Lecture 11: Probability Examples

Eric Vigoda
Discrete Mathematics for Computer Science

February 18, 2026

11 Probability Examples

In this lecture we'll look at more probability examples, focusing on our familiar example of poker hands. We'll repeatedly use complement counting inside probability calculations. Then we'll introduce conditional probabilities and Bayes Law.

11.1 Poker Hands

Let's return to our running example of poker hands. We have 52 cards, divided into 4 suits – clubs, diamonds, hearts, and spades – and each suit has cards numbered 1 to 13. Two suits – diamonds and hearts – are red cards, and the other two suits are black cards.

Let Ω denote all possible poker hands which are all sets of 5 cards. The set Ω is our sample space. Any subset $S \subset \Omega$ is an event, and the probability of an event S is the following:

$$\Pr[S] = \sum_{\omega \in S} \Pr[\omega].$$

In this case, since every poker hand is equally likely and hence we are considering the uniform distribution over Ω, which are all poker hands, we then have:

$$\Pr[S] = \sum_{\omega \in S} \Pr[\omega] = \sum_{\omega \in S} \frac{1}{|\Omega|} = \frac{|S|}{|\Omega|}.$$

Let's consider a few examples.

Example 1: You are dealt a random poker hand, which are 5 cards chosen at random from the deck. What is the probability that all 5 cards are red?

Let A denote the event that all 5 cards are red. Our first task is to calculate $|\Omega|$, which is the number of possible poker hands. That is easy to do as it is

$$|\Omega| = \binom{52}{5}.$$

Our main task then is to compute $|A|$, which is the number of poker hands with all red cards. This again is fairly straightforward to calculate using our combinatorics techniques. Since there are 26 red cards we have that

$$|A| = \binom{26}{5}.$$

Combining these bounds together, we have the following result:

$$\Pr[\text{random poker hand is all red}] = \Pr[A] = \frac{|A|}{|\Omega|} = \frac{\binom{26}{5}}{\binom{52}{5}}.$$

If you compute this quantity numerically it's approximately .02531, and thus there's roughly a 2.5% chance that a random poker hand is all red.

Example 2: In a random poker hand, what is the probability that there's at least one black card? Let $\overline{A} = \Omega \setminus A$. Observe that $\overline{A}$ are those hands which contain at least one black card. Since A and $\overline{A}$ are disjoint (which means $A \cap \overline{A} = \emptyset$) and $A \cup \overline{A} = \Omega$, we have

$$\Pr[A] + \Pr[\overline{A}] = 1.$$

Therefore, we can use complement counting to conclude the following:

$$\Pr[\text{random poker hand has at least one black card}] = \Pr[\overline{A}] = 1 - \Pr[A] = 1 - \frac{|A|}{|\Omega|} = 1 - \frac{\binom{26}{5}}{\binom{52}{5}} \approx .97469,$$

so there's a very good chance of getting at least one black card.

Example 3: In a random poker hand, what is the probability of getting at least one black card and at least one red card? Our first guess might be $1 - 2 \times .02531 = .94938$, is that correct?

Let R denote the event that at least one card is red, and let B denote the event that at least one card is black. We are interested in the event $R \cap B$, which is the event that at least one card is red and at least one card is black. How do we compute $|R \cap B|$?

Recall Lecture 8 where we computed the number of hands with at least one diamond and at least one heart. Let's again apply complement counting and compute $|\overline{R \cap B}$. Applying DeMorgan's law we have:

$$\overline{R \cap B} = \overline{R} \cup \overline{B}.$$

This is an easier task to compute $|\overline{R} \cup \overline{B}|$ as we can apply the inclusion-exclusion formula:

$$|\overline{R} \cup \overline{B}| = |\overline{R}| + |\overline{B}| - |\overline{R} \cap \overline{B}|.$$

