Lecture 1: Graphs and Set Operations

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Discrete Mathematics for Computer Science

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1.1 Course Overview

Welcome to the Discrete Mathematics for Computer Science course. This course has several complementary goals:

- Learn some fun, interesting mathematics.
- Gain a strong background to enable you to succeed in CS 130A/B Data Structures and Algorithms courses.
- Introduce discrete mathematics topics, including the basics of graph theory, combinatorics, probability, and number theory.

At the same time, you will get extensive exposure to proofs (via examples in lectures), and a lot of practice writing proofs. Writing proofs is like writing essays, you can always improve, so you're not expected to be an expert at the end of the course. But, after finishing this course, you will be much more comfortable writing proofs, and at the same time you will have a much better mathematical background so that you can better understand proofs.

This course was inspired and influenced by CS 70 (Discrete Mathematics and Probability Theory) at UC Berkeley, and my lecture notes are often based on the CS 70 course notes. I'm also using the Discrete Mathematics textbook by Oscar Levin [Lev25], which is a free, open access resource so check it out.

1.2 Introduction to Graphs

One of the key tools in this course, and more generally in computer science, are graphs. These are not graphs as you make a diagram to show for example a histogram of some data.

In computer science a graph is a representation of set of objects and connections between these objects. The objects are called the **vertices** of the graph. Some fields, such as those in engineering and even some CS fields, use the term nodes instead of **vertices**.

For example, the vertices may be a set of cities, a set of books, a set of people, a set of webpages, or a set of genes. The **edges**, which are sometimes called links, will represent connections between pairs of vertices.

Below is an example graph in Fig. 1. This graph might represent a group of 8 students where the edges are pairs of students who have taken a class together.

Before continuing, notice that we referred to the vertices as a set of objects. **What is a set?** A set is an unordered collection of unique objects. For example, here is the set of names in the example graph:

$$\{\textit{Bill, Chen, Jen, Heng, Dave, Alan, Liz, Nina}\}.$$

Moreover, I can write it as follows:

$$V = \{\textit{Bill, Chen, Jen, Heng, Dave, Alan, Liz, Nina}\}.$$

and then I can use V to refer to this particular set. We typically use a capital letter, such as V or S, to refer to a set, the exact letter depends on what the set represents: if it's a set of vertices then I'll use V, if it's a set of edges then I'll use E, or I might just use S to denote that it's a set if I can't think of a more

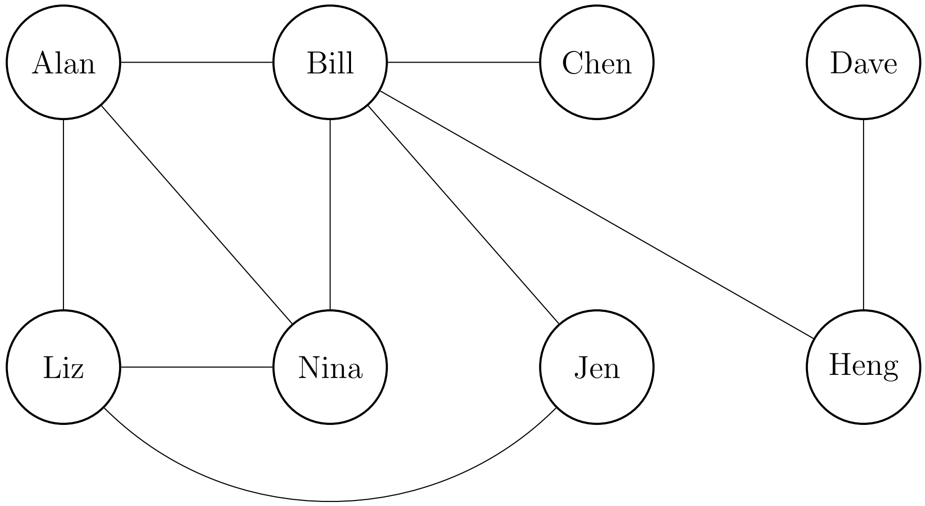

Figure 1: Example graph with 8 vertices and 10 edges.

appropriate letter. By thinking a bit about the notation we use later in our proofs (in this case the letter used to represent a set), it can help the reader understand it easier.

Each of these names is an element of the set, e.g., Chen is an element of the set V, and also Jen is an element of the set V; there are 8 elements in this set V. We denote set membership as: $x \in S$ for element x is in the set S or $y \notin T$ for element y is not in the set T. In our earlier example, I'd say: $Nina \in V$ and $Eric \notin V$.

