Lecture 15: Lovász Local Lemma

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15.1 Lovasz Local Lemma

Definition 15.1 *For events $\mathcal{E}$ and $\mathcal{F}$, $\mathcal{E}$ is mutually independent of $\mathcal{F}$ if:*

$$\Pr(\mathcal{E}|\mathcal{F}) = \Pr(\mathcal{E}).$$

Moreover, for a set of events $(\mathcal{F}_i) = \{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_l\}$, then $\mathcal{E}$ is mutually independent of the set $(\mathcal{F}_i)$ if for all subsets $\mathcal{F} \subset (\mathcal{F}_i) = \{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_l\}$,

$$\Pr(\mathcal{E}|\mathcal{F}) = \Pr(\mathcal{E}).$$

Definition 15.2 *We denote good event as $\mathcal{G}$, for example, in k-SAT problem f, $\mathcal{G}$ is event that f is satisfied by random assignment. If $\Pr(\mathcal{G}) > 0$, then f is satisfiable.*
In k-SAT, $\mathcal{G} = \cap_{i=1}^{n} \mathcal{G}_i$, where $\mathcal{G}_i$ = clause i is satisfied. Given that $Pr(\mathcal{B}_i) \leq p < 1$, where $\mathcal{B}_i = \overline{\mathcal{G}_i}$, and $\mathcal{G}_i$'s are independent, we have that

$$\Pr(\mathcal{G}_i) \geq 1 - p$$

15.1.1 Lemma1

Lemma 15.3 *For Ek-SAT (=CNF) problem f, where clause has size k. With random assignment ($p = \frac{1}{2}$, and $e(d+1)p \leq 1$ where $d \geq e$), if no variable appears in more than $\frac{2^{k-2}}{k}$ clauses, then $\mathcal{F}$ is satisfiable.*
***Observation**: Let $\mathcal{G}_i$ be the i-th clause being satisfied, and $\mathcal{B}_i = \overline{\mathcal{G}_i}$ be the i-th clause not being satisfied. We have*

$$\Pr(\mathcal{B}_i) = 2^{-k}$$

and

$$d \leq k \times \frac{2^{k-2}}{k} = 2^{k-2}$$

therefore, we have

$$\Pr(\mathcal{B}_i) = 2^{-k} = \frac{1}{4d} \leq p \leq \frac{1}{e(d+1)}.$$

15.1.2 Lemma2

Lemma 15.4 *$\forall S \subset \{1, 2, \cdots, n\}$, if $e(d+1)p \leq 1$, then $\Pr(\mathcal{B}_i | \cap_{j \in S} \mathcal{G}_j) \leq \frac{1}{d+1}$. That is, $\Pr(\mathcal{G}_i | \cap_{j \in S} \mathcal{G}_j) \geq 1 - \frac{1}{d+1}$.*

Proof: We prove this lemma by using the principle of mathematical induction. The induction here is applied on the size (cardinality) of the set S. Let $m = |S|$.

Base case $m = 0$: Obviously, we have that $\Pr(\mathcal{B}_i) \leq p \leq \frac{1}{e(d+1)} \leq \frac{1}{d+1}$.

For $m > 0$: let $D_i = \{j : \mathcal{B}_i \text{ depends on } \mathcal{B}_j\}$. Then we have $|D_i| \leq d$. Then we partition S into two parts: $S_1 = S \cap D_i$ and $S_2 = S \backslash S_1$. Note that $S_1 \leq d$ since $\mathcal{B}_i$ depends on $\leq d$ other events.

$$\begin{aligned}\Pr(\mathcal{B}_i | \cap_{j\in S} \mathcal{G}_i) &= \Pr(\mathcal{B}_i | (\cap_{j\in S_1} \mathcal{G}_j) \cap (\cap_{l\in S_2} \mathcal{G}_l)) \\ &= \frac{\Pr(\mathcal{B}_i \cap (\cap_{j\in S_1} \mathcal{G}_j) | \cap_{l\in S_2} \mathcal{G}_l)}{\Pr(\cap_{j\in S_1} \mathcal{G}_j | \cap_{l\in S_2} \mathcal{G}_l)} \\ &\leq \frac{\Pr(\mathcal{B}_i | \cap_{l\in S_2} \mathcal{G}_l)}{\Pr(\cap_{j\in S_1} \mathcal{G}_j | \cap_{l\in S_2} \mathcal{G}_l)} \\ &\leq \frac{\Pr(\mathcal{B}_i)}{\Pr(\cap_{j\in S_1} \mathcal{G}_j | \cap_{l\in S_2} \mathcal{G}_l)}, \text{since } \mathcal{B}_i \text{ is independent of } S_2\end{aligned}$$

Now we need to lower bound the denominator $\Pr(\cap_{j\in S_1} \mathcal{G}_j | \cap_{l\in S_2} \mathcal{G}_l)$ and show that $\Pr(\cap_{j\in S_1} \mathcal{G}_j | \cap_{l\in S_2} \mathcal{G}_l) \geq \frac{1}{e}$.

