Probabilistic method:

Let $\mathcal{G}$ = good event & $\mathcal{B} = \overline{\mathcal{G}}$ = bad event.

For example, for CNF formula f, for a random assignment let $\mathcal{G}$ be the event that f is satisfied.

If we prove that $Pr(\mathcal{G}) > 0$ (or equivalently, $Pr(\mathcal{B}) < 1$) then this shows there exists a satisfying assignment & so f is satisfiable.

A natural approach is to break $\mathcal{G}$ into smaller events $\mathcal{G}_1, \ldots, \mathcal{G}_n$ where $\mathcal{G} = \bigcap_{i=1}^{n} \mathcal{G}_i$

If these $\mathcal{G}_i$ are independent of each other & for each i, $Pr(\mathcal{B}_i) \leq p$ (equivalently, $Pr(\mathcal{G}_i) \geq 1-p$)

then we have that:

since they are indpt.

$$Pr(\mathcal{G}) = Pr\left(\bigcap_{i=1}^{n} \mathcal{G}_i\right) \geq (1-p)^n > 0 \quad \text{if } p<1.$$

equivalently,

$$Pr(\mathcal{B}) = 1 - Pr(\mathcal{G}) < 1-(1-p)^n < 1 \quad \text{if } p<1.$$

$$= Pr\left(\bigcup_i \mathcal{B}_i\right)$$

Often we don't have independence between the events.

LLL = Lovász Local Lemma allows some dependencies.

<u>Definition:</u> For events $\mathcal{E}$ & $\tilde{\mathcal{F}}$,

$\mathcal{E}$ is mutually independent of $\tilde{\mathcal{F}}$ if:

$$\Pr(\mathcal{E} \mid \tilde{\mathcal{F}}) = \Pr(\mathcal{E}).$$

Moreover, for a set of events $(\tilde{\mathcal{F}}_i) = \{\tilde{\mathcal{F}}_1, \dots, \tilde{\mathcal{F}}_\ell\}$

then $\mathcal{E}$ is mutually independent of the set $(\tilde{\mathcal{F}}_i)$

if for all subsets $\tilde{\mathcal{F}} \subset (\tilde{\mathcal{F}}_i) = \{\tilde{\mathcal{F}}_1, \dots, \tilde{\mathcal{F}}_\ell\}$,

$$\Pr(\mathcal{E} \mid \tilde{\mathcal{F}}) = \Pr(\mathcal{E}).$$

<u>Lovász Local Lemma:</u>

Let ~~&~~ $\mathcal{B}_1, \dots, \mathcal{B}_n$ be a set of "bad" events where

for each i:

$$\Pr(\mathcal{B}_i) \le p < 1$$

& $\mathcal{B}_i$ is mutually independent of all but $\le d$ of the other $\mathcal{B}_j$.

Then if $ep(d+1) \le 1$

then: $\Pr(\mathcal{G}) = \Pr\left(\bigcap_{i=1}^{n} \mathcal{G}_i\right) = \Pr\left(\bigcap_{i=1}^{n} \overline{\mathcal{B}_i}\right) > 0$

(or equivalently, $\Pr(\mathcal{B}) = 1 - \Pr\left(\bigcap_i \overline{\mathcal{B}_i}\right) < 1$.

Note, a union bound says $pn < 1$ yields $\Pr(\mathcal{G}) > 0$ so this is much stronger.

Can replace $ep(d+1)$ by $4pd$ which is better for $d \leq 2$ but worse as $d \uparrow$.

Here's an application of LLL:

Lemma: Ek-SAT input f in which no variable appears in more than $\frac{2^{k-2}}{k}$ clauses is satisfiable.

$\Rightarrow$ it doesn't say anything about the # of clauses.

Proof: Note, the LLL condition is $p \leq \frac{1}{e(d+1)}$ ~~$\leq \frac{1}{4d}$~~
we'll prove $p \leq \frac{1}{4d}$ which implies $\uparrow$ when $d \geq e$.

Let ~~B_i~~ B_i = clause i is not satisfied

$p = Pr(B_i) = 2^{-k}$

B_i & B_j only depend on each other if they share at least one variable.

Hence, B_i is dependent on $\leq k\left(\frac{2^{k-2}}{k}\right)$ other clauses since each of the k variables in ~~B~~ B_i appears in $\leq \frac{2^{k-2}}{k}$ other clauses

Thus, $d \leq k\left(\frac{2^{k-2}}{k}\right) = 2^{k-2} = \frac{2^k}{4}$

We have, $p = 2^{-k} \leq \frac{1}{4d} = 2^{-k}$ so LLL implies $Pr(f \text{ is satisfiable}) = Pr(\bigcap_i \overline{B_i}) > 0$ ∎

Lemma: $\forall$ subset $S \subsetneq \{1,\dots,n\}$ & any $i \in \{1,\dots,n\}$

$$\Pr\left(B_i \,\middle|\, \bigcap_{j \in S} \bar{G}_j\right) \le \frac{1}{d+1}$$

Proof: Let $m = |S|$. Induct on m.

Base case: $m=0$:

Thus we are looking at $\Pr(B_i)$ for which we know:

$$\Pr(B_i) \le p \le \frac{1}{e(d+1)} \le \frac{1}{d+1}.$$

For $m>0$:

Let D_i be those $l \in \{1,\dots,n\}$ where B_i depends on B_l.

Partition S into: $S_1 = S \cap D_i$

& $S_2 = S \setminus S_1$.

