energy U=U(**r**) and kinetic energy T= ½m($x'^2+y'^2+z'^2$) we have the **Lagrangian** (or **Lagrange function**): $\mathcal{L}$ = T–U, in this case $\mathcal{L}(x,y,z,x',y',z') = \frac{1}{2}m(x'^2+y'^2+z'^2) - U(x,y,z)$.

Note that $\partial\mathcal{L}/\partial x = -\partial U/\partial x = F_x$ and $\partial\mathcal{L}/\partial x' = mx' = p_x$, hence $\partial\mathcal{L}/\partial x - d(\partial\mathcal{L}/\partial x')/dt = 0$; similarly for y and z.

These are the three Lagrange equations, which follow from Newton's second law for q(t) = x(t), y(t), and z(t).

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}$$

Hamilton's Principle

Lagrange's equations indicate that Newton's second law for a single particle describes a path (x(t),y(t),z(t)) that obeys the Euler-Lagrange equations for the Lagrangian $\mathcal{L}$.

This is **Hamilton's Principle:** The path of a particle from time t_1 to t_2 is such that the **action integral**

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)\,dt$$ is stationary.

See Feynman's "QED: The Strange Theory of Light and Matter" for how this principle is crucial to modern physics.

Generalized Coordinates

Hamilton's principle remains true in cylindrical or spherical coordinates, and in fact for any reasonable kind of generalized coordinate system $\mathbf{r} = \mathbf{r}(q_1,q_2,q_3)$.

We call $\partial\mathcal{L}/\partial q_i$ is i-th the **generalized force** component, and $\partial\mathcal{L}/\partial q'_i$ the i-th **generalized momentum** component, with $\partial\mathcal{L}/\partial q_i = d(\partial\mathcal{L}/\partial q'_i)/dt$ for all i.

See Example 7.2 for new, simpler derivation of Newton's second law in polar coordinates. Note that as long as you remember your (polar) ∇U rules, you can avoid vector calculations like "$d\mathbf{e}_r/dt = d\varphi/dt\ \mathbf{e}_\varphi$" (cf. Section 1.7).

Further Generalizations

The Hamilton’s principle-approach can also be generalized to N particles with a joint Lagrangian $\mathcal{L}(\mathbf{r}_1,\ldots,\mathbf{r}_N,\mathbf{r}'_1,\ldots,\mathbf{r}'_N,t)$.

For N unconstrained particles in three dimensions, this gives 3N equations $\partial\mathcal{L}/\partial q_i = d(\partial\mathcal{L}/\partial q'_i)/dt$ for $i = 1,\ldots,3N$.

The real power of the Langrangian shows when dealing with constrained systems. Traditionally this requires us to write down the various interactions between the particles and their constraints (normal forces et cetera). Now we only have to worry about the potential and kinetic energy as a function of the generalized coordinates…

Constrained Systems

The Langrangian approach to mechanics works equally well for **constrained systems** where not all of the generalized coordinates can be chosen independently.

Typical example: a mass m that is limited in its mobility because it has to slide along a wire, or stick to a surface.

A system with N particles in positions $\mathbf{r}_\alpha$ for $\alpha=1,\ldots,N$ has a set of generalized coordinates $q_1,\ldots,q_n$ if each $\mathbf{r}_\alpha$ follows from $(q_1,\ldots,q_n,t)$ and each q_i follows from $(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)$.

If there is no time dependency, the system is **natural**.

Degrees of Freedom

The **degrees of freedom** of a system is the number of coordinates that can independently be varied in an infinitesimal displacement ("number of directions to go").

A system is **holonomic** if its degrees of freedom equals its number of generalized coordinates.

Nonholonomic systems have more coordinates than degrees of freedom, hence the chance of (some of) the coordinates is made "indirectly" (by following a path).

Example: a car has 4 coordinates (2 position, 1 orientation car and 1 orientation wheel), while when driving it has only two degrees of freedom (gas pedal + steering wheel). The art of (parallel) parking shows that those 2 degrees are sufficient to move in all 4 coordinates.

Holonomic Systems

Here we will only deal with holonomic systems.

With

- generalized coordinates $q_1,..,q_n$
- potential energy $U(q_1,\ldots,q_n,t)$
- kinetic energy T
- and Langrangian $\mathcal{L}$ = T–U

the time evolution of the system is described by

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad \text{for } i = 1,\ldots,n$$

Solving It the Lagrangian Way

When trying to solve the evolution of a holonomic system the “Lagrangian way” you should pay special attention to **picking the right coordinate system** $(q_1,\ldots,q_n)$:

• **Do not introduce too many coordinates**
Example 7.3 shows that Atwood’s machine has only one degree of freedom.

• **Pick your coordinates in an inertial system**
See the example of a pendulum hanging in an accelerating train.

• **Arrange the coordinates such that the Lagrangian $\mathcal{L}$ is independent from as many coordinates as possible.**
That is, use any kind of symmetry that you have. Example: When dealing with gravity, align the coordinates with the direction of g.

Advantage of Lagrangians

The Langrangian way of solving problems in mechanics has several advantages over the traditional way:

• Because it focuses on the potential energy U and the kinetic energy T, you get rid of a lot of vector notation. Instead your equations become 1d again.

• The notation is very economic; typically you only deal with degrees of freedom of the system, nothing more. This is especially helpful in constrained systems.

The Newtonian way often involves writing down forces that end up cancelling each other, and so on.

• It prepares you for modern physics (cf. Feynman's QED).