Lecture 5: Planar Graphs and Colorings

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Consider a map – for example, a map of USA, Europe, Africa, Asia, or even Narnia. Then, partition the map into regions as you like – these regions can represent countries or states, or they might have no actual meaning and just be what you felt like drawing. Now consider a set of 4 colors, choose whichever 4 colors you like, the color names don't matter. We can color each region with one of the 4 colors so that all pairs of neighboring regions have different colors, where neighboring regions share a common boundary (so they share a boundary segment, not just a single point). Maps are always drawn in this way so that you can distinguish the regions, but the fascinating fact is that, for any map, this can always be done using only 4 colors. This is the famous **Four Color Theorem**.

We will not prove the Four Color Theorem in this lecture, but we will prove a weaker version which states that 6 colors always suffice.

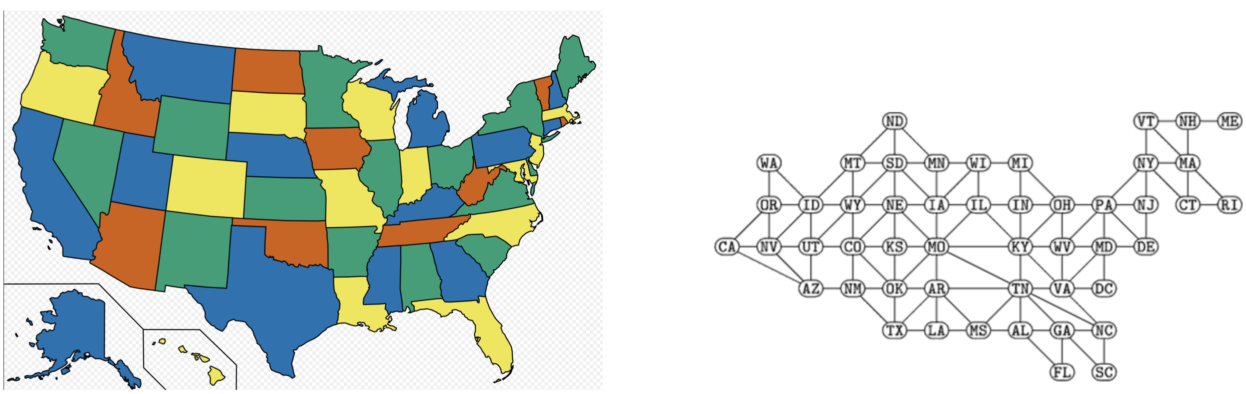
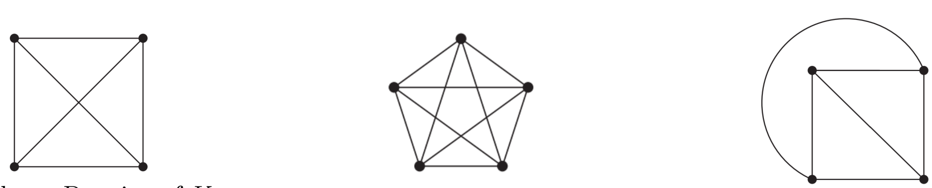


Figure 1: States of USA demonstrating the Four Color Theorem on the left, and the graphical representation of the continental USA (including DC) on the right.

5.1 Planar Graphs

In Figure 1 we can convert this map into a graph by creating a vertex for each state and connecting neighboring states by an edge. The corresponding graph is called a planar graph as there is a drawing with no crossing edges.



Non-planar Drawing of K_4 Non-planar Drawing of K_5 Planar Drawing of K_4

Figure 2: The first two images show non-planar drawings of K_4 and K_5, and the right image shows a planar drawing of K_4 where no edges cross.

More formally, for a graph $G = (V, E)$, a planar drawing of G (known as a planar embedding of G) is a drawing in the 2-dimensional plane $\mathbb{R}^2$ where there is a point $x_i \in \mathbb{R}^2$ for each $v_i \in V$, and for every edge $\{v_i, v_j\} \in E$ there is a curve connecting x_i and x_j. Moreover, in a planar drawing the curves for any two edges cannot intersect. For example, in Figure 2 we show two different drawings of K_4 which is the complete graph with 4 vertices, one is a non-planar drawing as two edges cross, and the other is a planar drawing.

