Today: Given a graph $G=(V,E)$, sample uniformly at random from $\mathcal{M}(G)$ where $\mathcal{M}(G)$ = all matchings of $G$.

MC on matchings:

From $X_t \in \Omega$,

1. Choose $e=(v,w) \in_R E$.
2. If $e \in X_t$, $X' = X_t \setminus e$.
3. If $v$ & $w$ are unmatched in $X_t$, then $X' = X_t \cup e$.
4. If $v$ is unmatched & $w$ is matched, say $(w,z) \in X_t$, then $X' = X_t \cup e \setminus (w,z)$.
5. Set $X_{t+1} = X'$ with prob. $1/2$ & $X_{t+1} = X_t$ otherwise.

Ergodic & symmetric: stationary is $\pi = \text{uniform}(\mathcal{M}(G))$.

Note, step 4 is not necessary. (can achieve by 2+3).

What if each $M \in \mathcal{M}(G)$ has a weight $w(M) > 0$ and we want $\pi(M) = \frac{w(M)}{Z}$ where $Z = \sum_{M} w(M)$.

Change (5) to:

Set $X_{t+1} = X'$ with prob. $\min\left\{1, \frac{w(X')}{w(X_{t+1})}\right\}$

This is the Metropolis filter & can easily check that $\pi(M)P(M, M') = \pi(M')P(M, M')$ for this

For example, if edges have weights $\lambda(e) > 0$, and we can assign $w(M) = \lambda(M) = \prod_{e \in M} \lambda(e)$.

How to prove rapid mixing, i.e., $T_{mix} = Poly(n)$?

Canonical paths:

For every pair $I, F \in \Omega$, define a path $\gamma_{IF}$ in the graph of the MC $(\Omega, P)$.

For transition $T = M \to M'$, let

$$\text{congestion } \rho(T) = \frac{|cp(T)|}{|\Omega|}$$

where $cp(T) = \{(I,F) : T \in \gamma_{IF}\}$ = set of canonical paths through $T$.

Then, let $\rho = \max_{T} \rho(T)$.

Since $P(M, M') = \Theta\left(\frac{1}{m}\right)$ where $m = |E|$

then conductance $\Phi = \Omega\left(\frac{1}{m\rho}\right)$

& $T_{mix} = O\left(m^2 \rho^2 \log\left(\frac{1}{\pi_{min}}\right)\right)$

What are the canonical paths $\gamma_{IF}$?

First, order the vertices $V = \{v_1, \ldots, v_n\}$.

For $I, F \in \Omega$, look at $I \oplus F$:

each component in $I \oplus F$ is

an alternating cycle,
an alternating path, or
augmenting path.

Can "unwind" each such component by seq. of transitions:

Possibly remove an edge of $I$,
and then sequence of slides (step 4),
and finally an add (possibly).

So order the components by lowest # vertex in each,
and then unwind in order.

Fix $T = M \to M'$.

How many $I, F$ have $\gamma_{IF} \ni T$? ($\gamma_{IF}$ goes through $T$)

Define a mapping $\eta : cp(T) \to \Omega \times E$

which is injective (can invert uniquely),

and hence $|cp(T)| \le |\Omega| \times m$ & $\rho = O(m)$.

Suppose $M' = M \cup e' \setminus e$ & thus let $\hat{M} = M \cup M'$

$= M' \cup e$

$= M \cup e'$.

Let $\eta = \underbrace{(I \cap F)}_{\text{common edges}} \cup \underbrace{(I \oplus F \setminus \hat{M})}_{\text{difference of } T \text{ on } I \oplus F}$

From $\eta$ plus $T = M \to M'$ (& possibly the 1st edge removed on the current cycle)

we can uniquely determine $I$ & $F$.

Let $\mathcal{P}$ = set of perfect matchings

Can we sample from $\mu = \text{Uniform}(\mathcal{P})$?

Appropriate Markov chain?

Need to be connected over $\mathcal{P}$.

Not sure how to do it.

Let $\mathcal{N} = \underbrace{\text{near-perfect matchings}}_{\text{have 2 holes = unmatched vertices}}$

Let $\Omega = \mathcal{P} \cup \mathcal{N}$.

Can we design chain that's ergodic over $\Omega$?

Yes, same chain as before, restricted to $\Omega$ (instead of $\mathcal{M}(G)$).

Can we prove rapid mixing?

Are the canonical paths valid?

For $I \in \mathcal{P}$, $F \in \mathcal{P}$, the path $\gamma_{IF}$ stays in $\Omega$ since $\leq 2$ holes at any time.

For $I \in \mathcal{N}$, $F \in \mathcal{P}$, can ~~u~~ unwind the augmenting path first & then the alternating cycles & this path stays in $\Omega$.

Similarly for $I \in \mathcal{P}$, $F \in \mathcal{N}$.