A further note on the definition of a set: a set needs to be well defined, this means that it needs to be clear and unambiguous which elements belong to a set or not. For example, the above set V is well-defined as it is clear that those 8 names are in the set V and those are the only elements of this set V. But if we let S be the set of best songs, that is not well defined as it is not clear what is the definition/criteria for best.

We use curly braces {} on the outside of the elements of the set, this is because sets are unordered. Hence, these two sets are the same:

$$\{\textit{Frog, Turtle, Dolphin}\} = \{\textit{Dolphin, Frog, Turtle}\}.$$

Also, repeats don't matter, an element is either in the set or it is not, hence these two sets are the same:

$$\{\textit{Green, Red, Orange, Orange, Green, Green}\} = \{\textit{Green, Orange, Red}\}.$$

In general, two sets A and B are **equal**, which we denote as $A = B$, if A and B contain the same elements, regardless of the order within each set. For example, for $A = \{5, 3, 7\}$ and $B = \{7, 5, 3\}$, then $A = B$.

1.3 Mathematical Notation

Here are some commonly referenced sets:

- The set of natural numbers is denoted as $\mathbb{N} = \{0, 1, 2, \dots\}$.
- The set of integers is denoted as $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
- The set of all real numbers is denoted as $\mathbb{R}$.

The set of all rational numbers is denoted as $\mathbb{Q}$. How do you define the set of rational numbers? These are all numbers which are obtainable as a fraction of a pair of integers (where the denominator is non-zero). We write that mathematically in the following manner:

$$\mathbb{Q} = \{x \in \mathbb{R} : x = \frac{a}{b} \text{ for some } a, b \in \mathbb{Z} \text{ where } b \neq 0\}.$$

How do we read the RHS (right-hand side) of the above expression? First, the colon is read as "such that", and sometimes the symbol | is used in place of a colon and it also means such that, which symbol you use is up to you, it's your stylistic choice. The expression, in words, says: the set $\mathbb{Q}$ is the set of all real numbers x such that x equals a fraction a over b where a and b are both integers and b is not zero. It's just saying there exists integers a and b where this holds.

The above expression defining the set $\mathbb{Q}$ is well-defined. If we wanted to express it in a slightly different way we could use the **quantifier $\exists$ which says "there exists"**. Then, we would state the definition of $\mathbb{Q}$ as follows (this is equivalent to the previous definition):

$$\mathbb{Q} = \left\{ x \in \mathbb{R} : \exists \, (a, b \in \mathbb{Z}) \text{ where } x = \frac{a}{b} \text{ and } b \neq 0 \right\}.$$

Here are some examples using the quantifier $\exists$ for there exists.

$$\exists x \in \mathbb{R}, \; x^2 = 4$$

This says: there exists a real number x such that $x^2 = 4$. This statement is true as $x = 2$ satisfies it.

$$(\exists n \in \mathbb{N}) \, (n = -1)$$

Note, this is slightly different formatting then the previous statement, but that is just stylistic choice; I want to expose you to different styles as different instructors and resources will vary. This statement says: there exists a natural number n such that $n = -1$. This statement is false as $n = -1$ is not a natural number; the natural numbers start from 0.

$$\exists x \in \mathbb{N}, \; x^2 = 2$$

This says: there is an integer x where $x^2 = 2$. That is a false statement. If we considered $x \in \mathbb{R}$ then that would be a true statement.

An alternative **quantifier** (complementary to "there exists") is **"for all" which is denoted by** $\forall$. Here is an example:

$$\forall x \in \mathbb{Z}, \; 2x + 2 \text{ is even.}$$

This says that for every integer x, then $2x+2$ is an even number. This is true since $2x$ is even and hence $2x + 2$ is also even.

Here are two mathematical expressions involving these two quantifiers:

$$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(y > x) \tag{1}$$

$$(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})(y > x) \tag{2}$$

The expression in Eq. (1) says: for every integer x there is some integer y where $y > x$. That is clearly true, since we can take $y = x + 1$ or $y = x + 100$ and then that expression is satisfied. Notice that we first set x and then y can depend on x.

The second expression, namely Eq. (2), says that there is an integer y so that for every integer x it holds that $y > x$. Here, we are first fixing y, and then for this particular y we need that for every possible integer x that $y > x$. But this is clearly not true since when we consider $x = y + 1$ it is false. Notice that in this case since we first choose y then y cannot depend on x (so we say it's a universal y), but x can depend on y in this case.

In summary, the first expression is true and the second expression is false. Each of these expressions is an example of a proposition. A proposition is an expression which is true or false. We will review propositions in the next lecture.