A graph is called a **planar graph** if there exists a planar drawing, and a graph is non-planar if it does not have a planar drawing. Hence, K_4 is a planar graph and we will show later that K_5 is a non-planar graph. How do we prove that a graph is non-planar?

For a planar graph, such as K_4, it seems that there are many possible planar drawings. Is there some inherent structure in all of these planar drawings of K_4? Yes, as we will now see.

Consider the planar drawing of K_4 in Figure 2; it contains 3 closed regions and 1 open region. These regions are called the **faces** of the planar drawing; the open region is called the **outer face** and is counted as one of the 4 faces in this planar drawing. It turns out that every planar drawing for a particular graph has the same number of faces.

5.2 Euler's Formula

There is a simple formula, known as **Euler's formula**, which relates the number of faces to the number of vertices and edges.

Theorem 5.1. *For a planar graph $G = (V, E)$, let $v = |V|$ and $e = |E|$, and let f denote the number of faces (including the outer face). Then, for every connected planar graph, we have:*

$$v + f = e + 2.$$

Let us check this formula for K_4: it has $v = 4$ and $e = 6$ and in our planar drawing it has $f = 4$, hence Euler's formula is satisfied in this particular case. We will now prove it in general.

Proof. Fix $v = |V|$. We will prove the theorem by induction on $e = |E|$.

Since G is connected then the base case is when $e = v - 1$ and hence G is a tree. In this case there are no closed regions, there is only one face (namely, the outer face), and hence $f = 1$. Thus, $v + f = v + 1$ and $e + 2 = (v - 1) + 2 = v + 1$, which satisfies Euler's formula in this case.

Assume for integer $k \geq v - 1$, that Euler's formula holds for any connected planar graph on v vertices with $e = k$ edges.

Now consider a connected planar graph $G = (V, E)$ on v vertices and $e = k + 1$ edges. Let f denote the number of faces in G. Since G has $\geq v$ edges then it is not a tree and so there is at least one cycle in G. Consider any cycle C in G, and let $\{v, w\} \in C$ be an edge in this cycle.

Consider the graph $G' = (V, E \setminus \{v, w\})$ obtained by removing the edge $\{v, w\}$. Since the edge $\{v, w\}$ lies on a cycle, then after removing this edge the graph G' is still connected, and hence we can apply Euler's formula by the inductive hypothesis to this new graph G'.

Let v', e', f' denote the number of vertices, edges, and faces in this new graph G'. We have $v' = v$ and also $e' = k$ edges since we removed one edge. Notice that by removing an edge from a cycle we merged two faces together and hence $f' = f - 1$. Since G' has k edges, by the inductive hypothesis we know that:

$$v' + f' = e' + 2, \text{ and thus } v' + f' = k + 2.$$

Plugging in our earlier observations that $v' = v, e = k + 1 = e' + 1$ and $f' = f - 1$ we then have:

$$v + (f - 1) = (e - 1) + 2, \text{ which is the same as } v + f = e + 2,$$

which proves Euler's formula for the original graph G. □

5.3 Upper Bounding Number of Edges in Planar Graphs

Consider a planar graph such as Figure 3, which has $v = 11$ vertices, $e = 13$ edges, and $f = 4$ faces where f_{ext} is the outer-face. The red colored edges distinguish the so-called *bridge edges*. A bridge edge is an edge whose removal will disconnect the graph, and it lies within one face. Every non-bridge edge is on at least one cycle, and therefore separates two distinct faces.

Each face is bounded by a *closed walk* where the term closed means that the walk starts and ends at the same vertex. For example in Figure 3, the walks defining each face are as follows:

$$\begin{aligned} f_3\text{-walk: } & v_8, v_3, v_2, v_7, v_8 \\ f_2\text{-walk: } & v_6, v_8, v_7, v_5, v_9, v_5, v_6 \\ f_1\text{-walk: } & v_5, v_1, v_4, v_6, v_1 \\ f_{ext}\text{-walk: } & v_1, v_4, v_{10}, v_{11}, v_{10}, v_4, v_6, v_8, v_3, v_2, v_7, v_5, v_1 \end{aligned}$$

