Coupling:

For an ergodic MC with transition matrix P, state space Ω & stationary distribution π,

a coupling is a joint evolution $(X_t, Y_t) \to (X_{t+1}, Y_{t+1})$

where

$$\Pr(X_{t+1}=j \mid X_t=i, Y_t=k) = P(i,j)$$

$$\& \Pr(\cancel{X}\, Y_{t+1}=l \mid X_t=i, Y_t=k) = P(k,l).$$

Glauber Dynamics for colorings:

Ω = all proper vertex k-colorings of $G=(V,E)$

From $X_t \in \Omega$,

1. Choose random $v \in_R V$ & $c \in_R \{1,\dots,k\}$.
2. For $w \neq v$, $X_{t+1}(w) = X_t(w)$
3. If $c \notin X_t(N(v))$ then $X_{t+1}(v) = c$

   else $X_{t+1}(v) = X_t(v)$.

Identity coupling:

1. Choose same v & c for both chains.

Alternative view:

1. Choose $v \in_R V$ & $r \in_R [0,1]$.
2. For $i \in \{1, \dots, k-1\}$, if $r \in \left[\frac{i-1}{k}, \frac{i}{k}\right)$ then set $c = i$

Hence, can view this coupling as a function

$$f : \Omega \times [0,1] \to \Omega$$

Need that for $r \in_R [0,1]$, for all $x, y \in \Omega$,

$$\Pr(f(x,r) = y) = P(x,y).$$

So the random seed r defines the evolution for all states.

Hence, this is a global coupling.

Another example:

for input graph $G=(V,E)$,

let Ω = all independent sets of G.

Glauber Dynamics:

From $X_t \in \Omega$,

1. Choose $v \in_R V$.
2. Let $X' = \begin{cases} X \cup v & \text{w. prob. } 1/2 \\ X \setminus v & \text{w. prob. } 1/2 \end{cases}$
3. If $X' \in \Omega$ then $X_{t+1} = X'$
   else $X_{t+1} = X_t$

Note, it's symmetric & ergodic so π = uniform(Ω).

Coupling:

From $X_t \in \Omega$,

1. Choose $v \in_R V$ & $r \in_R [0,1]$.
2. Let $X' = \begin{cases} X \cup v & \text{if } r \le 1/2 \\ X \setminus v & \text{if } r > 1/2 \end{cases}$
3. If $X' \in \Omega$ then $X_{t+1} = X'$
   else $X_{t+1} = X_t$.

For bipartite graphs $V = E \cup O$, then it's <u>monotone</u>.

for $X, Y \in \Omega$, say $X \geq Y$ if

$$X \cap E \supseteq Y \cap E \ \& \ X \cap O \subseteq Y \cap O.$$

Note • $E \geq X$ for all $X \in \Omega$

& $X \geq O$ for all $X \in O$,

So it's a partial order.

And if $X_t \leq Y_t$ then the above coupling

ensures that $X_{t+1} \leq Y_{t+1}$.

Intuition: once we've coupled from all pairs of initial states, then we've reached the stationary distribution.

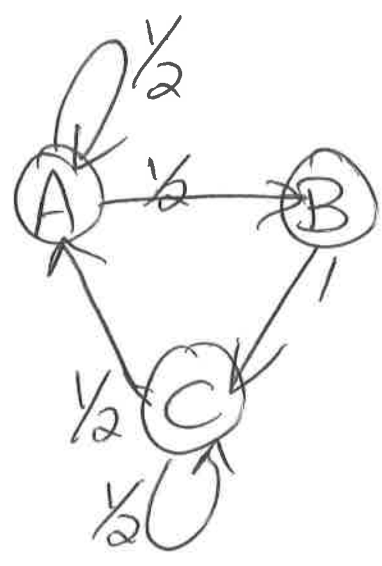
So have $|\Omega|$ chains with distinct starting states.

Run with the global coupling.

Once they've all coupled, output the coupled state. Is this a sample from the stationary dist.?

No.

Simple example:



Let X_T be the final state, when they 1st couple.

Note, $X_T \neq B$ b/c if $X_T = B$ then $X_{T-1} = \cancel{B}A$

So never 1st couple at B.

However $\pi(B) > 0$, so $X_T \not\sim \pi$.

Instead of coupling forward, we couple backwards

For each time t, let $r_t \in_R [0,1]$,

and let $f_t(\cdot\,; r_t)$ be the coupling at time t.

Thus, $X_{t+1} = f(X_t)$.

For $t_1 < t_2$,

let $$F_{t_1}^{t_2}(x) = f_{t_2-1}(\cdots(f_{t_1+1}(f_{t_1}(x))))$$

$$= (f_{t_2-1} \circ f_{t_2-2} \circ \cdots \circ f_{t_1})(x)$$

Note, for $x, y \in \Omega$,

$$\Pr(F_{t_1}^{t_2}(x) = y) = P^{t_2-t_1}(x,y)$$

Forward alg.: (which doesn't always work)

Let T be 1st time when $|F_0^T(\Omega)| = 1$

Backward alg:

Let M be the 1st time when $|F_{-M}^0(\Omega)| = 1$.

Theorem: $F_{-M}^0(\Omega)$ has distribution π.

This is coupling from the past by [Propp-Wilson '96]

<u>Proof</u>: Fix $t>0$, for $x, y \in \Omega$,

$$\Pr\left(F_0^t(x)=y\right)=\Pr\left(F_{-t}^0(x)=y\right)$$

Thus, $\lim_{t\to\infty}\Pr\left(F_{-t}^0(x)=y\right)=\lim_{t\to\infty}\Pr\left(F_0^t(x)=y\right)$

$$=\pi(y)$$

Note, if $\left|F_{-m}^0(\Omega)\right|=1$, ~~then~~ say $F_{-m}^0(\Omega)=x$

then for all $t>m$, for $x\in\Omega$,

$$F_{-t}^0(x) = \left(F_{-m}^0\circ F_{-t}^{-m-1}\right)(x) = F_{-m}^0(x).$$

So, $F_{-m}^0(x)$ is the same as $F_{-\infty}^0(x)$.

For monotone system (such as Bipartite independent sets)

let T_{mix} be the mixing time

& M be the coupling from the past time.

Then, $E[M] \leq 2 T_{mix} \ln(4n)$.