What about $I \in \mathcal{N}$ & $F \in \mathcal{N}$?

Choose a random $P \in \mathcal{P}$, and go $\gamma_{IP}$ then $\gamma_{PF}$.

How much does this increase the congestion?

For $A \in \mathcal{N}$, $B \in \mathcal{P}$ the expected increase is

$$\frac{|\mathcal{N}|}{|\mathcal{P}|}$$

$|\mathcal{N}|$: choice of the other endpoint

$|\mathcal{P}|$: choice of B.

So we need $\alpha = \dfrac{|\mathcal{N}|}{|\mathcal{P}|} \leq \mathrm{poly}(n)$.

Hence, if $\alpha = \frac{|N|}{|P|} \leq Poly(n)$ then the mixing time is $Poly(n)$.

But stationary distribution is $u = \text{Uniform}(N \cup P)$ and we are interested in $P$?

With prob. $\frac{1}{\alpha}$, ~~the~~ a sample from $u$ is in $P$ & if it is in $P$ it is uniform($P$).

Suppose $deg(v) > \frac{n}{2}$ $\forall v \in V$.

Then $\alpha = O(n^2)$ & so we can generate a random perfect matching in poly($n$) time for dense graphs.

General ~~bipartite~~ graphs?

Corresponds to permanent of 0-1 matrix $\overset{n\times n}{A}$:

$$\mathrm{Per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

permutations of $\{1,\dots,n\}$

Give each matching $M \in \Omega = P \cup N$ a weight based on its hole pattern.

Let $V = L \cup R$ ~~|L|=|R|=n.~~ $|L| = |R| = n$.

For $y \in E$, $z \in O$,

$$\text{let } w(y,z) = \frac{|P|}{|N(y,z)|}$$

where $N(y,z)$ = # of near-perfect matchings with $y$ & $z$ as the unmatched.

For $M \in N(y,z)$, let $\omega(M) = \omega(y,z) = \frac{|\mathcal{P}|}{|N(y,z)|}$

For $M \in \mathcal{P}$, let ~~$\omega(\mathcal{P})$~~ $\omega(M) = 1$

Note, $\omega(N(y,z)) = \sum_{M \in N(y,z)} \omega(y,z) = |\mathcal{P}|$

& $\omega(\mathcal{P}) = |\mathcal{P}|$

Hence, if the stationary distribution is Prop. to $\omega()$ then $\pi(\mathcal{P}) = 1/(n^2+1)$.

Turns out that with these weights then $T_{mix} = poly(n)$.

Why?

Need $\omega(I)\omega(F) \leq \omega(T)\omega(\eta)$

Consider $I, F \in \mathcal{P}$.

Then, $T = M \to M'$ where $M \in N(y,z)$

& $\eta \in N(a,b)$ where $\underbrace{(a,z) \in E, (b,y) \in E}$

Hence, one can show $\omega(y,z)\omega(a,b) \geq 1$ when ↑.

But how to get these weights?

Suppose we have weights $\hat{\omega}$ where:

$$\frac{\omega(y,z)}{\alpha} \le \hat{\omega}(y,z) \le \alpha\,\omega(y,z)$$

Run MC wrt $\hat{\omega}$ to get $\hat{\pi}$.

Note, $\hat{\pi}(N(y,z)) = |N(y,z)|\,\hat{\omega}(y,z)$

& $\hat{\pi}(P) = |P|$

Thus,
$$\frac{\hat{\pi}(N(y,z))}{\hat{\pi}(P)} = \frac{|N(y,z)|}{|P|}\,\hat{\omega}(y,z) = \frac{\hat{\omega}(y,z)}{\omega(y,z)}$$

Therefore,
$$\omega(y,z) = \hat{\omega}(y,z)\,\frac{\hat{\pi}(P)}{\hat{\pi}(N(y,z))}$$

So rough weights $\hat{\omega}$ can be boosted to close-approx. weights $\omega$.

Since can use samples from $\hat{\pi}$ ~~[illegible]~~ to estimate this ratio.

For input graph $G=(L \cup R, E)$

assign activities $\lambda(y,z)=\begin{cases} 1 & \text{if } (y,z)\in E \\ \lambda_i & \text{o/w} \end{cases}$

Start with $\lambda_0=1$ so that it's $K_{n,n}$

Slowly go from $\lambda_0=1>\lambda_1>\cdots>\lambda_N<\frac{1}{n!}\approx 0$

So that the final graph $\lambda_N \approx G$.

Set $\lambda_i=\lambda_{i-1}e^{-1/2n}$

so that for matching $M$, $\lambda_i(M)\geq \frac{\lambda_{i-1}(M)}{2}$.

then $N=O(n^2\log n)$

& $w_{i-1}$ is a 2-approx. of $w_i$

& we can use samples from $\pi_{\lambda_i,w_{i-1}}$ to get a good approx. of $w_i$.