Considering $|\overline{R}|$, which is the number of hands with no red cards, that means all 5 cards are black; we computed this earlier and saw that it's $\binom{26}{5}$ since there are 26 black cards. Similarly, $|\overline{B}|$ is the same since there are 26 red cards. Hence, we have:

$$|\overline{R}| = |\overline{B}| = \binom{26}{5}.$$

It remains to compute $|\overline{R} \cap \overline{B}|$. These are hands with no red cards and no black cards – that seems very difficult to achieve! Clearly,

$$|\overline{R} \cap \overline{B}| = 0.$$

Putting it all together we have the following:

$$\begin{aligned}
\Pr[\geq 1 \text{ black and} \geq 1 \text{ red card}] &= \Pr[R \cap B] \\
&= 1 - \Pr[\overline{R \cap B}] \\
&= 1 - \frac{|\overline{R \cap B}|}{|\Omega|} && \text{by complement counting} \\
&= 1 - \frac{|\overline{R} \cup \overline{B}|}{|\Omega|} && \text{by DeMorgan's law} \\
&= 1 - \frac{|\overline{R}| + |\overline{B}| - |\overline{R} \cap \overline{B}|}{|\Omega|} && \text{by inclusion-exclusion} \\
&= 1 - \frac{2 \times \binom{26}{5}}{\binom{52}{5}} \\
&\approx 1 - 2 \times .02531 = .94938,
\end{aligned}$$

which confirms our original guess!

11.2 Conditional Probability

Example 4: Suppose I tell you that your poker hand contains at least one red card. What is the probability that it contains at least one black card?

Recall, R is the event that at least one card is red and B is the event that at least one card is black. In our current example, we want to compute "the conditional probability of B given R". That is if we condition on the event R occurring, what is the probability of the event B in this conditional space? We denote this as:

$$\Pr[B \mid R],$$

which we read as the conditional probability of B given R.

By conditioning on the event R occurring then we've changed our sample space from Ω to R. Then for each $\omega \in R$, its new probability is denoted by $\Pr[\omega \mid R]$ and it's equal to:

$$\Pr[\omega \mid R] = \frac{\Pr[\omega]}{\Pr[R]};$$

this is referred to as the conditional probability of ω given R. Conditioning restricts our universe from Ω to the smaller sample space R, and then renormalizes probabilities within that space.

It is important that the conditional probabilities sum to one (so we have a well-defined probability distribution), which we can verify now:

$$\sum_{\omega \in R} \Pr[\omega \mid R] = \sum_{\omega \in R} \frac{\Pr[\omega]}{\Pr[R]} = \frac{1}{\Pr[R]} \sum_{\omega \in R} \Pr[\omega] = \frac{\Pr[R]}{\Pr[R]} = 1.$$

Going back to the conditional probability of B given R we now have:

$$\Pr[B \mid R] = \sum_{\omega \in B \cap R} \Pr[\omega \mid R] = \sum_{\omega \in B \cap R} \frac{\Pr[\omega]}{\Pr[R]} = \frac{\Pr[B \cap R]}{\Pr[R]}.$$

Let's state the general formula for conditional probabilities. For events S and T with $\Pr[T] > 0$,

$$\Pr[S \mid T] = \frac{\Pr[S \cap T]}{\Pr[T]}.$$

In this case, since our probability distribution is the uniform distribution we can simplify it further as follows:

$$\Pr[B \mid R] = \frac{\Pr[B \cap R]}{\Pr[R]} = \frac{|B \cap R|/|\Omega|}{|R|/|\Omega|} = \frac{|B \cap R|}{|R|}.$$

Notice this is exactly the same formula as ordinary probability — except the denominator is now $|R|$ instead of $|\Omega|$.

