Boolean formula f on n variables $x_1, \dots, x_n$

CNF = conjunctive normal form
conjunction (AND) of clauses &
each clause is the disjunction (OR) of literals

DNF = disjunctive normal form
OR of clauses & each clause is the AND of literals
(so need to satisfy all literals in ≥ 1 clause)

Easy to find a satisfying assignment for a formula f in DNF.

But how many satisfying assignments?

#DNF:

input: formula f in DNF with n variables $x_1, \dots, x_n$ & m clauses $C_1, \dots, C_m$.

output: the # of satisfying assignments.

#P-complete to compute exactly for all f in time $poly(n, m)$.

Can we approximate it?

Strongest form of approximation:

For input f, let $N(f)$ = # of satisfying assignments.

FPRAS = fully polynomial randomized approximation scheme.

Input: formula f, accuracy $\epsilon > 0$, error $\delta > 0$.

Algorithm is an FPRAS if it outputs OUT s.t.

$$\Pr\left(\mathrm{OUT}(1-\epsilon) \leq N(f) \leq \mathrm{OUT}(1+\epsilon)\right) \geq 1-\delta$$

in time $\mathrm{poly}\left(n, \frac{1}{\epsilon}, \log(1/\delta)\right)$.

Suffices to do so with:

$$\Pr\left(\mathrm{OUT}(1-\epsilon) \leq N(f) \leq \mathrm{OUT}(1+\epsilon)\right) \geq 3/4$$

Then can run $O(\log(1/\delta))$ trials,

& take the median of their outputs.

By Chernoff bounds, this is an FPRAS.

Monte Carlo approach:

Set S that we want to estimate $|S|$.

Find trivial set Ω where $\Omega \supset S$ & $|\Omega|$ is known.

Sample from Ω & look at Prob. that it lies in S.

Let $X_1, \dots, X_t$ be uniform, random samples from Ω,

& let $Y_i = \begin{cases} 1 & \text{if } X_i \in S \\ 0 & \text{if not} \end{cases}$

Note, $E[Y_i] = Pr(X_i \in S) = \frac{|S|}{|\Omega|} := \mu$

Look at $Y = \frac{1}{t} \sum_i Y_i$

Note, $|S| = \mu |\Omega|$

Let $\hat{Y} = |\Omega| Y = \frac{|\Omega|}{t} \sum_i Y_i$

How large of t to get a good estimate of $|S|$?

Apply Chernoff bounds:

$$\Pr\left(\left| \frac{|\Omega|}{t} \sum_i Y_i - |S| \right| \geq \epsilon |S| \right)$$

$$= \Pr\left(\left| \frac{1}{t} \sum_i Y_i - \frac{|S|}{|\Omega|} \right| \geq \epsilon \frac{|S|}{|\Omega|} \right)$$

$$= \Pr\left(\left| \frac{1}{t} \sum_i Y_i - \mu \right| \geq \epsilon \mu \right)$$

$$\cancel{\leq 2e}$$

$$= \Pr\left(\left| \sum_i Y_i - t\mu \right| \geq \epsilon t \mu \right)$$

$$= \Pr\left(\left| \sum_i Y_i - \hat{\mu} \right| \geq \epsilon \hat{\mu} \right) \quad \text{where } \hat{\mu} = t\mu = E[Y] = E[\textstyle\sum_i Y_i]$$

$$\leq 2e^{-\hat{\mu}\epsilon^2/3} \quad \text{(by Chernoff bounds)}$$

$$= 2e^{-t\frac{|S|}{|\Omega|}\epsilon^2/3} = 2e^{-t\mu\epsilon^2/3}$$

$$\leq \delta$$

$$\text{for } t \geq \frac{\cancel{4}3\ln(2/\delta)}{\mu\epsilon^2}$$

So only efficient if $\mu = \Omega\left(\frac{1}{\mathrm{Poly}(n)}\right)$

Back to #DNF.

Naive approach:

Generate random assignments $X_1, \dots, X_t$.

