Classical Mechanics

Phys105A, Winter 2007

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Formalities

- Latest news and course slides always found on the Phys105A site at http://www.cs.ucsb.edu/~vandam/…

- Homework 4 has been posted.
  It is due Monday February 12, 11:30 am.

- Midterm is scheduled for Week 6, Thursday Feb 15.
  The material will be Chapters 1–4; no electronics are allowed, it is not open book, but you are allowed a letter sized, double sided ‘cheat sheet’.

One Dimensional Systems

For one dimensional systems every possible position dependent force **F**(x) will be conservative and the potential $U(x_1)$ is simply the 'straight' integral $-\int F_x(x)\,dx$ between the reference point x_0 and the point x_1.

This allows us to plot the potential U on a line and reason about it in a intuitive "hills and valleys" kind of way.

Solving the equations of motion of such a system is relatively easy (although the integrals involved might still require a computer to approximate).

Solving 1D Systems

For a 1D conservative system in x with energy E, we have $T = \frac{1}{2}mv^2 = E - U(x)$ and hence $v(x) = \pm\sqrt{(2/m)}\sqrt{(E - U(x))}$.

As $dt = dx/v$, we get for the Δt time between x_0 and x_1:

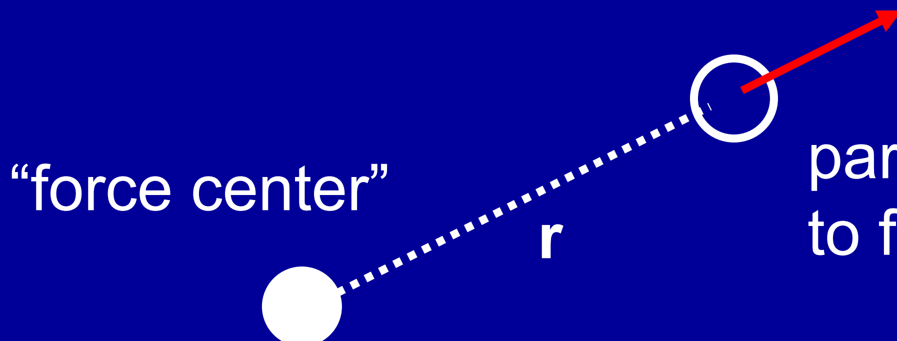
$$\Delta t = \int_{x_0}^{x_1} \frac{dx}{v(x)} = \pm\sqrt{\frac{m}{2}} \int_{x_0}^{x_1} \frac{dx}{\sqrt{E - U(x)}}$$

(The +/– is determined by the direction of the initial speed.)

Very similar evaluations are possible for other systems with only one degree of freedom such as curvilinear systems and Atwood systems. We will see much more of this in Chapter 7 on *Lagrange's Equations.*

Central Forces

A particle subjected to a central force from a “force center” in O can be viewed as a 1d system (in r):



particle at **r** subjected to force $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\,\mathbf{e}_r$.

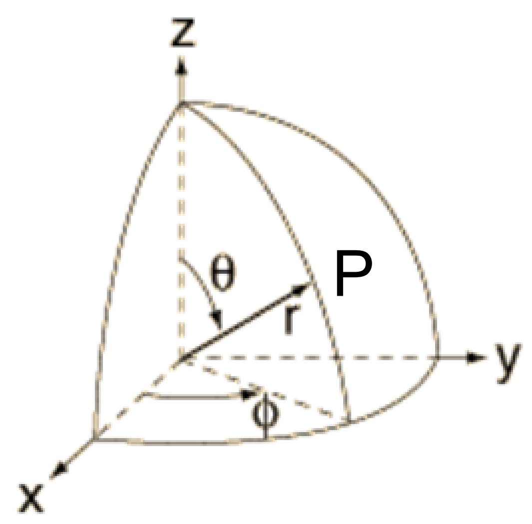
<u>Remarkable fact about central forces</u>:
F is conservative if and only if **F** is spherical (= rotationally) invariant such that $f(\mathbf{r}) = f(r)$ depends only on r.

Spherical Coordinates (I)

3D systems that are rotationally invariant are often best described in *spherical polar coordinates*.

A vector **r** (pointing from the origin to a point P) has three spherical coordinates (r,θ,φ) with

- distance r = |**r**| from O.
- angle θ between **r** and $\mathbf{e}_z$
- angle φ (azimuth) between $\mathbf{e}_x$ and **r** projected in XY plane.



In earth coordinates we would say that r is the radius of the earth, θ gives the co-latitude (N-S) and φ gives the longitude (E-W).