Here are further examples of various sets:

$$\text{Set of all even numbers} = \{x \in Z : x = 2k \text{ for some } k \in \mathbb{Z}\} = \{x : \exists k \in \mathbb{Z}, x = 2k\}.$$

Here is another set: let

$$A = \left\{ x^2 : x \in \mathbb{Z}, x \geq 1 \right\}.$$

Note, in this case $A = \{1, 4, 9, 16, \ldots, \}$.

1.4 Sets: Basic Operations

Consider the set

$$A = \{\textit{5.3, apple, orange, } \frac{7}{11}, \pi\ \}.$$

There are 5 elements of the set A. We denote this as

$$|A| = 5,$$

and we refer to this as the cardinality of the set, or more informally as the size of the set. A set might have size 0, in which case we say the set is the empty set. We denote the empty set by the symbol $\emptyset$. For example, consider the following set:

$$B = \{x \in \mathbb{N} : x < 0\}.$$

Notice that there are no natural numbers less than 0. Hence, $|B| = 0$ and we say $B = \emptyset$ which we read as B is the empty set. For any set A where $|A| = 0$, we use the notation $A = \emptyset$ to indicate that this set A is empty.

We denote a **subset** of a set by $\subseteq$. Thus, $D \subseteq A$ if $(\forall x \in D)(x \in A)$, that is every element of D is also an element of A. For example, for $A = \{3, 5\}$, then $\{3, 5\} \subseteq A, \{3\} \subseteq A, \{5\} \subseteq A, \emptyset \subseteq A$. Note, D might contain all elements of A, in which case $|A| = |D|$, or D might be the empty set and then D is a subset of every other set. You can also flip the symbol to indicate a **superset**, thus we write $D \supseteq A$ to indicate that D is a superset of A which is equivalent to $A \subseteq D$.

Note, that if $A \subseteq D$ and $D \subseteq A$ then $A = D$.

Also, the empty set is always a subset of another set, thus for any set A, $\emptyset \subseteq A$. Hence, if a set A has cardinality $|A| = n$, then there are 2^n subsets of A. For example, for the set $A = \{3, 5\}$, there are 4 subsets: $\{3, 5\}, \{3\}, \{5\}, \emptyset$. Finally, when we write $A = \emptyset$ that means $|A| = 0$ since A has no elements in it. But if we write $B = \{\emptyset\}$ that means $|B| = 1$ where $\emptyset \in B$ (so the empty set is an element of B). For example, if we let $C = \{\{3, 5\}, \{3\}, \{5\}, \emptyset\}$ denote all subsets of A, then $\emptyset \in C$ but $C \neq \emptyset$. (In case you encounter it in a future course, note, C is called the power set of A, since it contains all subsets of A.)

We can also indicate a **proper subset** by $D \subset A$ or $D \subsetneq A$; both notations mean that D is a subset of A and $D \neq A$. Both notations $\subset$ and $\subsetneq$ are OK (and they mean the same thing), but I prefer the notation $\subsetneq$ to stress that the sets cannot be equal because it's easy to overlook it otherwise (and because sometimes people use $\subset$ when they really mean $\subseteq$).

Two basic operations on sets are the union operation and the intersection operation. The union operation is denoted by $\cup$ and is used to combine two sets. For example, let $A = \{5, 7, 11.1\}$ and $B = \{7, 2\}$, and let $C = A \cup B$. Then, $C = \{5, 2, 11.1, 7\}$ (remember the order doesn't matter, and an element is either a member of a set or not – multiplicity doesn't matter). Formally, for sets A and B, their **union** is defined as:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

In contrast, the intersection of two sets are the elements in both sets. Thus, for $A = \{5, 7, 11.1\}$ and $B = \{7, 2\}$, then $C = A \cap B = \{7\}$. Formally, for sets A and B, their **intersection** is defined as:

$$A \cap B = \{x \in A : x \in B\}.$$

For two sets A and B whose intersection is the empty set (i.e., $|A \cap B| = 0$) then we say the sets A and B are **disjoint**. For example, the sets $A = \{5, 7, 11.1\}$ and $D = \{8, 6\}$ are disjoint, which we denote by $A \cap D = \emptyset$.

For sets A and B, the **set difference** between A and B is denoted as

$$A \setminus B = \{x \in A : x \notin B\}.$$

Sometimes the notation $A - B$ is used instead but that risks confusion with subtraction if the sets contain numbers. For example, if $A = \{5, 7, 11.1\}$ and $B = \{7, 2\}$, then $A \setminus B = \{5, 11.1\}$ and $B \setminus A = \{2\}$.