In Example 2 we calculated the number of hands with at least one red card using complement counting:

$$|R| = |\Omega| - \binom{26}{5}.$$

And in Example 3 we calculated the number of hands with at least one black card and at least one red card:

$$|B \cap R| = |\Omega| - 2 \times \binom{26}{5}.$$

Putting it all together we have the following:

$$
\begin{aligned}
\Pr[\geq 1 \text{ black card} \mid \geq 1 \text{ red card}] &= \Pr[B \mid R] \\
&= \frac{|B \cap R|}{|R|} \\
&= \frac{|\Omega| - 2 \times \binom{26}{5}}{|\Omega| - \binom{26}{5}} \\
&= \left(\frac{|\Omega| - \binom{26}{5}}{|\Omega| - \binom{26}{5}} \right) - \left(\frac{\binom{26}{5}}{|\Omega| - \binom{26}{5}} \right) \\
&= 1 - \frac{\binom{26}{5}}{\binom{52}{5} - \binom{26}{5}} \\
&\approx 1 - .02596 = .97404.
\end{aligned}
$$

Recall in Example 2, we calculated the probability of at least one black card is roughly .97469, which is slightly higher than the probability of at least one black card when we condition on having at least one red card. This smaller probability makes sense because conditioning on having at least one red card slightly reduces the number of cards available to be black.

Going back to our example, we saw that:

$$
\Pr[\geq 1 \text{ black card} \mid \geq 1 \text{ red card}] = 1 - \frac{\binom{26}{5}}{\binom{52}{5} - \binom{26}{5}}
$$

This last term has a clear interpretation. Note, $\binom{26}{5}$ is the number of hands with all red cards, and $\binom{52}{5} - \binom{26}{5}$ is the number of hands with at least one red card. Note, for B denoting the hands with at least one black card, then $\overline{B}$ are the hands with no black cards which is the same as having all red cards. Hence, we have the following:

$$
\Pr[B \mid R] = 1 - \Pr[\overline{B} \mid R],
$$

which is the analog of complement counting once again.

In general for events S and T where $\Pr[T] > 0$, we have:

$$
\Pr[S \mid T] = 1 - \Pr[\overline{S} \mid T] = 1 - \frac{\Pr[\overline{S} \cap T]}{\Pr[T]}. \tag{1}
$$

In our case, where the probability distribution is the uniform distribution we can simplify it further as follows:

$$
\Pr[B \mid R] = 1 - \Pr[\overline{B} \mid R] = 1 - \frac{\Pr[\overline{B} \cap R]}{\Pr[R]} = 1 - \frac{\binom{26}{5}}{\binom{52}{5} - \binom{26}{5}};
$$

this gives us a much more straightforward way of computing the conditional probability of at least one black card given at least one red card, using the analog of complement counting.

11.3 Bayes Law

For events S and T where $\Pr[T] > 0$ we have the following conditional probability of S given T:

$$
\Pr[S \mid T] = \frac{\Pr[S \cap T]}{\Pr[T]}.
$$

Suppose $\Pr[S] > 0$ and let's look at the conditional probability of T given S:

$$
\Pr[T \mid S] = \frac{\Pr[T \cap S]}{\Pr[S]}.
$$

If we solve both of these equations for $\Pr[S \cap T] = \Pr[T \cap S]$ we obtain the following:

$$\begin{aligned}\Pr[S \cap T] &= \Pr[S \mid T] \Pr[T] \\ &= \Pr[T \mid S] \Pr[S].\end{aligned}$$

Therefore,

$$\Pr[S \mid T] Pr[T] = \Pr[T \mid S] \Pr[S].$$

Now solving for $\Pr[S \mid T]$ we have:

$$\Pr[S \mid T] = \frac{\Pr[T \mid S] \Pr[S]}{\Pr[T]};$$

this is known as **Bayes Law** (it is also called Bayes Theorem or Bayes Rule).

Notice that conditional probability introduces asymmetry: $\Pr[S \mid T]$ need not equal $\Pr[T \mid S]$. Conditioning changes the sample space, and different conditionings generally produce different probability distributions. Bayes Law tells us precisely how these two conditional probabilities are related. We can use the relationship between the two quantities to simplify our calculations.

11.4 Example Application of Bayes Law

Now let's see a nice application of Bayes Law to our poker setting.

Example 5: You're told that your poker hand contains at least one Ace, what's the conditional probability that it contains exactly one Ace?