For face f, let $d(f)$ be the length of the walk bounding f, and we will refer to $d(f)$ as the *degree* of face f (analogous to the degree of a vertex). Hence, $d(f_1) = 4, d(f_2) = 6, d(f_3) = 4, d(f_{ext}) = 12$. Observe that $d(f_1) + d(f_2) + d(f_3) + d(f_{ext}) = 26 = 2e$. In general, we have the following *face-edge formula.*

Lemma 5.2 (Face-edge formula)**.** *Let $G = (V, E)$ be a planar graph, and let F denote the set of faces. Then, the following holds:*

$$\sum_{f_i \in F} d(f_i) = 2|E|.$$

This lemma is analogous to the handshaking lemma which says that for any graph: $\sum_{v \in V} d(v) = 2|E|$. The handshaking lemma followed from the fact that every edge is counted in exactly 2 degrees – one per endpoint. In the face-edge formula, observe that every edge is counted twice in the sum over the degrees of every face, this corresponds to the two “sides” of an edge. For the non-bridge edges (black edges in Figure 3), each such edge is on the border of two faces and hence is included in the bounding walk for each of these two faces. For the bridge edges (red edges in Figure 3), each such edge is traversed twice for the bounding walk corresponding to the one face containing this edge; for example, the f_2-walk traverses $\{v_5, v_9\}$ twice (once per direction), and the f_{ext}-walk traverses edges $\{v_4, v_10\}$ and $\{v_{10}, v_{11}\}$ twice each.

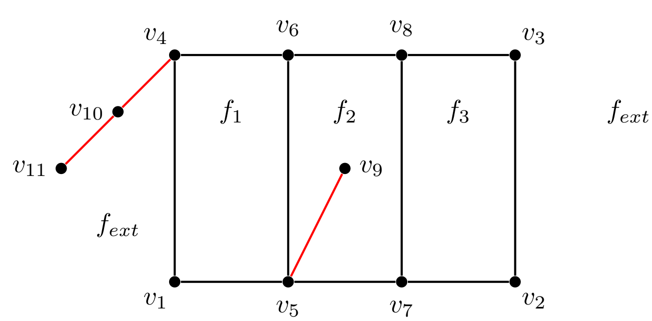


Figure 3: Example planar graph with $v = 11, e = 13, f = 4$.

In a planar graph with ≥ 3 vertices, every face f must satisfy $d(f) \geq 3$. To see this fact, first suppose a face is bounded by a cycle (for example f_1 and f_3 in Figure 3), then the shortest possible cycle in a graph is a triangle, so $d(f) \geq 3$. If a face contains a bridge edge (for example, the red edges in Figure 3), the 'out-and-back' traversal of the red edges ensures the degree is even larger than 3.

Since every face $f \in F$ has degree $d(f) \geq 3$, thus $\sum_{f_i \in F} d(f_i) \geq 3f$ where $f = |F|$, and then combining that with the face-edge formula we obtain:

$$2|E| = \sum_{f_i \in F} \geq 3f, \text{ or in other words } f \leq 2|E|/3.$$

Now let's plug this bound back into Euler's formula and we obtain:

$$e - v + 2 = f \leq 2e/3 \text{ which implies that } 2e \geq 3e - 3v + 6.$$

Solving for e we have the following for any planar graph.

Lemma 5.3. *In any planar graph $G = (V, E)$ with v vertices and e edges, the following holds:*

$$e \leq 3v - 6. \tag{1}$$

5.4 Non-Planar Graphs

Consider K_5, which is the complete graph on 5 vertices. It has $v = 5$ vertices and $e = \binom{5}{2} = 10$ edges. Plugging in $v = 5$ into the upper bound in Equation (1), we see that in any planar graph on 5 vertices, there are at most 9 edges. But K_5 has 10 edges, hence it is not a planar graph.

Now consider $K_{3,3}$. This is the graph depicted in Figure 4. Note the graph is bipartite, which means that the vertices can be partitioned into two sets $L = \{\ell_1, \ell_2, \ell_3\}$ and $R = \{r_1, r_2, r_3\}$, and all edges go between L and R (there are no edges with both endpoints in L and no edges with both endpoints in R).

Observe that this graph $K_{3,3}$ has no triangles; in fact, every bipartite graph has no triangles (and no cycles of odd-length). Hence, in a planar graph which is bipartite, every face $f_i \in F$ has degree $d(f_i) \geq 4$ (instead of the previous bound of $d(f_i) \geq 3$); this is because the shortest cycle length is 4 edges in a bipartite graph.