Note $|S_1| \le d$ since B_i depends on $\le d$ other events

$$\Pr\left(B_i \,\middle|\, \bigcap_{j\in S} G_j\right)$$

$$= \Pr\left(B_i \,\middle|\, \left(\bigcap_{j\in S_1} G_j\right) \cap \left(\bigcap_{l\in S_2} G_l\right)\right)$$

$$= \frac{\Pr\left(B_i \cap \bigcap_{j\in S_1} G_j \,\middle|\, \bigcap_{l\in S_2} G_l\right)}{\Pr\left(\bigcap_{j\in S_1} G_j \,\middle|\, \bigcap_{l\in S_2} G_l\right)}$$

$$\leq \frac{\Pr\left(B_i \,\middle|\, \bigcap_{l\in S_2} G_l\right)}{\Pr\left(\bigcap_{j\in S_1} G_j \,\middle|\, \bigcap_{l\in S_2} G_l\right)}$$

$$\leq \frac{\Pr(B_i)}{\Pr\left(\bigcap_{j\in S_1} G_j \,\middle|\, \bigcap_{l\in S_2} G_l\right)}$$ since B_i is indep't. of S_2.

Need to lower bound the denominator

we'll show the denominator is $> \frac{1}{e}$

$$\leq \frac{\Pr(B_i)}{1/e} \leq ep \leq \frac{1}{d+1}. \quad \blacksquare$$

Let $S_1 = \{j_1, \ldots, j_r\}$

If $r=0$ then $S_1 = \emptyset$ so we know that the denominator is 1 in this case.

Hence we can assume $r>0$ & we know that $r \le d$ since $|S_1| \le d$ as we just pointed out.

Let $\mathcal{F} = \bigcap_{l \in S_2} \mathcal{G}_l$

$$\Pr\left(\bigcap_{j \in S_1} \mathcal{G}_j \,\middle|\, \bigcap_{l \in S_2} \mathcal{G}_l\right) = \Pr\left(\bigcap_{j \in S_1} \mathcal{G}_j \,\middle|\, \mathcal{F}\right)$$

$$= \Pr(\mathcal{G}_{j_1} \mid \mathcal{F}) \times \Pr(\mathcal{G}_{j_2} \mid \mathcal{F}\mathcal{G}_{j_1}) \times \Pr(\mathcal{G}_{j_3} \mid \mathcal{F}, \mathcal{G}_{j_1}, \mathcal{G}_{j_2}) \times \cdots \times \Pr(\mathcal{G}_{j_r} \mid \mathcal{F}, \mathcal{G}_{j_1}, \ldots, \mathcal{G}_{j_{r-1}})$$

$$= \prod_{k=1}^{r} \Pr\left(\mathcal{G}_{j_k} \,\middle|\, \mathcal{F} \cap \bigcap_{k'<k} \mathcal{G}_{j_{k'}}\right)$$

$$= \prod_{k} \left(1 - \Pr\left(B_{j_k} \,\middle|\, \mathcal{F} \cap \bigcap_{k'<k} \mathcal{G}_{j_{k'}}\right)\right)$$

$$\ge \left(1 - \frac{1}{d+1}\right)^{r} \text{ by induction}$$

$$\ge \left(1 - \frac{1}{d+1}\right)^{d} \text{ since } r \le d$$

$$\ge \left(1 - \frac{1}{d} + \frac{1}{2d^2}\right)^{d} > \frac{1}{e}$$ which is our desired lower bound on the denominator.

Now we can prove the Lovász Local Lemma using the lemma we just proved.

We want to prove $\Pr(\mathcal{G})>0$:

$$\Pr(\mathcal{G}) = \Pr\left(\bigcap_{i=1}^{n} \mathcal{G}_i\right)$$

(by the chain rule)

$$= \Pr(\mathcal{G}_1)\times\Pr(\mathcal{G}_2|\mathcal{G}_1)\times\Pr(\mathcal{G}_3|\mathcal{G}_1,\mathcal{G}_2)\times\cdots\times\Pr(\mathcal{G}_n|\mathcal{G}_1,\dots,\mathcal{G}_{n-1})$$

$$= \prod_{i=1}^{n}\Pr\left(\mathcal{G}_i \,\Big|\, \bigcap_{j<i}\mathcal{G}_j\right)$$

$$= \prod_{i=1}^{n}\left(1-\Pr\left(\mathcal{B}_i \,\Big|\, \bigcap_{j<i}\mathcal{G}_j\right)\right)$$

$$\geq \left(1-\frac{1}{\delta+1}\right)^{n} \quad \text{(by the Lemma)}$$

$$>0. \quad \blacksquare$$

but note that this is exp. small so it's unclear how to find such a solution.

Asymmetric Lovász Local Lemma:

For event B_i, let $D_i \subseteq \{B_1, \dots, B_n\}$ denote the dependencies for B_i
(i.e., B_i is independent of $\{B_1, \dots, B_n\} \setminus D_i$)

Note, the original form of LLL required that $|D_i| \le d$.

Theorem: If there exists $x_1, \dots, x_n \in [0,1)$ s.t. for all i,

$$\Pr(B_i) \le x_i \prod_{j \in D_i} (1 - x_j)$$

then,

$$\Pr(\mathcal{G}) = \Pr\left(\bigcap_{i=1}^{n} \overline{B_i}\right) \ge \prod_{i=1}^{n} (1 - x_i) > 0.$$

Proof: Same proof as the original one except for in the lemma replace $\frac{1}{d+1}$ in the RHS by x_i

Note, the original form follows from the asymmetric one by setting $x_i = \frac{1}{d+1}$