Lecture 9: Pascal's Triangle and Combinatorial Identities

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9 Pascal's Triangle and Combinatorial Identities

We'll begin with a slew of practice problems, then we'll introduce Pascal's triangle and a few combinatorial identities.

9.1 Counting with sequences/permutations

Permutations of people: In the following problem, there are n people labeled $1, 2, \ldots, n$.

1. How many ways of placing the n people in a line?

2. They are going on the Round Up ride at Six Flags. They stand in a circle (and then it rotates and tilts – fast). How many ways of arranging the n people around the circle which has exactly n positions? Two standing arrangements are the same if every person has the same person to their right in both arrangements, and every person has the same person on their left in both arrangements.
(Note: clockwise and counterclockwise orders are considered different.)

   Answer: There are $n!$ permutations of the n people as we saw in the previous problem. But now that they are placed on a circle, it doesn't matter who is in the first position, thus there are n permutations which are the same. Therefore, the answer is $n!/n = (n-1)!$.

3. Alice knows her brother Bob is likely to get sick so she wants to not stand next to him; how many standing arrangements where Alice and Bob are not next to each other?

   Answer: From the previous problem we saw that there are $(n-1)!$ seating arrangements if we ignore the constraint on Alice and Bob. Let's use complement counting. How many arrangements where Alice and Bob sit next to each other? Treat Alice and Bob as one mega-person so we now have $n-1$ "people", and there are then $(n-2)!$ ways to arrange these $n-1$ "people" around the circle. For each such arrangement, we can break up the mega-person in 2 ways: Alice on the left and Bob on the right, or Bob on the left and Alice on the right. Thus, the total number of arrangements with Alice and Bob not sitting next to each other is the following:

   $$(n-1)! - 2 \times (n-2)!$$

   If you do a little algebra you'll notice that our answer can be simplified a bit to the following (not required, just for fun):

   $$(n-1)! - 2 \times (n-2)! = [(n-1) - 2] \times (n-2)! = (n-3)(n-2)!$$

4. From the n people, we want to form a committee of k people, where $1 \leq k \leq n$, and one of those k people must be designated as the "Chairperson." How many ways can this be done?

Answer: There are $\binom{n}{k}$ ways of choosing the committee and then there are k ways to choose the chairperson; hence the number of ways is $k \times \binom{n}{k}$. Alternatively, we can first choose the chairperson – there are n choices – and then we choose the remaining $k-1$ committee members from the $n-1$ remaining people, hence this yields $n \times \binom{n-1}{k-1}$. In summary the answer is

$$k \times \binom{n}{k} = n \times \binom{n-1}{k-1}$$

Rearranging words:

1. Given the word $\mathbf{w} = uncopyrightable$, which has 15 distinct letters, how many sequences can be formed by rearranging the letters of the word $\mathbf{w}$?

   Answer: 15!

2. How many ways can you rearrange the letters in the word $\mathbf{w} = uncopyrightable$ such that *no two vowels are adjacent*? There are **6 vowels**: u, o, y, i, a, e and 9 consonants $n, c, p, r, g, h, t, b, l$.

   Answer: First, arrange the 9 consonants in 9! ways. This creates 10 possible "gaps" (including the ends) – 9 consonants create 10 gaps: _ C _ C _ C _ C _ C _ C _ C _ C _ C _. We choose 6 of these 10 gaps for the 6 distinct vowels, there are $\binom{10}{6}$ ways to do that, and then arrange the vowels in 6! ways. Total:

   $$9! \times \binom{10}{6} \times 6!$$

3. Let $\mathbf{w} = aaaaaabbbb$, which has 6 occurrences of the letter a and 4 occurrences of b. How many sequences can be formed by rearranging the letters of the word $\mathbf{w}$?

   Answer: Here are 3 ways to obtain the answer. Choose the positions for the letter a; there are $\binom{10}{6}$ ways to do that. A second approach is to choose the positions for the letter b; there are $\binom{10}{4}$ ways to do so. Finally, there are 10! permutations of the 10 letters, and we overcounted 6! times due to the 6 copies of a and 4! times due to the letter b, and hence there are 10!/(6!4!) distinct orderings. Note, these 3 answers are all the same:

   $$\binom{10}{6} = \binom{10}{4} = \frac{10!}{6!4!}.$$

4. Let $\mathbf{w} = abracadabra$, which has 11 total letters but the letter a appears 5 times, b appears 2 times, and r appears 2 times (d and c appear once). How many sequences can be formed by rearranging the letters of the word $\mathbf{w}$?

   Answer: There are 11! permutations of the 11 letters but we overcounted by 5! due to the letter a, 2! due to b, and 2! times due to r. Thus, the answer is

   $$\frac{11!}{5! \times 2! \times 2!}.$$

5. The n people from the first problem decide to order 10 donuts: 6 glazed and 4 chocolate. (Oh my!) How many ways can the donuts be distributed to the n people? (Assume each person can receive multiple donuts and the donuts of the same flavor are identical).