Consequently, for a planar, bipartite graph we then have that $f \leq e/2$ (instead of the previous bound of $f \leq 2e/3$). Plugging this into Euler's formula and then solving for e as we did before, we now obtain that in any planar, bipartite graph we have:

$$e \leq 2v - 4.$$

Going back to the specific case of $K_{3,3}$, it has $v = 6$ vertices and $e = 9$ edges. But the above bound says that any planar, bipartite graph on 6 vertices has ≤ 8 edges. Therefore, $K_{3,3}$ is not planar.

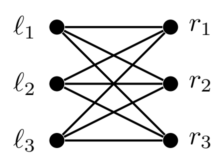


Figure 4: The complete bipartite graph $K_{3,3}$. It has $v = 6$ and $e = 9$. In any drawing, at least two edges must cross.

We have shown that the graphs K_5 and $K_{3,3}$ are not planar. Consider one of these two graphs, say K_5 for concreteness. If we take any edge of K_5 and replace it by a path of length 2 - this operation is called *subdividing* the edge – then the resulting graph is still non-planar as a planar drawing of this new graph would yield a planar drawing of the original K_5. Similarly, if we subdivide any edges and then add additional edges, it is still non-planar. Hence, any graph that contains K_5 or $K_{3,3}$, or which contains a subgraph which is a subdivision of one of these two graphs, is non-planar.

Here is another way to view the above statement for non-planar graphs. What is the opposite operation of subdividing an edge? It's called *contracting* an edge, where we "merge" the endpoints of the edge together into a new vertex which replaces the previous two. More formally, to contract an edge $\{v, w\}$, we replace the vertices v and w by a new vertex z, and the edges $\{v, y\}$ and $\{w, y'\}$ are replaced by $\{z, y\}, \{z, y'\}$, respectively, while maintaing at most one copy of each edge in the new graph. A graph $G = (V, E)$ is non-planar if one can obtain a copy of K_5 or $K_{3,3}$ by removing edges or contracting edges in G.

The converse also holds: if a graph does not contain K_5 or $K_{3,3}$, and one cannot obtain a copy of either graph by removing edges or contracting edges, then it is planar. This is known as Kuratowski's Theorem, which was published in 1930.

5.5 Graph Coloring

Given a graph $G = (V, E)$ and an integer $k \geq 2$, a k-coloring of G is an assignment of a color to each vertex, from a pallette of k colors, such that neighboring vertices get different colors. More formally, a k-coloring is a function $\sigma : V \to \{1, \ldots, k\}$ such that for all $\{v, w\} \in E$, $\sigma(v) \neq \sigma(w)$.

As an example, a bipartite graph has a 2-coloring: use one color for all the vertices in L, and the other color for all of R, and since all edges go between L and R then the endpoints will have different colors. One can also show the converse: if there is a 2-coloring then the graph is bipartite: put all of the vertices with color 1 in L, and all of the vertices with color 2 in R.

For a planar graph there is a 4-coloring. This implies the 4-color theorem mentioned at the beginning of the lecture notes. From any map, we can create a planar graph by creating a vertex in each face, and connecting two vertices by an edge if their corresponding faces share a boundary. The resulting graph is planar and hence a 4-coloring of this planar graph (called the dual of the original map) yields a 4-coloring of the original map by using the vertex colors to color the corresponding faces of the map.

The Four Color Theorem, which states that every planar graph has a 4-coloring, was proved by Kenneth Appel and Wolfgang Haken in 1976 using a computer-assisted proof to check roughly 2,000 cases. Later a simpler proof, still computer-assisted but only roughly 600, was proved in 1996. Interestingly, there was a claimed proof in 1879 by Kempe, which was shown to be flawed in 1890. Kempe's proof does establish the Five Color Theorem. For more on Kempe's failed proof, see this Math Horizons article.

We will show a simpler result: the Six Color Theorem which says that every planar graph has a 6-coloring. We'll point out how to extend this weaker result to obtain a 5-coloring. Further reducing it to 4 colors is much more difficult.