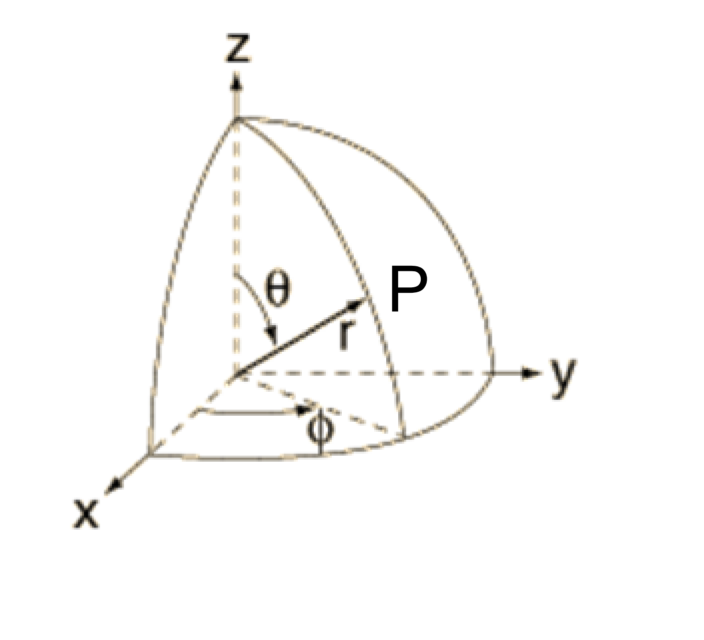
Spherical Coordinates (II)

3D systems that are rotationally invariant are often best described in *spherical polar coordinates*.

A vector **r** (pointing from the origin to a point P) has three spherical coordinates (r,θ,φ).

Getting back to Cartesian coordinates is easy:

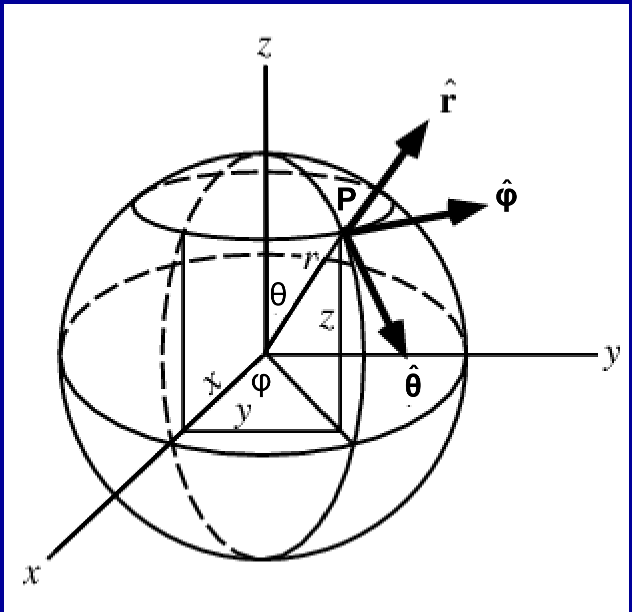
- $x = r \cos\varphi \sin\theta$
- $y = r \sin\varphi \sin\theta$
- $z = r \cos\theta$.



Spherical Coordinates Basis

Given the vector **r**, we can use the basis defined by the orthogonal vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$ such that $\mathbf{a} = a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_\varphi\mathbf{e}_\varphi$ and so on.

For a scalar field $f(r,\theta,\varphi)$, calculating the gradient ∇f such that $df = \nabla f \cdot d\mathbf{r}$ is a bit harder to do.



The displacement $d\mathbf{r}$ has three components dr, $d\theta$ and $d\varphi$ with $d\mathbf{r} = dr\,\mathbf{e}_r + r\,d\theta\,\mathbf{e}_\theta + r\sin\theta\,d\varphi\,\mathbf{e}_\varphi$.

As $df = \partial f/\partial r\,dr + \partial f/\partial\theta\,d\theta + \partial f/\partial\varphi\,d\varphi$ we have

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\mathbf{e}_\varphi$$

$\mathbf{F} = -\nabla U$ Again

Claimed fact about central forces:
$\mathbf{F} = f(\mathbf{r})\, \mathbf{e}_r$ is conservative $\Leftrightarrow$ $f(\mathbf{r}) = f(r)$ depends only on r.

Proof of " $\Rightarrow$" using spherical coordinates:
As **F** is conservative, let U be the potential of **F**, with
$\mathbf{F} = -\nabla U = -\partial U/\partial r\, \mathbf{e}_r - 1/r\, \partial U/\partial\theta\, \mathbf{e}_\theta - (1/r \sin\theta)\partial U/\partial\varphi\, \mathbf{e}_\varphi$.
Because **F** is central, $\partial U/\partial\theta = \partial U/\partial\varphi = 0$ (U is rotationally invariant), hence $\mathbf{F} = -\partial U/\partial r\, \mathbf{e}_r$ is rotationally invariant.

Proof of " $\Leftarrow$" using spherical coordinates:
Using the "spherical curl" on $\mathbf{F} = f(r)\, \mathbf{e}_r$, (see back of book) one gets $\nabla\times\mathbf{F} = \mathbf{0}$, so **F** is conservative.