Let $Y_i = \begin{cases} 1 & \text{if assignment } X_i \text{ satisfies } f \\ 0 & \text{otherwise.} \end{cases}$

$$E[Y_i] = Pr(Y_i = 1) = \frac{N(f)}{2^n}$$

Let $Z = \frac{2^n}{t} \sum_i Y_i$

& thus $E[Z] = \frac{2^n}{t} \times t \times \frac{N(f)}{2^n} = N(f).$

Note, $\mu = \frac{N(f)}{2^n}$ & thus for $t \geq \frac{3 \ln(2/\delta)}{\mu \epsilon^2}$

$$= \frac{2^n \times 3 \times \ln(2/\delta)}{N(f) \epsilon^2}$$

we have an (ϵ, δ)-approximation scheme.

But it may be that $\frac{2^n}{N(f)}$ is huge.

What to do when:

$$N(f) \ll 2^n ?$$

For clauses $C_1, \ldots, C_m$,

let S_i = assignments which satisfy C_i

& let $\Omega = \bigcup_i S_i$

Note, $N(f) = |\Omega|$.

We can easily sample uniformly at random from S_i by satisfying the literals in C_i & then choosing a random assignment for the remaining variables.

Moreover, if k variables appear in C_i (i.e., $|C_i| = k$) then $|S_i| = 2^{n-k}$.

Consider: $\Lambda = \{(i, \sigma) : \sigma \in S_i\}$

which is the multiset union of the S_i's.

Note, $|\Lambda| \leq m \times |\Omega|$.

We can sample uniformly at random from Λ & then use that to estimate $|\Omega| = N(f)$

For $j = 1 \to t$:

1. Choose $i \in \{1, \dots, m\}$ with prob. $\frac{|S_i|}{Z}$

   where $Z = \sum_i |S_i|$.

2. Choose σ u.a.r. from S_i

3. Let $Y_j = \frac{Z}{N(\sigma)}$ where $N(\sigma)$ = # of clauses satisfied by σ.

Output $Y = \frac{1}{t} \sum_{j=1}^{t} Y_j$

$$E[Y_j] = \sum_i \frac{|S_i|}{Z} \sum_{\sigma \in S_i} \frac{1}{|S_i|} \frac{Z}{N(\sigma)}$$

$$= \sum_i \sum_{\sigma \in S_i} \frac{1}{N(\sigma)} \cancel{= |\Omega| = N(f)}.$$

$$= \sum_{\sigma \in \Omega} \sum_{i : \sigma \in S_i} \frac{1}{N(\sigma)} = \sum_{\sigma \in \Omega} 1 = |\Omega| \cancel{= N(f)}$$

$$= N(f).$$

~~Now by Chernoff bounds,~~

$$\cancel{\Pr\left(|Y - E[Y]| \geq \epsilon E[Y] \right) \leq 2e^{-E[Y]\epsilon^2/3}}$$

Note, Y_j is possibly huge so not clear how to apply Chernoff bounds.

$$E[Y_j^2] = \sum_i \frac{|S_i|}{Z} \sum_{\sigma \in S_i} \frac{1}{|S_i|} \frac{Z^2}{N(\sigma)^2}$$

$$= \sum_i \sum_{\sigma \in S_i} \frac{Z}{N(\sigma)^2}$$

$$= \sum_{\sigma \in \Omega} \sum_{i : \sigma \in S_i} \frac{Z}{N(\sigma)^2}$$

$$= \sum_{\sigma \in \Omega} \frac{Z}{N(\sigma)}$$

$$\leq \sum_{\sigma \in \Omega} Z = Z \sum_{\sigma \in \Omega} 1 = Z \times |\Omega|$$

$$= Z \times N(f).$$

$$\mathrm{Var}(Y_j) = E[Y_j^2] - E[Y_j]^2$$

(because $Z = \sum_i \sum_{\sigma \in S_i} 1 = \sum_{\sigma \in \Omega} \sum_{i : \sigma \in S_i} 1 \leq m \sum_{\sigma \in \Omega} 1 = m N(f)$).

$$\leq Z \times N(f) - N(f)^2$$

$$= N(f)^2 \left(\frac{Z}{N(f)} - 1 \right) \leq N(f)^2 (m-1)$$

~~because $Z \leq m N(f)$~~.

