Classical Mechanics

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Euler-Lagrange Equation

Let y(x) be the path that minimizes/maximizes the integral

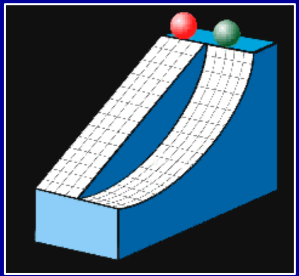
$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx$$

The Euler-Lagrange equation tells us that S is extremal, or *stationary*, when y(x) obeys

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

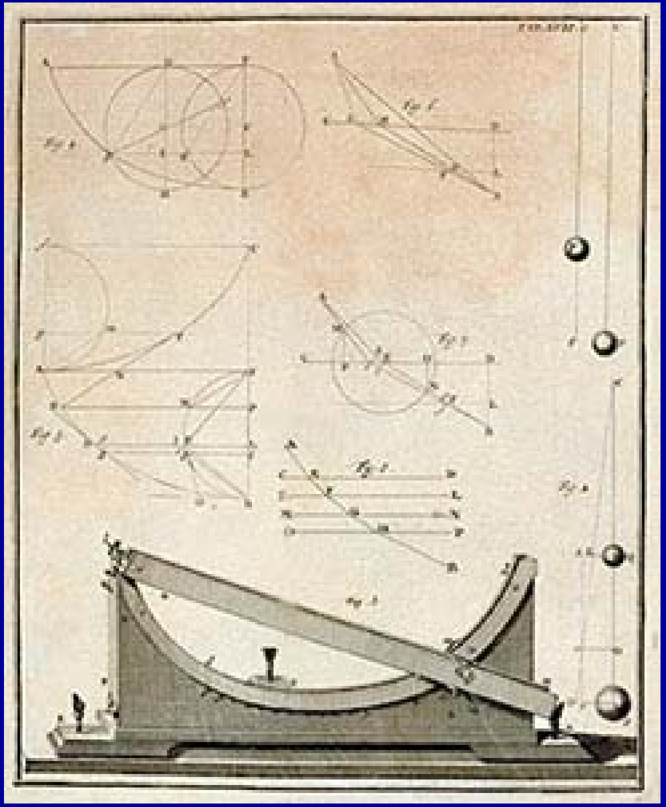
Fastest Path

Given two points P,Q in $\mathbb{R}^2$ in a gravitational field $g\mathbf{e}_x$, what is the fastest path y(x)?



Find y(x) that minimizes

$$\frac{1}{\sqrt{2g}}\int_{x_1}^{x_2}\frac{\sqrt{1+y'^2}}{\sqrt{x}}dx$$



The answer to this classic *brachistochrone problem* is that y(x) is (part) of a cycloid
$x(\theta) = a(1-\cos\theta)$
$y(\theta) = a(\theta-\cos\theta)$

Brachistochrone Problem

For the shortest path we look for the extremal values of the integral $\int f(x)\,dx$ with $f(x) = \sqrt{(1+y'^2)}/\sqrt{x}$.

Euler-Lagrange says $\partial f/\partial y - d(\partial f/\partial y')/dx = 0$, hence $\partial f/\partial y'$ is constant (in x).

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{x}} \frac{y'}{\sqrt{1+y'^2}} = c$$ gives the equation $$y' = \sqrt{\frac{x}{2a-x}}$$

$$y(x_2) = \int_{x=0}^{x_2} \sqrt{\frac{x}{2a-x}}\,dx$$ has solution $$x = a(1-\cos\theta)$$ $$y = a(\theta - \sin\theta)$$

E-L for Several Variables

For an arbitrary number of variables $q_1(t),\ldots,q_n(t)$, with t the independent variable, what are the extremals of:

$$S = \int_{t_1}^{t_2} L[q_1,\dot{q}_1,\ldots,q_n,\dot{q}_n,t]dt\ ?$$

The Euler-Lagrange equation tells us (again) that the dependent variables have to obey for all j:

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0$$

Notation

Note the following shorthand conventions:

The derivative of y with respect to x can be denoted using the apostrophe: dy/dx = y'.

Derivatives with respect to time are often denoted with a dot above the coordinate. Hence

$$y' = \frac{dy}{dx} \quad \text{and} \quad \dot{y} = \frac{dy}{dt}$$

The "∂ versus d" notation in $\partial L/\partial q - d(\partial L/\partial \dot{q})/dt = 0$ should remind you that t is the independent variable and that d/dt concerns all variables in L (unlike the partial derivatives such as ∂L/∂q).

Chapter 7

Lagrange's Equations

Langrangian

For an unconstrained system in 3 dimensions subject to the a conservative field with potential energy U=U(**r**) and kinetic energy $T = \frac{1}{2}m(x'^2+y'^2+z'^2)$ we have the **Langrangian** (or **Lagrange function**): $\mathcal{L} = T - U$, in this case $\mathcal{L}(x,y,z,x',y',z') = \frac{1}{2}m(x'^2+y'^2+z'^2) - U(x,y,z)$.

Note that $\partial\mathcal{L}/\partial x = -\partial U/\partial x = F_x$ and $\partial\mathcal{L}/\partial x' = mx' = p_x$, hence $\partial\mathcal{L}/\partial x - d(\partial\mathcal{L}/\partial x')/dt = 0$; similarly for y and z.

These are the three Lagrange equations, which follow from Newton's second law for q(t) = x(t), y(t), and z(t).

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}$$