Let $S_1 = \{j_1, j_2, \cdots, j_r\}$. If $r = 0$, then $S_1 = \emptyset$. So $\Pr(\emptyset | \cap_{l\in S_2} \mathcal{G}_l) = 1 \geq \frac{1}{e}$. Hence, we can assume that $r > 0$ and we know that $r \leq d$ since $|S_1| \leq d$ as we just pointed out.

$$\begin{aligned}\Pr(\cap_{j\in S_1} \mathcal{G}_j | \cap_{l\in S_2} \mathcal{G}_l) &= \Pr(\mathcal{G}_{j_1} | \cap_{l\in S_2} \mathcal{G}_l) \times \Pr(\mathcal{G}_{j_2} | \mathcal{G}_{j_1} \cap (\cap_{l\in S_2} \mathcal{G}_l)) \times \cdots \times \Pr(\mathcal{G}_{j_k} | (\cap_{k'<k} \mathcal{G}_{j_{k'}}) \cap (\cap_{l\in S_2} \mathcal{G}_l)) \\ &= \prod_{k=1}^{r} \Pr(\mathcal{G}_{j_k} | (\cap_{k'<k} \mathcal{G}_{j_{k'}}) \cap (\cap_{l\in S_2} \mathcal{G}_l)) \\ &\geq (1 - \frac{1}{d+1})^r, \text{by induction} \\ &\geq (1 - \frac{1}{d+1})^d \\ &\geq (1 - \frac{1}{d} + \frac{1}{2d^2})^d, \text{since } e^{-\frac{1}{d}} \leq 1 - \frac{1}{d} + \frac{1}{2d^2} \\ &\geq \frac{1}{e}\end{aligned}$$

Therefore, we have that

$$\begin{aligned}\Pr(\mathcal{B}_i | \cap_{j\in S} \mathcal{G}_i) &\leq \frac{\Pr(\mathcal{B}_i)}{\Pr(\cap_{j\in S_1} \mathcal{G}_j | \cap_{l\in S_2} \mathcal{G}_l)} \\ &\leq \frac{\Pr(\mathcal{B}_i)}{\frac{1}{e}} \\ &= \frac{p}{\frac{1}{e}} \\ &= ep \\ &\leq \frac{1}{d+1}\end{aligned}$$

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15.1.3 Lovasz Local Lemma

Lemma 15.5 ***(Lovasz Local Lemma)****: Bad events $\mathcal{B}_1, ..., \mathcal{B}_n$ where: for all i, $\Pr(\mathcal{B}_i) \leq p$, and $\mathcal{B}_i$ is independent of all but less than d other $\mathcal{B}_j$'s.*
If $e(d+1)p \leq 1$, then we have

$$\Pr(\mathcal{G}) = \Pr(\cap_{i=1}^{n} \overline{\mathcal{B}_i}) > 0$$

Proof: By Lemma1 and Lemma2.

$$\begin{aligned} Pr(\mathcal{G}) &= Pr(\cap_{i=1}^{n}\mathcal{G}_i) \\ &= Pr(\mathcal{G}_1) \times Pr(\mathcal{G}_2|\mathcal{G}_1) \times ... \times Pr(\mathcal{G}_n|\mathcal{G}_1, ..., \mathcal{G}_{n-1}) \\ &\geq (1-p) \times (1 - \frac{1}{d+1})^{n-1} \\ &\geq (1 - \frac{1}{d+1})^n \\ &> 0 \end{aligned}$$

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15.2 Asymmetric Lovasz Local Lemma

For event $\mathcal{B}_i$, let $D_i \subseteq \{\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_n\}$ denote the dependencies for $\mathcal{B}_i$, i.e., $\mathcal{B}_i$ is independent of $\{\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_n\} \backslash D_i$.

Theorem 15.6 *If there exists* $x_1, x_2, \cdots, x_n \in [0, 1)$ *such that for all* i, $\Pr(\mathcal{B}_i) \leq x_i \prod_{j \in D_i}(1 - x_j)$, *then* $\Pr(\mathcal{G}) = \Pr(\cap_{i=1}^{n}\overline{\mathcal{B}_i}) \geq \prod_{i=1}^{n}(1 - x_i) > 0$.

Proof: Same proof as the original one except for in the lemma replace $\frac{1}{d+1}$ in the RHS by x_i. Note, the original form follows from the asymmetric one by setting $x_i = \frac{1}{d+1}$. ■

References

[1] Erdős, Paul and Lovász, László Problems and results on 3-chromatic hypergraphs and some related questions. In A. Hajnal et al., editor, *Infinite and Finite Sets*, pages 609-628. North-Holland, Amsterdam, 1975.