In Equation (1) we showed that in any planar graph $G = (V, E)$,

$$|E| \leq 3|V| - 6 < 3|V|.$$

Recall, that by the handshaking lemma, $\sum_{v \in V} d(v) = 2|E|$. Hence, combining these two facts we have the following for any planar graph $G = (V, E)$,

$$\sum_{v \in V} d(v) < 6|V|. \tag{2}$$

As a consequence, we obtain that a planar graph always has at least one vertex of degree ≤ 5.

Lemma 5.4. *Let $G = (V, E)$ be a planar graph.*

$$\exists v \in V, d(v) \leq 5.$$

Proof. In Equation (2), we showed the following:

$$\text{For a graph } G = (V, E)\text{, if } G \text{ is a planar graph, then } \sum_{v \in V} d(v) < 6|V|.$$

The contrapositive of this statement is the following:

$$\text{For a graph } G = (V, E)\text{, if } \sum_{v \in V} d(v) \geq 6|V|\text{, then } G = (V, E) \text{ is not planar.} \tag{3}$$

We are trying to show the following:

$$G = (V, E) \text{ is planar} \implies \exists v \in V, d(v) \leq 5.$$

The contrapositive is the following:

$$\forall v \in V, d(v) \geq 6 \implies G = (V, E) \text{ is } \textbf{not} \text{ planar.}$$

If every vertex has $d(v) \geq 6$, then $\sum_{v \in V} d(v) \geq 6|V|$ and then by Equation (3) we know that G is not planar, which proves the contrapositive and hence establishes the lemma. □

5.6 The Six Color Theorem

Now we can easily find a 6-coloring of a planar graph using a recursive algorithm, which is really just an inductive proof. Consider an arbitrary planar graph $G = (V, E)$ and apply the following recursive algorithm to G:

1. If $|V| = 1$ then color the one vertex of G using an arbitrary color.
2. If $|V| \geq 2$ then proceed as follows:
   (a) Take a vertex v with degree $d(v) \leq 5$; by the above argument we know there is at least one such vertex.
   (b) Let G' the graph obtained by deleting v and all edges incident to v.
   (c) Recursively run this same 6-coloring algorithm on G' to obtain a 6-coloring of G', and let σ' denote the 6-coloring of G'.
   (d) Now we need to color v to complete the 6-coloring of G. Since $d(v) \leq 5$, then v has ≤ 5 neighbors and hence there is at least one color not used by any neighbor of v. Let c be one of these unused colors by the neighbors of v and set $\sigma(v) = c$, and for all $w \neq v$ set $\sigma(w) = \sigma'(w)$ (so all other vertices maintain the same color).
   (e) We output σ which is a 6-coloring of G.

For those curious about how to extend the argument to obtain a 5-coloring we point you to the lecture notes for CS70 at UC Berkeley.

5.7 Chromatic Number

The chromatic number of a graph is the minimum number of colors needed to color it; it is denoted as $\chi(G)$ for a graph $G = (V, E)$. The Four Color Theorem says that for any planar graph $G = (V, E)$ then $\chi(G) \leq 4$. For a bipartite graph $G = (V, E)$, $\chi(G) \leq 2$ (and it's equality if there is at least one edge).

Some other examples of the chromatic number:

- For the complete graph K_n then $\chi(K_n) = n$.
- For the cycle C_n, then:

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd, } n \geq 3 \end{cases}$$

Here is an illustration of a well-known graph $P = (V, E)$ known as the Petersen graph. Notice that the Petersen graph P has chromatic number 3. To prove that $\chi(P) = 3$, we need to show that $\chi(P) \leq 3$, which we do by showing a 3-coloring of P – try to find a 3-coloring of it. Then we need to show $\chi(P) \geq 3$, which means that there is no 2-coloring of the Petersen graph. How do we prove that? It contains an odd cycle, e.g., $7 - 10 - 5 - 4 - 9 - 7$, and an odd cycle requires at least 3 colors.

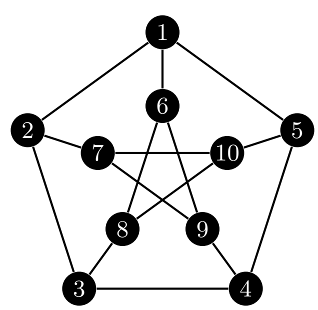


Figure 5: The Petersen graph.