   Answer: This is a "Stars and Bars" problem. For the glazed donuts, we distribute 6 items to n people: $\binom{6+n-1}{n-1}$. For the chocolate, we distribute 4 items to n people: $\binom{4+n-1}{n-1}$. Since these are independent choices then we multiply them (this is a key step – independent choices then multiply):

   $$\binom{n+5}{n-1} \times \binom{n+3}{n-1}$$

Chess problems:

Here is a chessboard, which is an 8×8 grid, with two rooks placed in opposite corners of the board. A rook is a type of chess piece. The rules of chess are not important for the following problems.

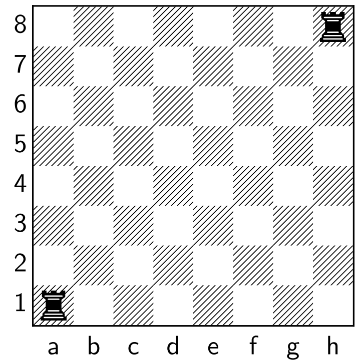

Figure 1: A standard 8x8 chessboard with algebraic notation.

1. How many ways are there to place the two identical rooks on the chessboard so that they are in different rows and different columns?

   Answer: There are 8^2 choices for the first rook. After placing the first rook, then the second rook has 7 choices for their row and 7 choices for their column. This gives $8^2 \times 7^2$ total choices, but we counted each configuration twice since the two rooks are identical. Therefore, the answer is

$$\frac{8^2 7^2}{2} = \frac{(8 \times 7)^2}{2}$$

2. How many ways can you place 8 identical rooks on the chessboard so that no two rooks are in the same row and no two rooks are in the same column.

   Answer: We need to place one rook in each row so that each rook is in a different column. For the first row there are 8 choices. For the second row there are now 7 choices as it must be a different column from the first row. Continuing we see that the answer is 8!.

9.2 Pascal's Triangle

Pascal's triangle is a useful tool for visualizing the binomial coefficients, see Figure 2. The numbers in the triangle are assigned in the following recursive manner: in the top triangle a 1 is placed, then working downwards each triangle adds the numbers in the triangles above-left and above-right, where an empty triangle is treated as 0. For example, in the fourth row, the two middle triangles have a 1 and a 2 above and hence they each have a 3 assigned, whereas the triangles at the end only have a 1 above it.

The topmost row is considered row $n = 0$, the second row is $n = 1$, and so on. The entries in row n are equal to $\binom{n}{k}$ for $k = 0, 1, \ldots, n$ (from left to right).

9.3 Properties of Binomial Coefficients

Here we will explore some nice properties of binomial coefficients. From this point on, many identities will be proven by showing two different ways to count the same collection of objects – this is what we refer to as a combinatorial proof. Many (probably all) of these lemmas can be proved in an algebraic way – which means expanding out each of the binomial coefficients and then proving they're equal, probably by induction

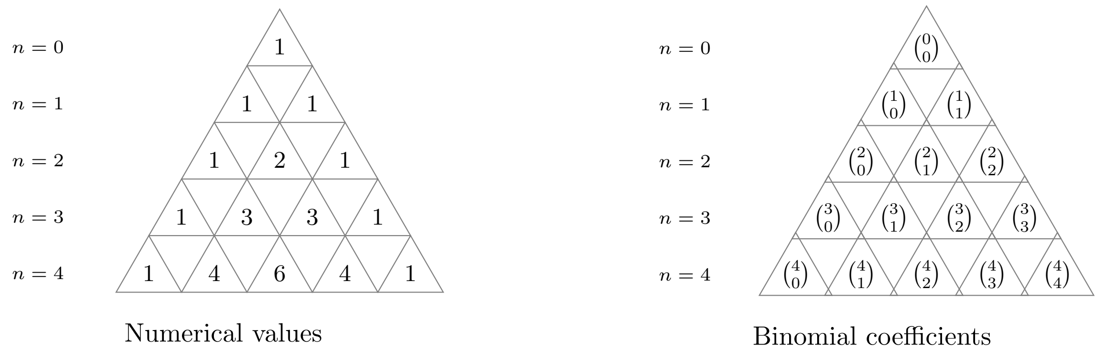

Figure 2: Pascal's Triangle shown with numerical values (left) and binomial coefficients (right).

– but we instead use a combinatorial proof – which is often more intuitive and simpler, we just phrase two different counting approaches for the same problem.

Recall the definition of the binomial coefficients,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{(n)(n-1)\dots(n-k+1)}{(k)(k-1)\dots(1)}.$$

Note, $\binom{n}{0} = 1$ since $0! = 1$. Similarly $\binom{n}{n} = 1$. This hints at a nice symmetry in the binomial coefficients which is stated in the following lemma.