Let S denote the event that it contains exactly one Ace, and let T denote the event that it contains at least one Ace. Hence, our goal is to compute $\Pr[S \mid T]$. Applying Bayes Law we have:

$$\Pr[S \mid T] = \frac{\Pr[T \mid S] \Pr[S]}{\Pr[T]}.$$

This is a bit easier since $\Pr[T \mid S]$ is trivial as we'll see now. If our poker hand has exactly one Ace, then it also has at least one Ace; hence, $S \subset T$, and therefore: $\Pr[S \cap T] = \Pr[S]$, which implies the following:

$$\Pr[T \mid S] = \frac{\Pr[T \cap S]}{\Pr[S]} = \frac{\Pr[S]}{\Pr[S]} = 1.$$

Therefore,

$$\Pr[S \mid T] = \frac{\Pr[T \mid S] \Pr[S]}{\Pr[T]} = \frac{\Pr[S]}{\Pr[T]} = \frac{|\, S \,|}{|\, T \,|}.$$

Computing each of the remaining quantities is straightforward. For $|S|$, which is exactly one Ace, we have the following:

$$|S| = \binom{4}{1}\binom{48}{4} = 4 \times \binom{48}{4}.$$

For $|T|$, which is at least one Ace, we use complement counting:

$$|T| = \binom{52}{5} - \binom{48}{5}.$$

Putting these calculations together we obtain:

$$\Pr[S \mid T] = \frac{\Pr[T \mid S] \Pr[S]}{\Pr[T]} = \frac{\Pr[S]}{\Pr[T]} = \frac{|S|}{|T|} = \frac{4 \times \binom{48}{4}}{\binom{52}{5} - \binom{48}{5}}.$$

If we compute this numerically, it's approximately .8776, so if your hand contains at least one Ace then there's roughly 87.8% chance that it contains exactly one Ace.

11.5 Additional Exercise

Compute, using Bayes Law, the conditional probability that a poker hand contains at least 2 aces given it contains at least 1 ace.

Let R denote the poker hands with at least 2 Aces, and let T denote the poker hands with at least 1 Ace. Then our goal is to compute $\Pr[R \mid T]$. Applying Bayes Law we obtain the following:

$$\Pr[R \mid T] = \frac{\Pr[T \mid R] \Pr[R]}{\Pr[T]}.$$

Since $R \subset T$, we have $R \cap T = R$. Thus,

$$\Pr[T \mid R] = \frac{\Pr[T \cap R]}{\Pr[R]} = \frac{\Pr[R]}{\Pr[R]} = 1.$$

Therefore we only need to compute $\Pr[R] = |R|/|\Omega|$, and $\Pr[T] = |T|/|\Omega|$. Since T are those hands with at least one Ace, we use complement counting as in the last example:

$$|T| = \binom{52}{5} - \binom{48}{5}.$$

And since R are those hands with at least 2 Aces, we again use complement counting. In this case, $|\overline{R}|$ are those hands with at most 1 Ace, which is the same as the number of hands with no Aces or exactly 1 Ace. Hence, we have the following:

$$|R| = |\Omega| - |\overline{R}| = \binom{52}{5} - \binom{48}{5} - 4 \times \binom{48}{4}.$$

Finally, we have

$$\Pr[R \mid T] = \frac{\Pr[T \mid R] \Pr[R]}{\Pr[T]} = \frac{\Pr[R]}{\Pr[T]} = \frac{|R|}{|T|} = \frac{\binom{52}{5} - \binom{48}{5} - 4 \times \binom{48}{4}}{\binom{52}{5} - \binom{48}{5}} = 1 - \frac{4 \times \binom{48}{4}}{\binom{52}{5} - \binom{48}{5}}.$$

Notice that the event $T \setminus R$ consists of those hands with exactly 1 Ace, and the above shows that:

$$\Pr[R \mid T] = 1 - \Pr[T \setminus R \mid T],$$

which is the analog of complement counting in the setting of conditional probabilities. In general, for events A and B we have:

$$\Pr[A \mid B] = 1 - \Pr[B \setminus A \mid B] = 1 - \Pr[\overline{A} \cap B \mid B],$$

which matches our earlier statement in Equation (1).