Note, $Y = \frac{1}{t}\sum_{j=1}^{t} Y_j$

Clearly, $E[Y] = E[Y_j] = N(f)$.

& $Var(Y) = \frac{Var(Y_j)}{t} \leq \frac{(m-1)}{t} N(f)^2$

hence $\sqrt{Var(Y)} \leq \frac{\sqrt{m-1}}{\sqrt{t}} N(f)$

$N(f) \geq \sqrt{\frac{Var(Y) \times t}{m-1}}$

By Chebyshev's ineq.

$Pr\left(|Y - N(f)| \geq \epsilon N(f)\right)$

$\leq Pr\left(|Y - N(f)| \geq \epsilon\sqrt{\frac{Var(Y) \times t}{m-1}}\right)$

$\leq Pr\left(|Y - E(Y)| \geq \frac{\epsilon\sqrt{t}}{\sqrt{m-1}} \times \sigma\right)$ $\sigma = \sqrt{Var(Y)}$

$\leq \frac{m-1}{\epsilon^2 t} \leq 1/4$

for $t \geq \frac{4m}{\epsilon^2}$ & then take the median of $O(\log(\frac{1}{\delta}))$ trials.

Network unreliability problem:

input: undirected $G=(V,E)$, & parameter $0 \leq p \leq 1$.

For each edge $e \in E$, independently delete with prob. p. Let H be the resulting subgraph.

Let $FAIL_G(p) = Pr($resulting ~~s~~ subgraph H is disconnected$)$

output: FPRAS for $FAIL_G(p)$.

For graph H, let $c(H)$ = min-cut size in H (# of edges)

Note, $FAIL_G(p) = Pr(E(G) \setminus E(H)$ contains a cut of $G)$

removed a cut so H is disconnected.

Computing $FAIL_G(p)$ exactly is #P-complete.

Note, $FAIL = Pr(E(G) \setminus E(H)$ contains a cut$)$

$\geq Pr(E(G) \setminus E(H)$ contains a specific cut$)$

$\geq p^c$

FPRAs due to [Karger '99]:

Naive scheme:

Run the following experiment: ℓ times:

for each edge, delete w.p. p

let $X_i = \begin{cases} 1 & \text{if resulting graph is disconnected} \\ 0 & \text{if connected} \end{cases}$

$$E[X_i] = \mathrm{FAIL}_G(p)$$

$$E[X] = \cancel{\ell}\,\mathrm{FAIL}_G(p) \quad \text{for } X = \frac{1}{\ell}\sum_{i=1}^{\ell} X_i$$

assume $p^c \geq n^{-4}$ & thus $\mathrm{FAIL} \geq n^{-4}$!

Then for $\ell = \frac{3n^4}{\epsilon^2}\log(2/\delta)$, by Chernoff bounds

X is an (ϵ,δ)-approximation of $\mathrm{FAIL}_G(p)$.

What if p^c is tiny?

Karger's min-cut alg. implies $O(n^2)$ cuts of min-size c.

Moreover, for $\alpha \geq 1$, $O(n^{2\alpha})$ cuts of size $\leq \alpha c$.

(Run Karger's alg. down to 2α, instead of 2, vertices.)

And can enumerate all such small cuts in time $O(n^{2+2\alpha} \log n)$.

Choosing $\alpha = 2 + \ln(2/\epsilon)$ & then cuts larger than αc don't matter (i.e., have total prob. of failing $\leq \frac{\epsilon}{2}$.

Write a DNF with a variable x_e for each edge $e \in G$.
Make a clause for every cut of size $\leq \alpha C$.

~~Count satisf~~

Set each variable to true with prob. p
& false with $1-p$.

What's the probability the resulting formula is satisfied?

Corresponds to counting $\displaystyle\sum_{\text{sat. assig. } \sigma} p^{\text{positive}(\sigma)} (1-p)^{\text{negative}(\sigma)}$

where $\text{positive}(\sigma)$ = # of ~~positive~~ variables set to true in σ
& $\text{negative}(\sigma)$ = # of variables set to false in σ.

Our alg. for #DNF corresponded to $p = 1/2$,
& can easily be generalized to arbitrary p.