Lemma 9.1 (Symmetry of Binomial Coefficients)**.** *For all integer* $n \geq 0$*, and all integer* $0 \leq k \leq n$*,*

$$\binom{n}{k} = \binom{n}{n-k}.$$

This is easy to prove by simply examining the definition of the binomial coefficients but we will do a combinatorial proof, which shows that the LHS and RHS count the same number of objects, in a different manner.

Proof. Suppose there are n people and we want to choose a committee consisting of k people. Let A denote the collection of all possible sets of committees. More formally, $A = \{S \subset \{1, \dots, n\} : |S| = k\}$, and thus $|A| = \binom{n}{k}$.

Let B denote the collection of all possible non-committees (these are the people not chosen for the committee), thus, $B = \{S \subset \{1, \dots, n\} : |S| = n-k\}$ and $|B| = \binom{n}{n-k}$.

We will define a bijective function $f : A \to B$. Since f is a bijection that shows that $|A| = |B|$, and hence $|A| = |B| = \binom{n}{k} = \binom{n}{n-k}$. The bijection f is defined by, for $S \in A$, $f(S) = \{1, \dots, n\} \setminus S$. Similarly, for $T \in B$, then $f^{-1}(T) = \{1, \dots, n\} \setminus T$, and hence f is a bijection, which proves the lemma. ☐

The following lemma formalizes the recursion defining Pascal's triangle – to get the entry corresponding to $\binom{n}{k}$ we add the above-left $\binom{n-1}{k-1}$ and above-right $\binom{n-1}{k}$. This recurrence is exactly what allows binomial coefficients to be computed efficiently using dynamic programming (DP), which is a topic you'll explore in CS130B but you can experiment now with devising a fast algorithm to compute $\binom{n}{k}$.

Lemma 9.2 (Pascal's Identity)**.**

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof. Once again, let A denote the collection of all possible sets of committees of size k chosen from n people. Fix a particular person $i \in \{1, \dots, n\}$; let B denote the collection of all committees $S \in A$ where $i \in S$, and let B' denote the collection of all committees $S \in A$ where $i \notin S$. Clearly,

$$|A| = |B| + |B'|,$$

since we can define a bijection $f : A \to (B \cup B')$ by $f(S) = S$. The LHS counts the number of ways of choosing k committee members from n people, and thus $|A| = \binom{n}{k}$. On the RHS, the first term counts the number of committees which do not contain person i, and thus we are choosing the k committee members from $n-1$ people and hence $|B'| = \binom{n-1}{k}$. The second term on the RHS calculates $|B| = \binom{n-1}{k-1}$ by assigning person i to the committee and then choosing the other $k-1$ committee members from the remaining $n-1$ people. Therefore, since $|A| = |B'| + |B|$, we have that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. □

Previously, we saw the binomial theorem which we restate here for your convenience.

Theorem 9.3 (Binomial Theorem)**.** *For any integer $n \geq 0$,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

A nice consequence of the binomial theorem is the following result.

Lemma 9.4 (Sum of Binomials)**.**

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

We can obtain this lemma as a corollary of the binomial theorem by setting $x = y = 1$. Here is an alternative proof using a combinatorial approach.

Proof. Let A denote the set of all bit strings of length n, thus $A = \{b_1 b_2 \ldots b_n : \text{for all } 1 \leq i \leq n, b_i \in \{0,1\}\}$. For every $0 \leq j \leq n$, let B_j denote the n-bit strings which have exactly j 1's (and hence $n-j$ 0's). Clearly,

$$|A| = \sum_{j=0}^{n} |B_j|.$$

Let's begin by calculating $|A|$: since each bit b_i has 2 choices and there are n bits, then $|A| = 2^n$. Now for each $0 \leq j \leq n$, to calculate $|B_j|$, there are $\binom{n}{j}$ ways of placing the 1's and then the rest of the bit string is forced to be all 0's, and hence $|B_j| = \binom{n}{j}$. Therefore, $|A| = 2^n = \sum_{j=0}^{n} |B_j| = \sum_{j=0}^{n} \binom{n}{j}$, which proves the lemma. □

Lemma 9.5 (Vandermonde's Identity)**.** *For any non-negative integers m, n, r,*

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

Proof. Suppose we have a group of m men and n women, and we want to form a committee of r people.

- The LHS, $\binom{m+n}{r}$, directly counts the number of ways to choose r people from the total pool of $m+n$ people.
- For the RHS, we can partition the committees based on how many men (k) are chosen. For a fixed k, we choose k men in $\binom{m}{k}$ ways and the remaining $r-k$ committee members from the n women in $\binom{n}{r-k}$ ways.

Summing over all possible values of k from 0 to r gives the total number of ways to form the committee, matching the LHS. □