A closely related terminology is the **complement** of a set. For a set Ω and a subset $S \subseteq \Omega$, then the complement of S is denoted as:

$$\overline{S} = \Omega \setminus S = \{x \in \Omega : x \notin S\}.$$

A common example utilizing the complement of a set is the set of irrational numbers, which are defined to be $\mathbb{R} \setminus \mathbb{Q}$ where $\mathbb{R}$ are the real numbers and $\mathbb{Q}$ are the rational numbers.

1.5 Ordered Sets

Since a set is an *unordered* collection of objects denoted with $\{\}$, how do we denote and refer to an *ordered* collection? We use $(5, 7)$ to denote an **ordered pair**; this is also a vector of length 2. Similarly, we can have an ordered triple such as $(3, 1.2, 6)$. More generally, for n numbers $a_1, a_2, \ldots, a_n$ (these might be integers or real number or a mix), we refer to $(a_1, a_2, \ldots, a_n)$ as an n-tuple or a sequence of length n. If we have an infinite sequence $a_0, a_1, \ldots$ then we can refer to this sequence using $(a_i)_{i\geq 0}$ or as $(a_i)_{i\in\mathbb{N}}$.

We can also view a sequence as a function. For example, let $(f_n)_{n\geq 0}$ denote the sequence of Fibonacci numbers. Then, $f_0 = 0, f_1 = 1$ and for $i \geq 2$ then $f_i = f_{i-1} + f_{i-2}$. We can view f as a function from $f : \mathbb{N} \to \mathbb{N}$ and instead of referring to $f(n)$ for the n-th Fibonacci number, we prefer to use f_n for sequences but they are equivalent (just different styles).

1.6 Mathematical Notation: Sums and Products

We often want to write the sum or product of a sequence of numbers. For example,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n,$$

and for an infinite sum we write it as:

$$\sum_{i\geq 1}^{\infty} a_i = a_1 + a_2 + \ldots.$$

Note, for the summation, the starting index is in the subscript of the capital sigma and the ending index (if any) is in the superscript. Similarly, we can express the **product** as follows:

$$\prod_{i=1}^{n} a_i = a_1 \times a_2 \times \cdots \times a_n.$$

Here are a few simple examples:

$$\sum_{i=-5}^{5} i^2 = (-5)^2+(-4)^2+(-3)^2+(-2)^2+(-1)^2+0^2+1^2+2^2+3^2+4^2+5^2 = 2(25+16+9+4+1) = 110.$$

$$\prod_{i=2}^{5}(2i) = (2\times 2)\times(2\times 3)\times(2\times 4)\times(2\times 5) = 2^4 \times 120 = 1920.$$

We can similarly express sums or products of more complicated expressions, such as the following two examples:

$$\sum_{i\geq 2} \frac{3}{(i^2-1)} = \sum_{i=2}^{\infty} \frac{3}{(i^2-1)} = \frac{3}{2^2-1} + \frac{3}{3^2-1} + \frac{3}{4^2-1} + \ldots$$

$$\prod_{i=2}^{100}\left(1-\frac{1}{i^2}\right) = \left(1-\frac{1}{2^2}\right)\times\left(1-\frac{1}{3^2}\right)\times\left(1-\frac{1}{4^2}\right)\times\ldots\left(1-\frac{1}{100^2}\right)$$

We can also intersperse sums and products similar to what we did in our earlier examples for the quantifiers $\exists, \forall$ (see Eqs. (1) and (2)).

1.7 Back to Graphs

Finally, we can formally define graphs and introduce the appropriate notation. A graph consists of a set of vertices which we commonly denote by a set V. In this class (and in 130A and 130B) these graphs will always be finite, so we let $n = |V|$. We often label the vertices as $V = \{v_1, \ldots, v_n\}$ or as $V = \{1, \ldots, n\}$; this simplifies our algorithms or proofs so that we can refer to the i-th vertex as v_i or as i.

In our earlier example from Fig. 1, we can view V as an arbitrary labelling of the vertices as $\{1, \ldots, 8\}$ and then we have an additional function f where $f(1) = Alan, f(2) = Bill, f(3) = Chen, f(4) = Dave, f(5) = Liz, f(6) = Nina, f(7) = Jen$ and $f(8) = Heng$. Since Alan and Bill are connected by an edge, and Alan corresponds to vertex 1 and Bill corresponds to vertex 2, then we denote this edge between Alan and Bill by the set $\{1, 2\}$.

