Knapsack:

input: n objects with integer weights $w_1, \dots, w_n$
& integer values $v_1, \dots, v_n$
and a total weight W.

output: subset of objects $S \subset \{1, \dots, n\}$ where:

$$\sum_{i \in S} w_i \leq W \quad \text{(so } S \text{ fits in the knapsack)}$$

& which maximizes $\sum_{i \in S} v_i$ (max total value)

#Knapsack: is the counting version, instead of optimization.
How many feasible subsets?

Let $\Omega = \{ S \subset \{1, \dots, n\} : \sum_{i \in S} w_i \leq W \}$

input: integer weights $w_1, \dots, w_n$ & W
output: $|\Omega|$.

#Knapsack is #P-complete.

Today: FPRAS for #knapsack, due to [Dyer '03].

First an exact algorithm, but exponential time, Using Dynamic Programming.

Let $F(j,k) = $ # of subsets S of $\{1,\dots,j\}$ where $\sum_{i \in S} w_i \leq k$

Our goal is to compute $|\Omega| = F(n, W)$.

<u>Base case:</u> $F(0,k) = \begin{cases} 1 & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$

For $j \geq 1$:

$$F(j,k) = \underbrace{F(j-1,k)}_{\text{don't include } j} + \underbrace{F(j-1,k-w_j)}_{\text{include object } j}$$

Thus, it takes $O(nW)$ time to compute $F(n,W)$, which is a pseudo-polynomial time algorithm since the running time depends on W.

Let's get a faster algorithm which only approximates $|\Omega|$, by scaling the weights & rounding.

Let $w_i' := \left\lfloor \frac{n^2 w_i}{W} \right\rfloor$ & $W' = n^2$

& let Ω' be the set of feasible solutions for this new instance.

We can compute $|\Omega'|$ in $O(n^3)$ time using the DP algorithm.

Note $\Omega \subset \Omega'$.

Why? Consider $S \in \Omega$ so $\sum_{i \in S} w_i \leq W$.

Then, $\sum_{i \in S} w_i' \leq \frac{n^2}{W} \sum_{i \in S} w_i \leq \frac{n^2}{W} W = n^2 = W'$.

<u>Claim:</u> $|\Omega'| \leq |\Omega| \times (n+1)$.

Hence,

$$\frac{|\Omega'|}{n+1} \leq |\Omega| \leq |\Omega'|$$

& so $|\Omega'|$ gives a rough approximation of $|\Omega|$.

We'll then sample from Ω' & apply the standard Monte Carlo framework.

Consider the items sorted $w_1 \le w_2 \le \cdots \le w_n$.

Let b be the largest integer i s.t. $w_i \le \frac{W}{n}$

Note, any subset S of $\{1, \ldots, b\}$ is in Ω:

$$\sum_{i \in S} w_i \le \sum_{i \in S} \frac{W}{n} \le n \times \frac{W}{n} = W.$$

Thus, for $S \in \Omega' \setminus \Omega$ then $S \not\subset \{1, \ldots, b\}$.

Let h be the heaviest element in ~~such a~~ this S.

Let $f(S) = S \setminus \{h\}$.

<u>Claim:</u> $f : \Omega' \to \Omega$.

<u>Proof:</u> Let $\delta_i := \frac{w_i n^2}{W} - w_i'$ (this is the rounding error)

Therefore,

$$w_i = \frac{W}{n^2}\left(w_i' + \delta_i\right)$$

and note that $\delta_i \le 1$.

To see that $f(S) \in \Omega$:

$$\sum_{i \in f(S)} w_i = \frac{W}{n^2} \sum_{i \in f(S)} (w_i' + \delta_i)$$

$$= \frac{W}{n^2}\left(\left(\sum_{i \in S} w_i'\right) - w_k' + \left(\sum_{i \in S} \delta_i\right) - \delta_k\right)$$

$$\leq \frac{W}{n^2}\left(\left(\sum_{i \in S} w_i'\right) + n\right) - w_k$$

$$\leq \frac{W}{n^2} \sum_{i \in S} w_i' \qquad \text{since } w_k > \frac{W}{n}$$

$$\leq \frac{W}{n^2} \times n^2 \qquad \text{since } S \in \Omega'$$

$$= W. \qquad \blacksquare$$

For $S \in \Omega$, how many $S' \in \Omega'$ have $f(S') = S$?

Let ℓ ~~k~~ be the index of the heaviest element in S.

$(n-\ell)$ sets S' map to S by removing an element with weight $> w_\ell$.

& $f(S) = S$.

Thus, $\leq (n+1)$ sets S' have $f(S') = S$.

Now let's use the Monte Carlo approach to get a $(1 \pm \epsilon)$-approximation with prob. $\geq 1-\delta$.

First task: Sample uniformly at random from Ω':

1. Run the DP algorithm to build the table $F(j,k)$.
2. Let $T = \emptyset$, $j = n$, $k = n^2$.
3. While $j > 0$:
   a) with prob. $\dfrac{F(j-1, k-w_j)}{F(j,k)}$:

      set $T = T \cup \{j\}$ & $k = k - w_j$

   b) Set $j = j-1$:

The above alg. samples uniformly from Ω'.

Generate t samples $X_1, \ldots, X_t$ from Ω'.

Let $Y_i = \begin{cases} 1 & \text{if } X_i \in \Omega \\ 0 & \text{if not} \end{cases}$

Note, $E[Y_i] = \frac{|\Omega|}{|\Omega'|} = \mu$ & $\mu \geq \frac{1}{n+1}$

Let $Y = \frac{1}{t} \sum_i Y_i$

& output $\hat{Y} = |\Omega'| \times Y$.

By Chernoff bounds (as we saw last class), with $t \geq \frac{3\epsilon^2}{\mu} \log(2/\delta) \geq 3(n+1)\epsilon^2 \log(2/\delta)$

we have an FPRAS for $|\Omega|$.