Classical Mechanics

Phys105A, Winter 2007

Wim van Dam
Room 5109, Harold Frank Hall
vandam@cs.ucsb.edu
http://www.cs.ucsb.edu/~vandam/

Formalities

- New homework is posted,
  due Monday March 5, 11:30 am.

- Answers to Midterm/Homework 5 are posted.

- Questions?

Driven Damped Oscillations

A damped oscillator (with m,b,k) driven by a time dependent force F(t) is described by the equation

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

Rewriting with $2\beta = b/m$, $\omega_0 = \sqrt{(k/m)}$ and $f(t) = F(t)/m$ gives

$$(D^2 + 2\beta D + \omega_0^2)x = f(t)$$

This is an inhomogeneous differential equation, for which we know how to solve the homogeneous part.

We will describe a *particular solution* for $f = f_0 \cos \omega t$, where ω is the *driving frequency*.

Solving the Driven Oscillator

Solving the equation for the sinusoidal driving force

$$(D^2 + 2\beta D + \omega_0^2)x = f_0 \cos \omega t$$

gives…

$$x(t) = A\cos(\omega t - \delta) + C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

With

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \qquad \delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

The C_1, C_2, r_1, r_2 are determined by the homogeneous equation and their transient effect does not matter in the limit $t \to \infty$.

Resonance

The amplitude squared $A^2 = f_0^2/((\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2)$ of a driven oscillator depends not only on f_0 but also on the relation between the frequencies ω_0 and ω.

For small β (small drag), $\omega_0 \approx \omega$ will give a high response A: the frequencies are "in resonance".

Specifically, $\omega = \sqrt{(\omega_0^2 - 2\beta^2)}$ gives maximum A.

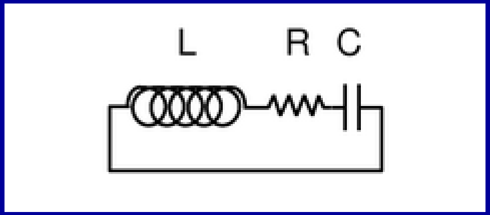
For small β, it gives $A \approx f_0/(2\beta\omega_0)$.

How critical a system depends on the driving frequency is expressed by the *quality factor* $Q = \beta/2\omega_0$ (cf. Section 5.6).

LRC Circuits

Besides for mechanical systems, the resonance of damped oscillations is also relevant for LRC circuits.

A circuit with inductor (L), resistor (R) and capacitor (C) is described by the differential equation:



$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$$

where q is the charge at the capacitor and we used Kirchoff's 2nd rule (which says that the sum of Voltage difference should be zero as we go around the circuit).

Fourier Analysis

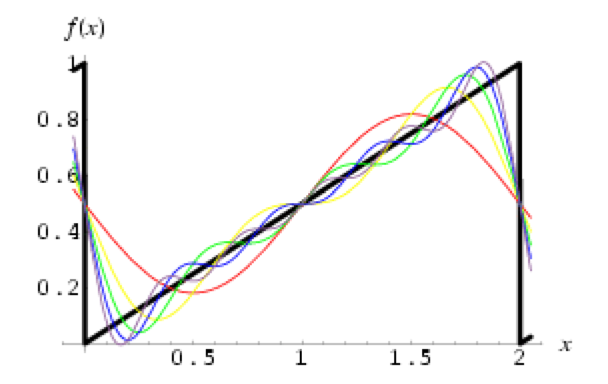
Observation: If one has solutions $x_j(t)$ to linear differential equations $Dx = f_j(t)$, then –by linearity– it holds that $x = c_1x_1 + \ldots + c_nx_n$ is a solution to the differential equation $Dx = c_1f_1(t) + \ldots + c_nf_n(t)$, where c_j are complex constants.

Fourier analysis tells us that each $2\pi/\omega$ periodic function f can be written as an (infinite) sum $f(t) = \Sigma_n c_ne^{n\omega it}$.

Using such a decomposition, and the known solutions to the equations $Dx = e^{n\omega it}$, we can solve the case $Dx = f(t)$.

This is very important when dealing with non sinusoidal pulses such as square waves or sawtooth waves.

A Fourier Decomposition



This sawtooth wave is defined as $f(t) = (t \bmod 2)/2$ for all t.

This function can be rewritten as

$f(t) = \frac{1}{2} - 1/\pi[\sin(\pi t) + \sin(2\pi t)/2 + \sin(3\pi t)/3 + \ldots]$

The coefficients $1/\pi n$ for the respective frequencies $n\pi$ make up the *spectrum* of the function f(t).