The edges are sets of pairs of vertices which we typically denote as E. In our earlier example from Fig. 1, we have the following collection of edges:

$$E = \{\{1, 2\}, \{8, 4\}, \{2, 8\}, \{3, 2\}, \{6, 1\}, \{5, 6\}, \{2, 6\}, \{5, 7\}, \{2, 7\}, \{4, 2\}\}.$$

The graph in Fig. 1 is then denoted by the ordered pair (V, E). For convenience, we write:

$$G = (V, E)$$

so that we can simply use G to refer to the graph defined by vertex set V and edge set E.

The graph defined by $G = (V, E)$ where $V = \{1, \ldots, n\}$ and $E = \{\{i, j\} : |i - j| = 1\}$ is known as the *path graph* P_n, which is depicted below for $n = 6$:

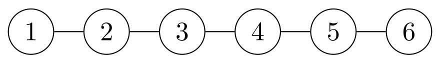

Figure 2: The path graph P_6.

1.8 More Graph Examples (optional)

(Optional sections of the notes cover material that will not be on graded work.)

Let $V = \{1, 2, 3, 4, 5, 6\}$. Then, $V \times V$ is the **Cartesian product** of the set V with itself, and is defined as

$$V \times V = \{(i, j) : i \in V, j \in V\} = \{(1, 1), (1, 2), \ldots, (1, 6), (2, 1), (2, 2), \ldots, (6, 6)\}.$$

Now consider the following graph, known as the grid graph. Let $G_{6,6} = (V', E')$ denote the grid graph on 6×6 vertices, and let $P_6 = (V, E)$ denote the path graph on 6 vertices. Then the vertex set for the grid graph is obtained by taking the Cartesian product of the path graph with itself, i.e., $V' = V \times V$ as illustrated in Fig. 3. Also, the grid graph $G_{6,6}$ is the Cartesian product of the graph P_6 with itself, but the Cartesian product of two graphs is denoted by $G_{6,6} = P_6 \square P_6$. The edges in $G_{6,6}$ are between pairs of vertices (i, j) and (k, ℓ) where $(i = k, \{j, \ell\} \in E)$ or $(\{i, k\} \in E, j = \ell)$. In contrast, if we use the symbol $\times$ as in $G \times G$, that refers what is known as the tensor product of G with itself, which is a different graph.

1.9 Undirected vs. Directed Graphs (optional)

All of the graphs we consider in this class are **undirected graphs**. In undirected graphs, the edges are a set of size two, thus a pair of vertices i and j are connected by an edge $\{i, j\}$. If you want to consider a directed pair (i, j) which we write as $\overrightarrow{ij}$ (which is different than $\overrightarrow{ji}$), then this is a **directed graph** $\overrightarrow{G} = (V, \overrightarrow{E})$; directed graphs are explored (literally with the Explore() procedure) in CS 130A.

1.10 Why Graphs?

Graphs are a great – and very natural – tool for creating simple models to study complex systems. An important example are probabilistic graphical models where the vertices are features of the data and the (weighted) edges model the strength of pairwise interactions. These models were introduced in the original work of Ackley, Hinton, and Sejnowski [AHS85] way back in 1985 where they were called Boltzmann machines. Graphical models or Boltzmann machines are a foundational tool in the theory of deep learning; in fact, Hinton was awarded the Nobel prize in physics in 2024, in part, for his work on Boltzmann machines.

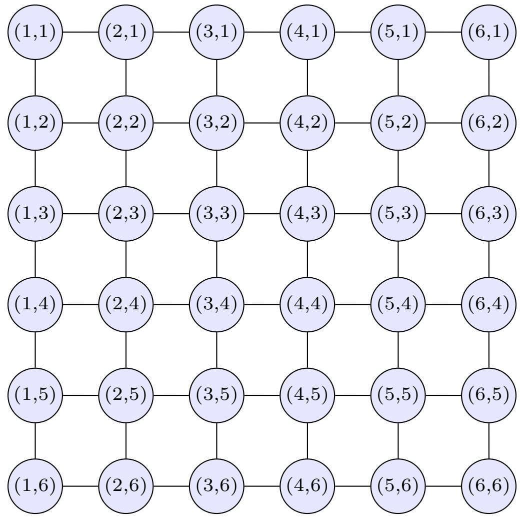

Figure 3: The grid graph on 6×6 vertices, which is obtained by taking the Cartesian product of P_6 with itself.

References

[AHS85] D. H. Ackley, G. E. Hinton, and T. J. Sejnowski. A learning algorithm for Boltzmann machines. *Cognitive Science*, 9(1):147–169, 1985.

[Lev25] Oscar Levin. *Discrete Mathematics: An Open Introduction*. CRC press, 4th edition, 2025. http://discrete.openmathbooks.org/.